Ill-posedness concepts and the distinguished role of smoothness in regularization for linear and nonlinear inverse problems

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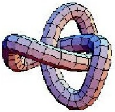

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Outline

1. Introduction
2. Two examples of nonlinear problems
3. Degree of ill-posedness for compositions with non-compact linear operators
4. The mystery of the Hausdorff moment operator in a composition with the compact integration operator

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Let X and Y be infinite dimensional Hilbert or Banach spaces.

We consider **operator equations** modelling inverse problems,

and distinguish **linear inverse problems**

$$\boxed{Ax = y \qquad (x \in X,\ y \in Y), \qquad (*)}$$

with a **bounded linear forward operator** $A \in \mathcal{L}(X, Y)$,

and **nonlinear inverse problems**

$$\boxed{F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y), \qquad (**)}$$

where $F : \mathcal{D}(F) \subseteq X \longrightarrow Y$ is a **nonlinear forward operator** with **domain** $\mathcal{D}(F)$.

The **forward operators** A and F are typically **'smoothing'**, i.e., information about the solution $x^\dagger$ of $(*)$ and $(**)$ is erased at the transition to y: **ill-posedness phenomenon.**

Considering for simplicity the **deterministic noise model**

$$\|y - y^\delta\|_Y \le \delta, \qquad (\textit{Noise})$$

regularization is to find **stable** approximations to $x^\dagger$ from y^δ, where **objective and subjective a priori information** helps to suppress the negative consequences of ill-posedness.

Local ill-posedness for nonlinear problems $(**)$

▷ B.H. AND O. SCHERZER: Factors influencing the ill-posedness of nonlinear problems. *Inverse Problems* **10** (1994), pp. 1277–1297.

Definition

The equation $(**)$ is called **locally well-posed** at the solution point $x^\dagger \in \mathcal{D}(F)$ if there is a ball $B_r(x^\dagger)$ around $x^\dagger$ with radius $r > 0$ such that for each sequence $\{x_n\}_{n=1}^\infty \subset B_r(x^\dagger) \cap \mathcal{D}(F)$

$$\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\|_Y = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} \|x_n - x^\dagger\|_X = 0$$

holds true. Otherwise $(**)$ is called **locally ill-posed** at $x^\dagger$.

For an application of this ill-posedness concept see:

▷ A. KIRSCH, A. RIEDER: Seismic tomography is locally ill-posed. *Inverse Problems* **30** (2014), 125001 (7pp).

Nashed's ill-posedness concept for linear problems $(*)$ in Hilbert spaces

▷ M. Z. NASHED: A new approach to classification and regularization of ill-posed operator equations. In: H. W. Engl and C. W. Groetsch (Eds.), Inverse and Ill-posed Problems (Sankt Wolfgang, 1986), volume 4 of Notes Rep. Math. Sci. Engrg., pp. 53–75. Academic Press, Boston, MA, 1987.

Definition

The linear operator equation $(*)$ is called **well-posed** if the range $\mathcal{R}(A)$ of A is a closed subset of Y. Consequently it is called **ill-posed** if the range is not closed, i.e. $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}^{Y}$. In the ill-posed case, the equation $(*)$ is called **ill-posed of type I** if the range $\mathcal{R}(A)$ contains an infinite dimensional closed subspace, and alternatively **ill-posed of type II** if A is compact.

Degree of ill-posedness is verified for type II (A compact) from **decay rate** of **singular values** $\sigma_i(A) \to 0$ as $i \to \infty$.

Definition

If there is a constant $C > 0$ and an exponent $\kappa > 0$ such that

$$\sigma_i(A) \ge C\, i^{-\kappa} \quad (i = 1, 2, ...)\,, \qquad (\$)$$

we call the operator equation $(*)$ **moderately ill-posed** of degree at most κ, and in particular for $\sigma_i(A) \asymp i^{-\kappa}$ of degree κ. If ($) does not hold for arbitrarily large $\kappa > 0$, we call the operator equation $(*)$ **severely ill-posed**.

Typical for severe ill-posedness is **exponential ill-posedness**.

