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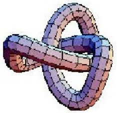
The distinguished role of smoothness in variational regularization for the solution of nonlinear inverse problems in Banach spaces

BERND HOFMANN

TECHNISCHE UNIVERSITÄT CHEMNITZ

TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz, GERMANY



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Mathematical description of nonlinear inverse problems

Let X, Y be infinite dimensional Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Y$, dual spaces X^*, Y^* and dual pairings $\langle\cdot,\cdot\rangle_{X^*\times X}$. Moreover we denote by τ_X, τ_Y topologies in X, Y which are weaker than the norm topology.

$F : \mathcal{D}(F) \subseteq X \longrightarrow Y$ **forward operator** with domain $\mathcal{D}(F)$.

We must treat the operator equation

$$F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y) \qquad (**)$$

with solution $x^\dagger \in \mathcal{D}(F)$ and exact right-hand side $y = F(x^\dagger)$, which is in most cases **ill-posed** and **nonlinear**.

Tikhonov-type regularization

For the stable approximate solution of $(**)$ we consider with convex and stabilizing functional $\mathcal{R} : \mathcal{D}(\mathcal{R}) \subseteq X :\to \mathbb{R}$ and for noisy data y^δ assuming a deterministic noise model

$$\|y - y^\delta\|_Y \leq \delta$$

variational regularization (Tikhonov-type regularization)

$$T_\alpha^\delta(x) := \frac{1}{p}\,\|F(x) - y^\delta\|_Y^p + \alpha\,\mathcal{R}(x) \to \min,$$

subject to $x \in \mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R})$, with exponents $1 \leq p < \infty$, regularization parameters $\alpha > 0$ and minimizers $x_\alpha^\delta \in \mathcal{D}(F)$.

Functional analysis for regularization in Banach spaces

Assumption 1

- X, Y are Banach spaces and $\mathcal{D}(F)$ is a convex subset of X.
- F is weak-to-weak τ_X-τ_Y-sequentially continuous and $\mathcal{D}(F)$ is τ_X-weakly closed, hence F weak-to-weak closed.
- $\mathcal{R}$ is **convex** and τ_X-weakly lower semi-continuous.
- $\mathcal{D} = \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R}) \neq \emptyset$.
- $\mathcal{R}$ is **stabilizing**, which means that for every $c \geq 0$ the sublevel sets

$$\mathcal{M}^{\mathcal{R}}(c) := \{x \in \mathcal{D}(F) : \mathcal{R}(x) \leq c\} \ ,$$

  are τ_X-weakly sequentially pre-compact in the sense that every sequence $\{x_k\}$ in $\mathcal{M}^{\mathcal{R}}(c)$ has a subsequence, which is τ_X-convergent in X to some element from X.

Stabilizing functionals and coercivity

a): For a **reflexive Banach space X** choose **weak convergence** $\rightharpoonup$ as τ_X-convergence.
If $\sup\limits_{x \in \mathcal{M}^{\mathcal{R}}(c)} \|x\|_X < \infty$ for all $c \geq 0$, then $\mathcal{R}$ is stabilizing since the closed unit ball in X is weakly sequentially pre-compact.

b): For a **non-reflexive Banach space $\mathbf{X} = \mathbf{Z}^*$ with predual separable Banach space Z** choose **weak* convergence** $\rightharpoonup^*$ as τ_X-convergence.
If $\sup\limits_{x \in \mathcal{M}^{\mathcal{R}}(c)} \|x\|_X < \infty$ for all $c \geq 0$, then $\mathcal{R}$ is stabilizing since the closed unit ball in X is weak* sequentially pre-compact (sequential Banach-Alaoglu theorem).

An element $x^\dagger \in \mathcal{D}$ is called an $\mathcal{R}$**-minimizing solution** to $(**)$ if

$$\mathcal{R}(x^\dagger) = \min\{\mathcal{R}(x) : x \in \mathcal{D},\ F(x) = y\}\,.$$

$\mathcal{R}$-minimizing solutions always exist under Assumption 1 and attainability, i.e. if, for given $y \in Y$, $(**)$ has a solution $x \in \mathcal{D}$.

