Ergänzungen III zur Vorlesung Inverse Probleme Sommersem. 2018

Moderne Konvergenzratentheorie für inkorrekte lineare Operatorgleichungen im Hilbertraum

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Linear inverse problems

X, Y infinite dimensional separable Hilbert spaces,

$A : X \longrightarrow Y$ **injective bounded** linear forward operator

with **non-closed range**, i.e., $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$.

We consider the **ill-posed** linear operator equation

$$\boxed{\quad A\,x \;=\; y \qquad (x \in X,\; y \in Y)\,. \qquad\qquad (*)\quad}$$

Let $Ax^\dagger = y$ and y^δ **noisy data** with $\|y - y^\delta\| \le \delta$.

General linear regularization methods

▷ see, e.g., VAINIKKO 1987, ENGL/HANKE/NEUBAUER 1996

We focus on linear regularization method for $Ax = y \quad (*)$ defined by a piecewise continuous function

$$g_\alpha(t) \quad (0 < t \le a := \|A^*A\|, \;\; 0 < \alpha \le \overline{\alpha} \le a)$$

and distinguish **regularized solutions**

$$x_\alpha \;=\; g_\alpha(A^*A)\, A^*y$$

in the **noiseless** case and

$$x_\alpha^\delta \;=\; g_\alpha(A^*A)\, A^*y^\delta$$

in the case of **noisy data**.

Most famous special case: Tikhonov regularization

$$g_\alpha(t) \;=\; 1/(t+\alpha)$$

Consider **residual (bias) function** for the method g_α

$$\boxed{r_\alpha(t) := 1 - t\, g_\alpha(t)}$$

Standing assumptions for all $0 < t \le a$

$$\begin{array}{lll} (i) & |r_\alpha(t)| \to 0 & \text{as } \alpha \to 0 \\ (ii) & |r_\alpha(t)| \le \gamma_1 & \text{for all } 0 < \alpha \le \overline{\alpha} \\ (iii) & \sqrt{t}\,|g_\alpha(t)| \le \gamma_*/\sqrt{\alpha} & \text{for all } 0 < \alpha \le \overline{\alpha} \end{array}$$

Satisfied for Tikhonov regularization with $\gamma_1 = 1$ and $\gamma_* = 1/2$.

Definition (index function)

A real function $\varphi(t)$ $(0 < t \le \bar{t})$ is called index function if it is continuous, strictly increasing and satisfies the limit condition $\lim\limits_{t \to 0} \varphi(t) = 0$.

▷ HEGLAND 1995, MATHÉ/PEREVERZEV 2003

Index functions play an important role in our studies.

Definition (profile function) ▷ H./MATHÉ 2007

We call an index function $f : (0, \overline{\alpha}] \to (0, \infty)$ profile function for the solution $x^\dagger$ of $(*)$ if it is a majorant of the noiseless error, i.e.,

$$\|x_\alpha - x^\dagger\| \;=\; \|r_\alpha(A^*A)x^\dagger\| \;\le\; f(\alpha) \quad \text{for all } \; 0 < \alpha \le \overline{\alpha}\,.$$

Noise level δ and profile function $f(\alpha)$ determine the total error

$$\|x_\alpha^\delta - x^\dagger\| \;\le\; f(\alpha) + \frac{\gamma_*\,\delta}{\sqrt{\alpha}} \qquad (0 < \alpha \le \overline{\alpha})$$

and with $\Theta(\alpha) := \sqrt{\alpha}\, f(\alpha)$ the order optimal convergence rate

$$\|x_{\alpha(\delta)}^\delta - x^\dagger\| \;=\; \mathcal{O}(f(\Theta^{-1}(\delta))) \qquad \text{as} \quad \delta \to 0$$

for the a priori parameter choice $\alpha(\delta) = \Theta^{-1}(\delta)$.