For decreasing sequences $s_i \ge 0$ and $t_i \ge 0$ we say that $s_i \asymp t_i$ ($i \in \mathbb{N}$) if there are constants $0 < \underline{c} \le \overline{c} < \infty$ such that $\underline{c}\, s_i \le t_i \le \overline{c}\, s_i$ ($i \in \mathbb{N}$).

Ill-posedness of type I: Hausdorff moment problem $(*)$ with non-compact forward operator $A: L^2(0,1) \to \ell^2$ defined as

$$[Ax]_j := \int_0^1 t^{j-1} x(t)\, dt \qquad (j = 1, 2, ...).$$

Ill-posedness of type II: r-times fractional differentiation $(*)$ with compact Volterra operator $A: L^2(0,1) \to L^2(0,1)$ as

$$[Ax](s) := \int_0^s \frac{(s-t)^{r-1}}{\Gamma(r)} x(t)\, dt \qquad (0 \le s \le 1).$$

For all $r > 0$, fractional differentiation is ill-posed of degree r.

Proposition

The linear operator equation $(*)$ is either **locally well-posed everywhere** on X (which is the case if the equation is well-posed in the sense of Nashed and if moreover the null-space of A is trivial), or the linear operator equation $(*)$ is **locally ill-posed everywhere** on X.

Proof: Evidently, by definition we see that $(*)$ is locally ill-posed everywhere if $\mathcal{N}(A) \neq \{0\}$. In the case $\mathcal{N}(A) = \{0\}$, local well-posedness at $x^\dagger$ is valid if and only if the implication

$$\|A(x_n - x^\dagger)\|_Y \to 0 \quad \Longrightarrow \quad \|x_n - x^\dagger\|_X \to 0 \quad \text{as} \quad n \to \infty$$

holds whenever $\|x_n - x^\dagger\|_X \le r$. This implication, however, is valid if and only if the inverse operator $A^{-1} : \mathcal{R}(A) \to X$ is bounded, which just characterizes the situation of a closed range $\mathcal{R}(A) = \overline{\mathcal{R}(A)}^Y$.

Further selected references on ill-posedness concepts:

▷ H. W. ENGL, M. HANKE AND A. NEUBAUER: *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

▷ O. SCHERZER, M. GRASMAIR, H. GROSSAUER, M. HALTMEIER, F. LENZEN: *Variational Methods in Imaging*. Springer, New York, 2009.

▷ T. SCHUSTER, B. KALTENBACHER, B. HOFMANN, K. S. KAZIMIERSKI: *Regularization Methods in Banach Spaces*. Walter de Gruyter, Berlin/Boston, 2012.

▷ B.H. AND R. PLATO: On ill-posedness concepts, stable solvability and saturation. *J. Inverse Ill-Posed Probl.* **18** (2018), pp. 287–297.

▷ P. MATHÉ, B.H. AND M. T. NAIR: Regularization of linear ill-posed problems involving multiplication operators. *Appl. Anal.* **101** (2022), pp. 714–732.

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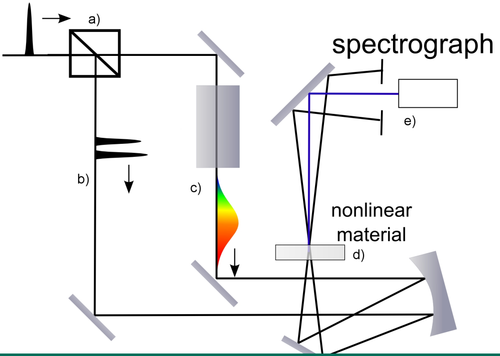
Example 1: A problem in short-term laser optics

SPIDER = Spectral Phase Interferometry for Direct Electric Field Reconstruction

Special version **Self-Diffraction (SD) SPIDER** was developed by **Max Born Institute for Nonlinear Optics, Berlin**