Results on **existence, stability and convergence** of $\mathcal{R}$-minimizing solutions $x^\dagger$ and **regularized solutions** x_α^δ for arbitrary $\alpha > 0$ can be found in

▷ H./KALTENBACHER/P./SCHERZER 2007, ▷ PÖSCHL 2008.

We introduce a general non-negative **error measure** $E(x, x^\dagger)$ applied to any approximate solution x for evaluating its quality.

The standard case is the **norm error**

$$E(x, x^\dagger) := \|x - x^\dagger\|_X \,,$$

but in (reflexive) Banach space regularization we often exploit

$$E(x, x^\dagger) := B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger)\,,$$

the **Bregman distance** (cf. ▷ BURGER/OSHER 2004) at $x^\dagger \in \mathcal{D}(\mathcal{R}) \subseteq X$ and $\xi^\dagger \in \partial\mathcal{R}(x^\dagger) \subseteq X^*$ for $\mathcal{R}$ with subdifferential $\partial\mathcal{R}$ defined as

$$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) := \mathcal{R}(x) - \mathcal{R}(x^\dagger) - \langle \xi^\dagger, x - x^\dagger \rangle_{X^* \times X}.$$

$\mathcal{D}_B(\mathcal{R}) := \{x \in \mathcal{D}(\mathcal{R}) : \partial\mathcal{R}(x) \neq \emptyset\}$ is called Bregman domain.

Partially we also need:

Assumption 2

Let F, $\mathcal{R}$, X, Y and $\mathcal{D}$ satisfy Assumption 1.

- There exists an $\mathcal{R}$-minimizing solution $x^\dagger$ which is an element of the Bregman domain $\mathcal{D}_B(\mathcal{R})$.
- There is a bounded linear operator $F'(x^\dagger) : X \to Y$ such that we have for the one-sided directional derivative at $x^\dagger$ and for every $x \in \mathcal{D}$ the equality

$$\lim_{t \to 0+} \frac{1}{t} \Big(F(x^\dagger + t(x - x^\dagger)) - F(x^\dagger) \Big) = F'(x^\dagger)(x - x^\dagger)\,.$$

The operator $F'(x^\dagger)$ has Gâteaux derivative like properties, and there is an adjoint operator $F'(x^\dagger)^* : Y^* \to X^*$

Example: Standard situation in Hilbert spaces

X, Y **Hilbert spaces**,

$\mathcal{R}(x) := \|x - \bar{x}\|_X^2, \qquad x^\dagger$ is called $\bar{x}$-minimum norm solution

$$T_\alpha^\delta(x) := \frac{1}{2}\,\|F(x) - y^\delta\|_Y^2 + \alpha\,\|x - \bar{x}\|_X^2$$

$\mathcal{D}(\mathcal{R}) = \mathcal{D}_B(\mathcal{R}) = X,$ since $\partial\mathcal{R}(x)$ is singleton

$$\xi^\dagger := \mathcal{R}'(x^\dagger) = 2(x^\dagger - \bar{x})$$

$$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) = \|x - x^\dagger\|_X^2.$$

Example: Regularization with differential operators

$$X, Y \ \textbf{Hilbert spaces}, \qquad p := 2$$

$\mathcal{R}(x) := \|Sx\|_X^2$ with unbounded s.a. operator $S : \mathcal{D}(S) \subset X \to X$

$$T_\alpha^\delta(x) := \frac{1}{2}\|F(x) - y^\delta\|_Y^2 + \alpha\,\|Sx\|_X^2$$

$\mathcal{D}(\mathcal{R}) = \widetilde{X}$ Hilbert space with stronger norm $\|x\|_{\widetilde{X}} := \|Sx\|_X$

$$\xi^\dagger := \mathcal{R}'(x^\dagger) = 2S^2x^\dagger$$

$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) = \|S(x - x^\dagger)\|_X^2$ with $\mathcal{D}_B(\mathcal{R}) = \mathcal{D}(S^2)$