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General source conditions and convergence rates

Due to ▷ TAUTENHAHN 1998, MATHÉ/PEREVERZEV 2003
convergence rates can be obtained by source conditions

$$\boxed{x^\dagger = \varphi(A^*A)\, w \qquad (w \in X,\ \|w\| \le K) \qquad\qquad (\$)}$$

with general index functions $\varphi(t)$ $(0 < t \le a)$. Then

$$\|x_\alpha - x^\dagger\| \;=\; \|r_\alpha(A^*A)\, \varphi(A^*A)w\|\,.$$

Definition (qualification being an index function)

$\triangleright$ MATHÉ/PEREVERZEV 2003, MATHÉ 2004

An index function $\varphi(t)$ $(0 < t \le a)$ is called a qualification with constant $1 \le \gamma < \infty$ of the regularization method g_α if

$$\sup_{0<t\le a} |r_\alpha(t)|\,\varphi(t) \;\le\; \gamma\,\varphi(\alpha) \qquad \text{for all} \quad 0 < \alpha \le \overline{\alpha}\,.$$

Proposition

Let $x^\dagger$ satisfy the general source condition (\$) and let the index function φ be a qualification with constant $1 \le \gamma < \infty$ of the regularization method g_α. Then

$$f(\alpha) = \gamma\,K\,\varphi(\alpha) \qquad\qquad (0 < \alpha \le \overline{\alpha})$$

is a profile function for the solution $x^\dagger$ of $(*)$.

This implies under ($) a best possible convergence rate

$$\|x_{\alpha(\delta)}^{\delta} - x^{\dagger}\| = \mathcal{O}(\varphi(\Theta^{-1}(\delta))) \qquad \text{as} \quad \delta \to 0$$

with $\Theta(t) = \sqrt{t}\varphi(t)$ $(t > 0)$ which is **order optimal** if

$$\varphi^2[(\Theta^2)^{-1}(t)] \quad (t > 0) \quad \text{is } \textbf{concave},$$

because then the function $K\varphi\left(\Theta^{-1}\left(\frac{\delta}{K}\right)\right)$ characterizes the modulus of continuity of A^{-1} restricted to elements with ($).

Not only for Tikhonov regularization the following is helpful:

Proposition (▷ BÖTTCHER/H./TAUTENHAHN/YAMAMOTO 2006)

The index function $\varphi(t)$ $(0 < t \le a)$ is a **qualification** with constant $\gamma = 1$ for the **Tikhonov regularization** if
(a) $\varphi(t)/t$ is monotonically **decreasing** on $(0, a]$ or
(b) $\varphi(t)$ is **concave** on $(0, a]$. If
(c) $\varphi(t)$ is **operator monotone** on $(0, a]$, then it is a qualification with some $1 \le \gamma < \infty$. If there exists a $\hat{t} \in (0, a)$ such that
(d) $\varphi(t)/t$ is monotonically **decreasing** on $(0, \hat{t}\,]$ or
(e) $\varphi(t)$ is **concave** on $(0, \hat{t}\,]$, then it is a qualification with constant $\gamma = \varphi(a)/\varphi(\hat{t}\,)$.

So we can apply this directly to main classes:

Special case 1 (Hölder convergence rates)

▷ NEUBAUER 1985

For $0 < r \le 1$ the index functions

$$\varphi(t) = t^r \qquad (0 < t < \infty)$$

are concave, where

$$\Theta(t) = t^{r+\frac{1}{2}},$$

and with the a priori parameter choice

$$\alpha(\delta) \sim \delta^{\frac{2}{2r+1}}$$

we have for Tikhonov regularization

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| = \mathcal{O}\left(\delta^{\frac{2r}{2r+1}}\right) \quad \text{as} \quad \delta \to 0\,.$$

Special case 2 (logarithmic convergence rates)

▷ HOHAGE 2000

For parameters $p > 0$ the index functions

$$\varphi(t) \;=\; \frac{1}{\left(\ln\left(\frac{1}{t}\right)\right)^p} \qquad\qquad (0 < t \le a \le 1/e)$$

are concave on $(0, \hat{t}\,]$ with $\hat{t} = e^{-p-1}$.

Using

$$\alpha(\delta) = c_0\,\delta^\kappa \quad (0 < \kappa < 2)$$

we obtain for Tikhonov regularization

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| \;=\; \mathcal{O}\left(\frac{1}{\left(\ln\left(\frac{1}{\delta}\right)\right)^p}\right) \quad \text{as} \quad \delta \to 0\,.$$

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How general are general source conditions?

First we state the insufficiency of Hölder source conditions.

In this context let us consider $H := A^*A$ and $a := \|H\|$.