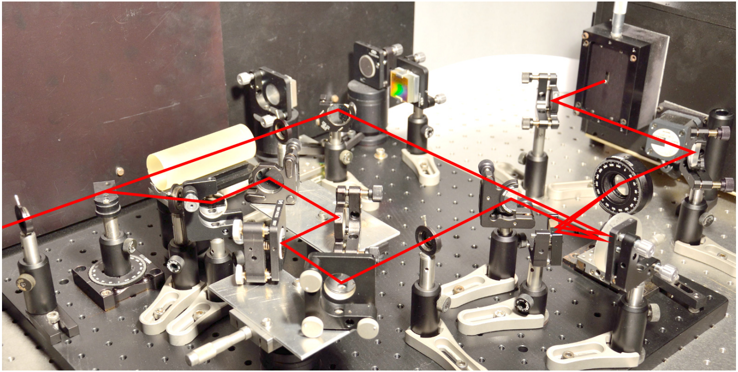


Figure: Measurement setup in self-diffraction spectral interferometry.

The physical model leads to an **autoconvolution problem**

$$\int_{\max(s-1,0)}^{\min(s,1)} k(s,t)x(s-t)x(t)dt = y(s) \quad (0 \le s \le 2) \quad (**)$$

with the corresponding **nonlinear forward operator**

$$F : X = L^2_{\mathbb{C}}(0,1) \to Y = L^2_{\mathbb{C}}(0,2).$$

We have to determine a **complex-valued** function x (characteristics of a short-term - femtosecond - laser pulse) from complex-valued measurement data of y, where the complex-valued continuous kernel k is available.

$\triangleright$ J. FLEMMING: *Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization. New Results in Modern Theory of Inverse Problems and an Application in Laser Optics*. Frontiers in Mathematics. Birkhäuser, Cham, 2018.

Let us consider for simplicity the case of a trivial kernel $k \equiv 1$ as

$$[F(x)](s) := \int_{\max(s-1,0)}^{\min(s,1)} x(s-t)x(t)dt = y(s) \quad (0 \le s \le 2) \quad (**).$$

Proposition

This equation $(**)$ is **locally ill-posed** everywhere on $L^2_{\mathbb{C}}(0,1)$.

Proof idea: We consider on $X = L^2_{\mathbb{C}}(0,1)$ the sequence $x_n = x^\dagger + \Delta_n$ for $\Delta_n(t) = r\, e^{i\, n^2 t^2}$ with $\|\Delta_n\|_X = r$, $\Delta_n \rightharpoonup 0$ and $\|F(\Delta_n)\|_Y \to 0$ as $n \to \infty$. The nonlinear operator F is **non-compact**, but its Fréchet derivative $F'(x^\dagger)$ is **compact** for all $x^\dagger \in L^2_{\mathbb{C}}(0,1)$. Hence, $\|F'(x^\dagger)\Delta_n\|_Y \to 0$ as $n \to \infty$ and thus $\|F(x_n) - F(x^\dagger)\|_Y = \|F(\Delta_n) + F'(x^\dagger)\Delta_n\|_Y \to 0$ as $n \to \infty$.

This shows the local ill-posedness everywhere.

We derive from of **Titchmarsh's convolution theorem**:

Proposition

If for given $y \in Y = L^2_{\mathbb{C}}(0,2)$ the function $x^\dagger \in X = L^2_{\mathbb{C}}(0,1)$ solves $(**)$, then $x^\dagger$ and $-x^\dagger$ are the only solutions of this operator equation.

Some more references:

▷ D. GERTH, B.H., S. BIRKHOLZ, S. KOKE AND G. STEINMEYER: Regularization of an autoconvolution problem in ultrashort laser pulse characterization. *Inverse Probl. Sci. Eng.* **22** (2014), pp. 245–266.

▷ S. W. ANZENGRUBER, S. BÜRGER, B.H. AND G. STEINMEYER: Variational regularization of complex deautoconvolution and phase retrieval in ultrashort laser pulse characterization. *Inverse Problems* **32** (2016), 035002 (27pp).

Example 2: A problem in inverse option pricing

Calibrating local volatility surfaces from market data is an ill-posed nonlinear inverse problem in finance.