Example: Power-type penalties in Banach spaces

$$X, Y \ \textbf{Banach spaces}, \qquad \mathcal{R}(x) := \frac{1}{q}\,\|x\|_X^q,$$

$$T_\alpha^\delta(x) := \frac{1}{p}\,\|F(x) - y^\delta\|_Y^p + \frac{\alpha}{q}\,\|x\|_X^q \qquad (p, q \geq 1)$$

$\mathcal{D}(\mathcal{R}) = \mathcal{D}_B(\mathcal{R}) = X$, since $\mathcal{R}(x)$ is differentiable with

$\xi^\dagger := \mathcal{R}'(x^\dagger) = J_q(x^\dagger)$ with $J_q : X \to X^*$ duality mapping

$$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) = \frac{1}{q}\,\|x\|_X^q - \frac{1}{q}\,\|x^\dagger\|_X^q - \langle J_q(x^\dagger), x - x^\dagger\rangle_{X^* \times X}.$$

Factors influencing the error and link conditions

We search for **convergence rates**

$$E(x_\alpha^\delta, x^\dagger) = \mathcal{O}(\varphi(\delta)) \quad \text{as} \quad \delta \to 0$$

with **index functions** φ.

We call a function $\varphi : (0, \infty) \to (0, \infty)$ index function if it is continuous and strictly increasing with $\lim_{t \to +0} \varphi(t) = 0$.

Rate results require

- Appropriate choices of the regularization parameter **a priori** as $\alpha = \alpha(\delta)$ and **a posteriori** as $\alpha = \alpha(\delta, y^\delta)$.
- The appropriate interplay of all model components.

The attainability of convergence rates will depend on the interplay of the following four relevant ingredients, as these are:

(i) the **smoothness of the solution** $x^\dagger$,
(ii) the **nonlinearity structure** of the forward operator F,
(iii) properties of the **penalty** $\mathcal{R}$,
(iv) and the character of the **error measure** $E(x, x^\dagger)$.

Link conditions are necessary for combining the four factors.

In Hilbert spaces solution smoothness can be expressed by variable Hilbert scales and **general source conditions** (see ▷ PEREVERZYEV, MATHÉ, HEGLAND).

Outline

Now we are back to Banach spaces X and Y from the introduction. For expressing solution smoothness we use **variational source conditions** (variational inequalities) in a form developed independently by FLEMMING and GRASMAIR 2010-11

Assumption 3 (variational source condition - VSC)

We assume to have a constant $0 < \beta \le 1$, *and a* **concave** *index function* φ *such that*

$$\beta\, E(x, x^\dagger) \le \mathcal{R}(x) - \mathcal{R}(x^\dagger) + \varphi(\|F(x) - F(x^\dagger)\|_Y) \quad \textit{for all}\;\; x \in \mathcal{M}.$$

The set $\mathcal{M}$ of the validity of (VSC) must be large enough such that it contains $x^\dagger$ and all regularized solutions x_α^δ under consideration for $0 < \delta \le \delta_{max}$. This is for example the case if $\mathcal{M} = \mathcal{M}^{\mathcal{R}}(\mathcal{R}(x^\dagger) + c)$ for some $c > 0$.

Namely, for any fixed parameter choice $\alpha_* = \alpha_*(\delta)$ or $\alpha_* = \alpha_*(y^\delta, \delta)$ satisfying

$$\alpha_* \to 0 \quad \text{and} \quad \frac{\delta^p}{\alpha_*} \to 0 \quad \text{as} \quad \delta \to 0 \qquad (+)$$

we have convergence for both

$$\mathcal{R}(x^\delta_{\alpha_*}) \to \mathcal{R}(x^\dagger) \quad \text{and} \quad \|F(x^\delta_{\alpha_*}) - F(x^\dagger)\|_Y \to 0 \quad \text{as} \quad \delta \to 0. \quad (++)$$

Moreover if $\delta_n \to 0$ then the regularized solutions $x^{\delta_n}_{\alpha_*}$ converge (in the sense of subsequences) with respect to the (weaker) topology τ_X of X to $\mathcal{R}$-minimizing solutions $x^\dagger$.