Proposition

For each injective, bounded, selfadjoint and nonnegative operator $H\colon X \to X$ with nonclosed range there is an element $x^\dagger \in X$ which does not satisfy a Hölder source condition

$$x^\dagger = H^p w \qquad (w \in X,\ \|w\| \le K)$$

for any $0 < p < \infty$ and $0 < K < \infty$.

In an analog manner one cannot expect an affirmative result concerning source conditions to hold if we replace the Hölder scale of source condition by other scales, e.g., the logarithmic scale with functions $\varphi(t) = \log^{-\mu}(2a/t)$ $(0 < t \le a)$ for $\mu \in (0, \infty)$.

In fact, then we could take double logarithmic functions for which failure of representability could be assured. The same would hold true for poly-log's, and so on.

Thus some **essentially different reasoning** must be used.

Theorem ▷ MATHÉ/H. 2008

Let $H : X \to X$ be a compact, injective, selfadjoint and non-negative operator. Then for every $x^\dagger \in X$ and $\varepsilon > 0$ there is an index function φ such that $x^\dagger = \varphi(H)w$, for some $w \in X$ with $\|w\| \le (1+\varepsilon)\|x^\dagger\|$.

We have extended the assertion to **non-compact** and even to **unbounded** selfadjoint operators H:

Theorem ▷ H./MATHÉ/V. WEIZSÄCKER 2009

Let $H : D(H) \subset X \to X$ be an injective, selfadjoint and non-negative operator. Then for every $x^\dagger \in X$ and $\varepsilon > 0$ there is a bounded index function φ such that $x^\dagger = \varphi(H)w$, for some $w \in X$ with $\|w\| \le (1+\varepsilon)\|x^\dagger\|$.

The theorems have various implications. We mention one:

Corollary

Given any $x^\dagger \in X$, there is no maximal smoothness with respect to H under all index functions φ with $x^\dagger \in \mathcal{R}(\varphi(H))$.

Proof: We iterate the argument of the theorem:
For $x^\dagger = \varphi_1(H)w_1,\ w_1 \in X$, we find an index function φ_2 with $v_1 = \varphi_2(H)w_2$ and $w_2 \in X$. But then $x^\dagger \in \mathcal{R}(\varphi_1(H)\varphi_2(H))$, where $\varphi_1(t)\varphi_2(t)$ is an index function with higher decay rate to zero than $\varphi_1(t)$ as $t \to 0$.

Outline

Approximate source conditions and convergence rates: An alternative based on distance functions

We choose a **benchmark index function** $\psi(t)$ $(0 < t \le a)$ being a **qualification** for g_α with constant γ and being **smoother than** $x^\dagger$ with respect to A, i.e., $x^\dagger \notin \mathcal{R}(\psi(A^*A))$.

$$\boxed{d_\psi(R) \;:=\; \inf\{\|x^\dagger - \psi(A^*A)\,v\| \;:\; v \in X,\; \|v\| \le R\} \quad (R \ge 0)}$$

is a **continuous, strictly positive and convex distance function decreasing to zero** as $R \to \infty$ and measuring for $x^\dagger$ the violation of the benchmark source condition.

Theorem ($\triangleright$ H./Mathé 2007)

Under the assumptions stated above we have

$$\|x_\alpha - x^\dagger\| \le \max(\gamma, \gamma_1)\,(d_\psi(R) + R\,\psi(\alpha))$$

for all $R \ge 0$ and $0 < \alpha \le \overline{\alpha}$. Hence we immediately find that

$$f(\alpha) = 2\max(\gamma, \gamma_1)\,\frac{\psi(\alpha)}{\theta^{-1}(\psi(\alpha))} \qquad (0 < \alpha \le \overline{\alpha})$$

is a profile function for the solution $x^\dagger$ noting that

$$\theta(t) := [\mathcal{S}(d_\psi)](t) := t\,d_\psi(1/t) \qquad (t > 0)$$

is an index function.

Proposition ($\triangleright$ HEGLAND/H. 2010)

Let $\Psi : (0,\infty) \to (0,\infty)$ be a convex (resp. concave) function. Then the function $\Phi : (0,\infty) \to (0,\infty)$ defined by

$$\Phi(\mu) := [\mathcal{S}(\Psi)](\mu) := \mu\,\Psi\left(\frac{1}{\mu}\right), \qquad 0 < \mu < \infty,$$

is also convex (resp. concave).