Consider the price process $P(t)$ for an asset

$$\frac{dP(t)}{P(t)} = \mu\, dt + \sigma(t)\, dW(t) \quad (t \geq 0,\ P(0) > 0).$$

A **benchmark problem** for studying phenomena is the calibration of time-dependent volatilities $\sigma(t),\ 0 \leq t \leq T$, from maturity-dependent option prices $u(t),\ 0 \leq t \leq T$, of European call options with a fixed strike $K > 0$.

For parameters $P > 0,\ K > 0,\ r \geq 0,\ t \geq 0$ and $s \geq 0$ we introduce the **Black-Scholes function** as

$$U_{BS}(P,K,r,t,s) := \begin{cases} P\Phi(d_1) - Ke^{-r\,t}\,\Phi(d_2) & (s > 0) \\ \max(P - Ke^{-r\,t}, 0) & (s = 0) \end{cases}$$

with

$$d_1 := \frac{\ln\left(\frac{P}{K}\right) + r\,t + \frac{s}{2}}{\sqrt{s}}, \quad d_2 := d_1 - \sqrt{s}$$

and the cumulative density function

$$\Phi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{\eta^2}{2}}\, d\eta.$$

of the standard normal distribution.

For

$$a(t) := \sigma^2(t) \qquad \text{and} \qquad S(t) = \int_0^t a(\tau)\, d\tau$$

the associated forward operator in $(**)$ is here $F:\ a \mapsto u$ with

$$[F(a)](t) := U_{BS}(P, K, r, t, S(t)) \quad (0 \le t \le T).$$

Hence, we have a composition $F = N \circ J$ with the nonlinear

Nemytskii operator $[N(S)](t) := k(t, S(t))\ (0 \le t \le T)$ for

$$k(t, v) = U_{BS}(P, K, r, t, v) \quad ((t, v) \in [0, T] \times [0, \infty)),$$

and with the linear **integral operator**

$$[J\,a](t) := \int_0^t a(\tau)\, d\tau \quad (0 \le t \le T).$$

This calibration problem can be written as $F(a) = u \quad (**)$.