On a posteriori choices $\alpha_* = \alpha_*(y^\delta, \delta)$: **discrepancy principles**

A **strong discrepancy principle** was used in the literature for Banach spaces and (VI) ($\triangleright$ RAMLAU ET AL.): For two constants $1 < \tau_1 < \tau_2 < \infty$ the regularization parameter α_* has to satisfy the condition

$$\tau_1 \delta \le \|F(x_{\alpha_*}^\delta) - y^\delta\|_Y \le \tau_2 \delta.$$

Duality gaps may destroy its applicability. To avoid this we suggest to use of the **sequential discrepancy principle (SDP)** for which the variational inequality (VSC) is also strong enough to ensure convergence rates.

Here we restrict the selection of the regularization parameter to a discrete exponential grid. Precisely, we select $0 < q < 1$, choose a parameter $\alpha_0 > 0$ large enough and consider the set

$$\Delta_q := \left\{ \alpha_j : \quad \alpha_j := q^j \alpha_0, \quad j = 1, 2, \ldots \right\}.$$

Definition

For prescribed $\tau > 1$ we say that the regularization parameter $\alpha_* \in \Delta_q$ is chosen according to the sequential discrepancy principle (SDP) if

$$\|F(x_{\alpha_*}^\delta) - y^\delta\|_Y \le \tau\delta < \|F(x_{\alpha_*/q}^\delta) - y^\delta\|_Y.$$

In ▷ ANZENGRUBER, H., MATHÉ 2013 we have proven:

Proposition

For $\alpha_* > 0$ from (SDP) we have

$$\alpha_* \to 0 \quad \text{and} \quad \frac{\delta^p}{\alpha_*} \to 0 \quad \text{as} \quad \delta \to 0 \qquad (+)$$

whenever the **exact penalization veto** is satisfied.

Definition

We say that the exact penalization veto is satisfied if, with the exception of singular cases, for arbitrary $\alpha > 0$ an $\mathcal{R}$-minimizing solution $x^\dagger$ cannot be a minimizer of

$$T_\alpha^0(x) := \frac{1}{p}\,\|F(x) - y\|_Y^p + \alpha\,\mathcal{R}(x) \to \min\,.$$

The veto is often failed in the case $p = 1$.

Convergence rates for a natural a priori parameter choice and for the sequential discrepancy principle

Theorem ▷ HOFMANN/MATHÉ 2012

Suppose that $x^\dagger$ obeys (VSC) for some concave index function φ and some set $\mathcal{M}$.

(i) For $p > 1$ let $\alpha_* = \alpha_*(\delta) > 0$ be selected according to the a priori parameter choice $\alpha_* := \frac{\delta^p}{\varphi(\delta)}$.

(ii) For prescribed $\tau > 1$ let $\alpha_* = \alpha_*(\delta, y^\delta) > 0$ be chosen according to the sequential discrepancy principle (SDP).

Provided that $x_{\alpha_*}^\delta \in \mathcal{M}$ for all $0 < \delta \le \delta_{max}$ and some $\delta_{max} > 0$ we have for both parameter choices (i) and (ii) the convergence rates

$$E(x_{\alpha_*}^\delta, x^\dagger) = \mathcal{O}(\varphi(\delta)), \quad \|F(x_{\alpha_*}^\delta) - F(x^\dagger)\|_Y = \mathcal{O}(\delta), \quad \text{and}$$

$$|\mathcal{R}(x_{\alpha_*}^\delta) - \mathcal{R}(x^\dagger)| = \mathcal{O}(\varphi(\delta)) \qquad \text{as } \delta \to 0.$$

Outline

I. The benchmark case

Here we assume that $x^\dagger \in \mathcal{D}_B(\mathcal{R})$ and the subdifferential $\xi^\dagger \in X^*$ fulfills the **benchmark source condition**

$$\xi^\dagger \;=\; F'(x^\dagger)^*\, v \in \partial\mathcal{R}(x^\dagger), \qquad \text{for some } v \in Y^*. \qquad (\$)$$

Such information allows us to bound for all $x \in X$

$$\begin{aligned} &\langle \xi^\dagger, x^\dagger - x\rangle_{X^*\times X} \\ = &\langle (F'(x^\dagger))^* v, x^\dagger - x\rangle_{X^*\times X} \quad = \langle v, F'(x^\dagger)(x^\dagger - x)\rangle_{Y^*\times Y} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \le \|v\|_{Y^*}\|F'(x^\dagger)(x - x^\dagger)\|_Y. \end{aligned}$$