The bijective **involution** $\mathcal{S} = \mathcal{S}^{-1}$ mapping between positive functions over the positive real half-axis **preserves convexity as well as concavity.**

Examples for $d_{1/2}(R)$ with medium benchmark $\psi(t) = \sqrt{t}$:

Situation 1 (logarithmic type decay)

If $d_{1/2}(R)$ **falls to zero very slowly** as $R \to \infty$, i.e.,

$$d_{1/2}(R) \le \frac{C}{(\ln R)^p} \qquad (R_0 \le R < \infty),$$

we have

$$f(\alpha) = \mathcal{O}\left(\frac{1}{\left(\ln\left(\frac{1}{\alpha}\right)\right)^p}\right) \quad \text{as} \quad \alpha \to 0.$$

Examples of PDE problems for that case in ▷ H./YAMAMOTO 2005

Situation 2 (power type decay)

If $d_{1/2}(R)$ **falls as a power** of R, i.e.,

$$d_{1/2}(R) \le \frac{C}{R^{\frac{\mu}{1-\mu}}} \qquad (R_0 \le R < \infty)$$

with $0 < \mu < 1$ and $C > 0$, then we obtain

$$f(\alpha) = \mathcal{O}\left(\alpha^{\frac{\mu}{2}}\right) \quad \text{as} \quad \alpha \to 0.$$

Case study for simple integration operator $\mathbf{X} = \mathbf{Y} = \mathbf{L^2(0,1)}$

$$[A\,x]\,(s) \;=\; \int_0^s x(t)\,dt \qquad\qquad (0 \le s \le 1)$$

$x^\dagger = A^* v$ is satisfied whenever $x^\dagger \in H^1(0,1)$ and $x^\dagger(1) = 0$.

For the family of solutions

$$x^\dagger(t) \;= (t-1)^2 + c\,t \qquad\qquad (0 \le t \le 1)$$

we have a jump in the convergence rate because of

$$\mu_{sup} := \sup\left\{\mu \in (0,1) : d_{1/2}(R) = \mathcal{O}\left(R^{\frac{-\mu}{1-\mu}}\right)\right\} \;=\; \begin{cases} 1 & \text{if } \; c = 0 \\ \frac{1}{2} & \text{if } \; c \neq 0 \end{cases} .$$

Situation 3 (exponential type decay)

Even if $d_{1/2}(R)$ **falls to zero exponentially**, i.e.,

$$d_{1/2}(R) \le C \exp\left(-c\, R^q\right) \qquad (R_0 \le R < \infty)$$

for $q > 0$, $C > 0$ and $c > \frac{1}{2}$, by constrction the benchmark rate $\mathcal{O}(\sqrt{\alpha})$ cannot be obtained. But we have at least

$$f(\alpha) = \mathcal{O}\left(\ln\left(\frac{1}{\alpha}\right)^{1/q} \sqrt{\alpha}\right) \quad \text{as} \quad \alpha \to 0.$$

which is only a little slower than $\mathcal{O}(\sqrt{\alpha})$.

Outline

Range inclusions for estimating the distance function

Specific Assumptions

$\triangleright$ H./YAMAMOTO 2005, DÜVELMEYER/H./YAMAMOTO 2006

- We assume the range inclusion

$$\mathcal{R}(\varrho_1(G)) \subset \mathcal{R}(\psi(A^*A))$$

– for some benchmark function $\psi(t)$ $(0 < t \le a)$ being a qualification of g_α
– and some index function $\varrho_1(t)$ $(0 < t \le \|G\|)$.

- With a selfadjoint linear operator $G : X \to X$ let

$$x^\dagger \in \mathcal{R}(\varrho_2(G))$$

– for some index function $\varrho_2(t)$ $(0 < t \le \|G\|)$.

Theorem

Let exist $0 < \varepsilon \le \|G\|$ such that the quotient function

$$q(t) := \left(\frac{\varrho_1}{\varrho_2}\right)(t) \qquad (0 < t \le \varepsilon)$$

is an index function.

Then we have positive constants C_1, C_2 and C_3 such that

$$d_\psi(R) \;\le\; C_1\, \varrho_2\left(\left(\frac{\varrho_2}{\varrho_1}\right)^{-1}(C_2\, R)\right) \qquad (R_0 \le R < \infty)$$

and

$$f(\alpha) \;=\; C_3\, \varrho_2\left(\varrho_1^{-1}(\psi(\alpha))\right) \qquad (0 < \alpha \le \overline{\alpha})$$

is a profile function for $x^\dagger$.