After adding the term $\mathcal{R}(x) - \mathcal{R}(x^\dagger)$ on both sides this yields that

$$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) \le \mathcal{R}(x) - \mathcal{R}(x^\dagger) + \|v\|_{Y^*}\|F'(x^\dagger)(x - x^\dagger)\|_Y,\; x \in \mathcal{M} := \mathcal{D}(\mathcal{R}).$$

The special case of Hilbert space regularization

X Hilbert space, $\quad \mathcal{R}(x) = \|x - \bar{x}\|_X^2, \quad B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) = \|x - x^\dagger\|_X^2.$

This implies that

$$\|x-x^\dagger\|_X^2 \le \|x-\bar{x}\|_X^2 - \|x^\dagger-\bar{x}\|_X^2 + \|v\|_{Y^*}\|F'(x^\dagger)(x-x^\dagger)\|_Y, \; x \in X,$$

and for a bounded linear operator $F := A : X \to Y$ we have (VSC)

with $\mathcal{M} = X, \; E(x, x^\dagger) = \|x - x^\dagger\|_X^2, \; \beta = 1$ and $\varphi(t) = \|v\|_{Y^*}\, t$.

In this Hilbert space setting for linear ill-posed problems solution smoothness can always be expressed by variational inequalities (VSC) with general index functions φ.

Also in Banach spaces we obtain for bounded linear operators such variational inequalities (VSC) with $\beta = 1$, $E(x, x^\dagger) = B^{\mathcal{R}}_{\xi^\dagger}(x, x^\dagger)$ and $\varphi(t) = \|v\|_{Y^*} t,\ t > 0$ on $\mathcal{M} = X$.

If the mapping F is nonlinear then we may use certain **structure of nonlinearity** to bound $\|F'(x^\dagger)(x - x^\dagger)\|_Y$ in terms of $\|F(x^\dagger) - F(x)\|_Y$.

Provided that

$$\|F'(x^\dagger)(x - x^\dagger)\|_Y \le \sigma(\|F(x) - F(x^\dagger)\|_Y), \quad x \in \mathcal{M}, \qquad (\&)$$

holds for some concave index function σ on some set $\mathcal{M} \subset \mathcal{D}(F)$, then we derive (VSC) on $\mathcal{M}$ with $\beta = 1,\ E(x, x^\dagger) = B^{\mathcal{R}}_{\xi^\dagger}(x, x^\dagger)$ and $\varphi(t) = \|v\|_{Y^*}\sigma(t),\ t > 0$.

An **alternative structural condition** is given in the form

$$\|F(x)-F(x^\dagger)-F'(x^\dagger)(x-x^\dagger)\|_Y \le \eta B^{\mathcal{R}}_{\xi^\dagger}(x,x^\dagger), \quad x\in\mathcal{M}, \quad (\&\&)$$

again for some set $\mathcal{M}\subset\mathcal{D}(F)$ (cf. ▷ RESMERITA, SCHERZER). This allows us to bound

$$\|F'(x^\dagger)(x-x^\dagger)\|_Y \le \eta B^{\mathcal{R}}_{\xi^\dagger}(x,x^\dagger) + \|F(x)-F(x^\dagger)\|_Y,\ x\in\mathcal{M}$$

and further as (VSC) under the **smallness condition**

$$\eta\|v\|_{Y^*} < 1 \qquad (\$\$)$$

with $0<\beta=1-\eta\|v\|_{Y^*}\le 1$, $E(x,x^\dagger)=B^{\mathcal{R}}_{\xi^\dagger}(x,x^\dagger)$ and $\varphi(t)=\|v\|_{Y^*}t,\ t>0$ on $\mathcal{M}$.

II. Violation of the benchmark

If the source condition (\$) is violated then we may use the **method of approximate source conditions** to derive variational using the **distance function**

$$d_{\xi^\dagger}(R) := \inf\{\|\xi^\dagger - \xi\|_{X^*} : \xi = F'(x^\dagger)^* v,\ v \in Y^*,\ \|v\|_{Y^*} \le R\},$$

which is nonincreasing, continuous and concace for all $R > 0$ and should obey the limit condition

$$d_{\xi^\dagger}(R) \to 0 \quad \text{as} \quad R \to \infty.$$

As mentioned in Boţ/Hofmann 2010 this is the case when $F'(x^\dagger)^{**} \colon X^{**} \to Y^{**}$ is injective.
Additionally this approach presumes q-coercivity

$$B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) \ge c_q \, \|x - x^\dagger\|_X^q \qquad \text{for all} \quad x \in \mathcal{M}, \quad q \ge 2,\ c_q > 0.$$

Such assumption is for example fulfilled if $\mathcal{R}(x) := \|x\|_X^q$ and X is a q-convex Banach space.

Then, for $R > 0$ one can find $v_R \in Y^*$ and $u_R \in X^*$ such that

$$\xi^\dagger = \left(F'(x^\dagger)\right)^* v_R + u_R \qquad \text{with} \qquad \|v_R\|_{Y^*} = R, \;\; \|u_R\|_{X^*} \le d_{\xi^\dagger}(R)\,,$$

and we can estimate for all $R > 0$ and $x \in \mathcal{M}$ as

$$\begin{aligned} -\langle \xi^\dagger, x - x^\dagger \rangle_{X^* \times X} &= -\langle \left(F'(x^\dagger)\right)^* v_R + u_R, x - x^\dagger \rangle_{X^* \times X} \\ &= -\langle v_R, F'(x^\dagger)(x - x^\dagger) \rangle_{Y^* \times Y} + \langle u_R, x^\dagger - x \rangle_{X^* \times X} \\ &\le R\,\|F'(x^\dagger)(x - x^\dagger)\|_Y + d_{\xi^\dagger}(R)\,\|x - x^\dagger\|_X\,. \end{aligned}$$

Adding again the difference $\mathcal{R}(x) - \mathcal{R}(x^\dagger)$ gives for $x \in \mathcal{M}$

$$B^{\mathcal{R}}_{\xi^\dagger}(x, x^\dagger) \le \mathcal{R}(x) - \mathcal{R}(x^\dagger) + R\,\|F'(x^\dagger)(x - x^\dagger)\|_Y + d_{\xi^\dagger}(R)\,\|x - x^\dagger\|_X.$$

Using Young's inequality and the q-coercivity, for the linear case $F'(x^\dagger) = A$, we equilibrate the second and the third term, depending of R and $d_{\xi^\dagger}(R)$, respectively, by means of the auxiliary continuous and strictly decreasing function

$$\Phi(R) := \frac{\left(d_{\xi^\dagger}(R)\right)^{q^*}}{R}\,, \quad R > 0, \quad \frac{1}{q} + \frac{1}{q^*} = 1\,.$$

By setting $R := \Phi^{-1}\left(\|A(x - x^\dagger)\|_Y\right)$ and introducing the index function

$$\zeta(t) := \left[d_{\xi^\dagger}(\Phi^{-1}(t))\right]^{q^*} \quad (t > 0)$$

we get again a variational inequality (VSC):

$$\beta B_{\xi^\dagger}^{\mathcal{R}}(x, x^\dagger) \le \mathcal{R}(x) - \mathcal{R}(x^\dagger) + \hat{K}\,\zeta(\|A(x - x^\dagger)\|_Y), \quad x \in \mathcal{M}.$$

Outline

No common source conditions but variational inequalities in ℓ^1-regularization when the sparsity assumption fails

▷ Burger/Flemming/H. 2012/2013 and ▷ Boţ/H. 2013

Under a sparsity expectation we consider for $X = \ell^1 = (c_0)^*$ with the weak*-topology as τ_X in ℓ^1 and $F : \mathcal{D}(F) \subseteq \ell^1 \to Y$ ℓ^1-regularized solutions x_α^δ as minimizers of

$$T_\alpha^\delta(x) := \frac{1}{p} \|F(x) - y^\delta\|_Y^p + \alpha \|x\|_{\ell^1} \to \min .$$

We are searching for convergence rates with respect to the ℓ^1-norm minimizing solution $x^\dagger$.