Example (backward heat equation) ▷ H./YAMAMOTO 2005

Find $x \in X$ from data $y \in Y$ with $X = Y = L^2(\Omega))$ satisfying

$$A : x \mapsto y : \quad \begin{cases} \partial_t u(\kappa, t) = \Delta u(\kappa, t) & (\kappa \in \Omega,\, 0 < t < T), \\ u(\kappa, t) = 0 & (x \in \partial\Omega,\, 0 < t \le T), \\ u(\kappa, 0) = x(\kappa) & (\kappa \in \Omega), \\ u(\kappa, T) = y(\kappa) & (\kappa \in \Omega)\,. \end{cases}$$

A priori solution smoothness: $\quad x^\dagger \in H_0^1(\Omega) \cap H^2(\Omega)\,.$

$G = -\Delta$ with homogeneous boundary conditions, $x^\dagger \in \mathcal{R}(G)$,

$\mathcal{R}(\varrho_1(G)) = \mathcal{R}(A^*)$ with $\varrho_1(t) = e^{-\frac{T}{t}}$ and $\rho_2(t) = t \;\; {\scriptstyle (0<t\le T)}\,.$

Above theorem applies. Profile function is:

$$\boxed{f(\alpha) = \frac{C}{\ln\left(\frac{1}{\alpha}\right)}\,.}$$

Outline

Convergence rates in Tikhonov regularization under different kinds of smoothness conditions

- BÖTTCHER/H./TAUTENHAHN/YAMAMOTO 2005

Now we consider the interplay of a

General solution smoothness condition

$$\boxed{x^\dagger = G\,w \qquad (w \in X;\ \mathcal{R}(G) \neq \overline{\mathcal{R}(G)}) \qquad (\circ)}$$

$G : X \to X$ bounded injective positive selfadjoint linear operator

and

Assumptions A1, A2, A3, A4, A5

that characterize the relative severity of ill-posedness of $(*)$ with respect to the operator G.

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Assumption A1

We have $G = \varphi(A^*A)$ with some index function φ on $[0, \|A\|^2]$.

Assumption A2

We have $\mathcal{R}(G) \subset \mathcal{R}(\varphi(A^*A))$ with some index function φ.

Assumption A3

We have $\|\varrho(G)x\| \le \|Ax\|$ for all $x \in X$ with some index function ϱ on $[0, \|G\|]$.

Assumption A4

We have $\|\varrho(G)x\| \le C\,\|Ax\|$ for all $x \in X$ with some index function ϱ and some constant $C > 0$.

Assumption A5

We have $\mathcal{R}(\varrho(G)) \subset \mathcal{R}(|A|) = \mathcal{R}((A^*A)^{\frac{1}{2}})$ for some index fct. ϱ.

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Interdependencies of assumptions on A with respect to G

$$
\begin{array}{ccccccc}
A1 & \Rightarrow & A3 & \begin{array}{c}\Leftarrow * \\ \Rightarrow\end{array} & A4 & \Leftrightarrow & A5 \\
\Downarrow & & \Downarrow * & & \Downarrow * & & \Downarrow * \\
A2 & & A2 & & A2 & & A2
\end{array}
$$

$\Rightarrow$ implication is always true

$\Rightarrow *$ implication is true under an additional condition

We recall that $x^\dagger = Gw$ $(\circ)$ and

Assumption A2

We have $\mathcal{R}(G) \subset \mathcal{R}(\varphi(A^*A))$ with some index function φ.

imply the general source condition $x^\dagger \in \mathcal{R}(\varphi(A^*A))$.

Then we have for some $K > 0$

$$\boxed{f(\alpha) = K\,\varphi(\alpha) \qquad (0 < \alpha \le \overline{\alpha})\,.}$$

Lemma

Let S and T be selfadjoint bounded linear operators on X and suppose T is injective.

(a) If $\mathcal{R}(S) \subset \mathcal{R}(T)$, then $T^{-1}S$ is bounded on X and we have $\|Sx\| \le C\|Tx\|$ for all $x \in X$ with $C = \|T^{-1}S\|$.