Benchmark source conditions and approximate source conditions are not applicable.

The situation of ℓ^1-regularization under consideration

Assumption 4

(a) $x^\dagger \in \ell^1$, **but the sparsity assumption fails**, i.e. $x^\dagger \notin \ell^0$;

(b) $F'(x^\dagger)\, e_k \rightharpoonup 0$ for all $k \in \mathbb{N}$;

(c) $e_k = (F'(x^\dagger))^* f_k$ for some $f_k \in Y^*$ and all $k \in \mathbb{N}$.

Theorem

Under the nonlinearity condition

$$\|F'(x^\dagger)(x - x^\dagger)\|_Y \le \sigma(\|F(x) - F(x^\dagger)\|_Y) \qquad (\&)$$

valid for all $x \in \mathcal{M} := \mathcal{M}^{\|\cdot\|_{\ell^1}}(c)$, *some concave index function* σ *and some* $c > \|x^\dagger\|_{\ell^1}$ *we have a variational inequality*

$$\|x - x^\dagger\|_{\ell^1} \le \|x\|_{\ell^1} - \|x\|_{\ell^1} + \varphi(\|F(x) - F(x^\dagger)\|_Y) \quad \textit{for all} \;\; x \in \mathcal{M}$$

with the concave index function

$$\varphi(t) = 2 \inf_{n \in N} \left(\sum_{k=n+1}^{\infty} |x_k^\dagger| + \left(\sum_{k=1}^{n} \|f_k\|_{Y^*} \right) \sigma(t) \right).$$

Example: Hölder rates

Consider a polynomial decay and growth $\sigma(t) \le K_3\, t^\kappa,\ t > 0,$

$$\sum_{k=n+1}^{\infty} |x_k^\dagger| \le K_1\, n^{-\mu}, \qquad \sum_{k=1}^{n} \|f_k\|_{Y^*} \le K_2\, n^\nu,$$

with exponents $0 < \kappa \le 1$, $\mu, \nu > 0$ and corresponding constants $K_1, K_2, K_3 > 0$. Then by setting $n^{-\mu} \sim n^\nu t^\kappa$ and hence $n \sim t^{\frac{-\kappa}{\nu+\mu}}$ we obtain the Hölder convergence rates

$$\|x_{\alpha_*}^\delta - x^\dagger\|_{\ell^1} = \mathcal{O}\left(\delta^{\frac{\mu\kappa}{\mu+\nu}}\right) \qquad \text{as} \qquad \delta \to 0$$

whenever the regularization parameter $\alpha_* = \alpha(\delta, y^\delta)$ is chosen according to the (SDP). The best possible rate arises from the limit case $\kappa = 1$ expressing the tangential cone condition.

Example: exponentially decaying solution components

In contrast to the last example we consider now an exponential decay of the solution components

$$\sum_{k=n+1}^{\infty} |x_k^\dagger| \le K_1 \, \exp\left(-n^\gamma\right), \qquad \sum_{k=1}^{n} \|f_k\|_{Y^*} \le K_2 \, n^\nu,$$

with exponents $\gamma, \nu > 0$ and corresponding constants $K_1, K_2 > 0$. For simplicity let $\sigma(t) \le K_3 \, t$, only.

By setting $n^\gamma \sim \log(1/t)$ and hence $\exp\left(-n^\gamma\right) \sim t$ the rate

$$\|x_{\alpha_*}^\delta - x^\dagger\|_{\ell^1} = \mathcal{O}\left(\delta \left(\log\left(\frac{1}{\delta}\right)\right)^{\frac{\nu}{\gamma}}\right) \qquad \text{as} \qquad \delta \to 0$$

holds for α_* from (SDP). The factor $\left(\log\left(\frac{1}{\delta}\right)\right)^{\frac{\nu}{\gamma}}$ prevents

$$\|x_{\alpha_*}^\delta - x^\dagger\|_{\ell^1} = \mathcal{O}\left(\delta\right) \qquad \text{as} \qquad \delta \to 0,$$

the rate which occurs for sparse solutions $x^\dagger \in \ell^0$.

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