(b) If $\|Sx\| \le C\|Tx\|$ for all $x \in X$ and some constant $C > 0$, then $\mathcal{R}(S) \subset \mathcal{R}(T)$ and $\|T^{-1}S\| \le C$.

Hence we have equivalence of

Assumption A4

We have $\|\varrho(G)x\| \le C\,\|Ax\|$ for all $x \in X$ with some index function ϱ and some constant $C > 0$.

and

Assumption A5

We have $\mathcal{R}(\varrho(G)) \subset \mathcal{R}(|A|) = \mathcal{R}((A^*A)^{\frac{1}{2}})$ for some index fct. ϱ.

Theorem

(a) If assumption A3 holds and the function φ defined by $\varphi(t) = \varrho^{-1}(\sqrt{t})$ for $t \in [0, \varrho^2(\|G\|)]$ has the property that φ^2 can be continued to an **operator monotone** index function on $[0, \|A\|^2]$, then assumption A2 is valid with this φ.
(b) If assumption A3 holds and the function ϱ^{-1} defined on $[0, \varrho(\|G\|)]$ can be continued to an **operator monotone** index function on $[0, \|A\|]$, then assumption A2 is valid with $\varphi(t) = \sqrt{\varrho^{-1}(\sqrt{t})}$ for $t \in [0, \|A\|^2]$.

Assumption A3

We have $\|\varrho(G)x\| \leq \|Ax\|$ for all $x \in X$ with some index function ϱ on $[0, \|G\|]$.

Assumption A2

We have $\mathcal{R}(G) \subset \mathcal{R}(\varphi(A^*A))$ with some index function φ.

Theorem

Let assumption A3 hold and set $\varphi(t) = \varrho^{-1}(\sqrt{t})$.
If moreover φ^2 is **concave** on $[0, \varrho^2(\|G\|)]$, then we have

$$f(\alpha) = \mathcal{O}(\varphi(\alpha)) \qquad \text{as} \quad \alpha \to 0\,.$$

Here we recall:

Assumption A3

We have $\|\varrho(G)x\| \le \|Ax\|$ for all $x \in X$ with some index function ϱ on $[0, \|G\|]$.

Theorem

Let assumption A5 hold and assume that the function $\varrho(t)/t$ is **an index function** in some interval $(0, \varepsilon]$. Then by setting $\varphi(t) = \varrho^{-1}(\sqrt{t})$ we have

$$f(\alpha) \;=\; \mathcal{O}(\varphi(\alpha)) \qquad \text{as} \quad \alpha \to 0\,.$$

Here we recall:

Assumption A5

We have $\mathcal{R}(\varrho(G)) \subset \mathcal{R}(|A|) = \mathcal{R}((A^*A)^{\frac{1}{2}})$ for some index fct. ϱ.

References I

- HOFMANN, B.: Approximate source conditions in Tikhonov-Phillips regularization and consequences for inverse problems with multiplication operators. *Mathematical Methods in the Applied Sciences* **29** (2006), 351–371.
- HOFMANN, B.; YAMAMOTO, M.: Convergence rates for Tikhonov regularization based on range inclusions. *Inverse Problems* 21 (2005), 805–820.
- BÖTTCHER, A.; HOFMANN, B.; TAUTENHAHN, U.; YAMAMOTO, M.: Convergence rates for Tikhonov regularization from different kinds of smoothness conditions. *Applicable Analysis* **85** (2006), 555–578.
- HOFMANN, B.; DÜVELMEYER, D.; KRUMBIEGEL, K.: Approximate source conditions in Tikhonov regularization – new analytical results and numerical studies. *Journal Mathematical Modelling and Analysis* **11** (2006), 41–56.
- HOFMANN, B; MATHÉ, P: Analysis of profile functions for general linear regularization methods. *SIAM J. Numer. Anal.* **45** (2007), 1122-1141.

References II

- HOFMANN, B; MATHÉ, P: Analysis of profile functions for general linear regularization methods. *SIAM J. Numer. Anal.* **45** (2007), 1122-1141.
- HOFMANN, B.; MATHÉ, P.; VON WEIZSÄCKER, H.: Regularization in Hilbert space under unbounded operators and general source conditions. *Inverse Problems* **25** (2009), 115013 (15pp).
- HEGLAND, M.; HOFMANN, B.:
  Errors of regularization under range inclusions using variable Hilbert scales. *Inverse Problems and Imaging* **5** (2011), 619-643.