It is split into an **ill-posed linear inner equation**

$$\boxed{J\,a = S \qquad (a \in D(F) \subset X,\; S \in Z) \qquad (in)}$$

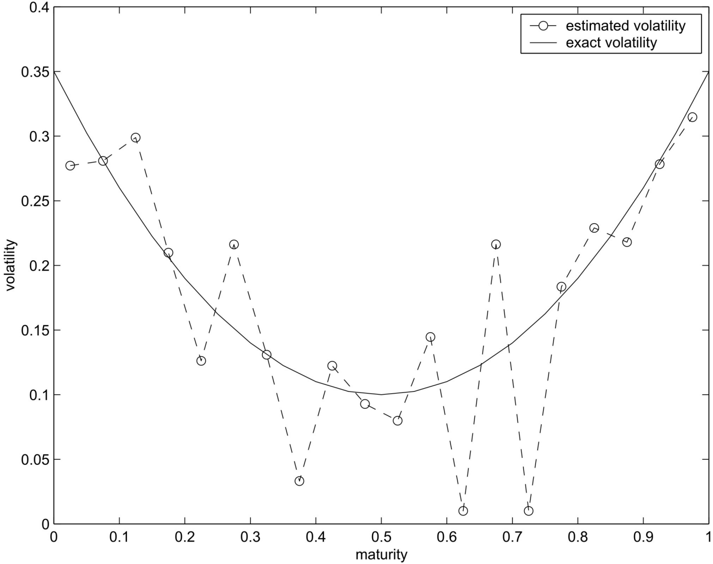
and a **nonlinear outer equation**

$$\boxed{N(S) = u \qquad (S \in Z,\; u \in Y), \quad (out)}$$

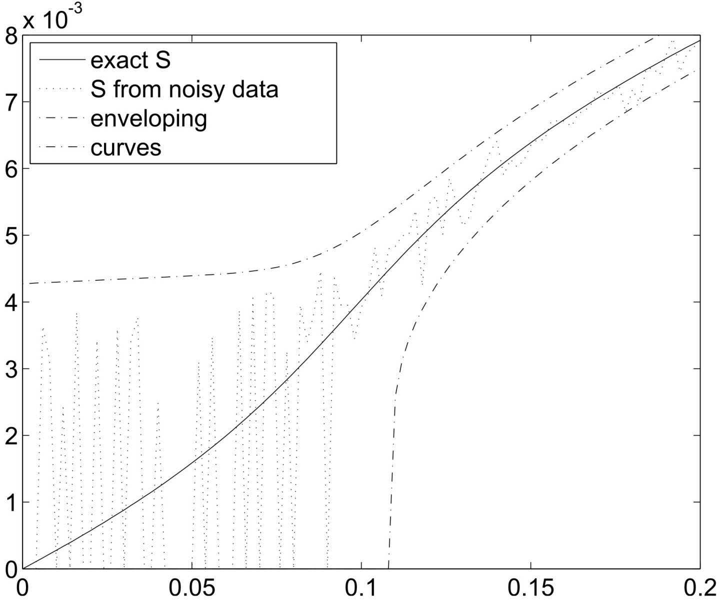
where X, Y, Z are Banach spaces of real functions over $[0, T]$.

The composition problem $(**)$ is **locally ill-posed** everywhere.
However, the character of the outer problem is not so clear.

Least-squares solution of $(**)$ after discretization with 20 grid points

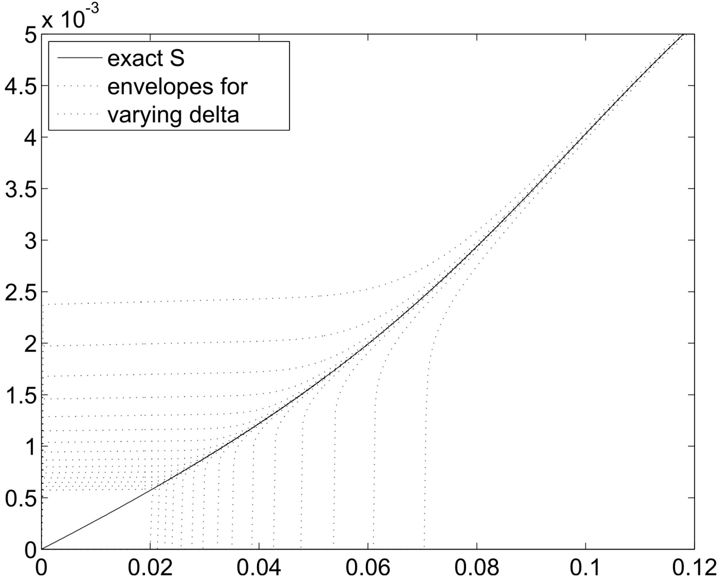


Oscillations near $t = 0$ in solving the outer equation



Reduction of oscillation areas for $\delta \to 0$:

Ill-conditioning but not ill-posedness of the outer equation in $C[0, T]$.



Some references:

▷ F. Black and M. Scholes: The pricing of options and corporate liabilities. *J. Political Econom.* **81** (1973), pp. 637–654.

▷ I. Bouchouev and V. Isakov: The inverse problem of option pricing. *Inverse Problems* **13** (1997), pp. L11–L17.

▷ T. Hein and B.H.: On the nature of ill-posedness of an inverse problem arising in option pricing. *Inverse Problems* **19** (2003), pp. 1319–1338.

▷ R. Krämer and P. Mathé: Modulus of continuity of Nemytskii operators with application to the problem of option pricing. *J. Inverse Ill-Posed Probl.* **16** (2008), pp. 435–461.

▷ A. De Cezaro, O. Scherzer and J. P. Zubelli: Convex regularization of local volatility models from option prices: convergence analysis and rates. *Nonlinear Anal.* **75** (2012), pp. 2398–2415.

▷ Y. F. Saporito, X. Yang, Xu and J. P. Zubelli: The calibration of stochastic local-volatility models: an inverse problem perspective. *Comput. Math. Appl.* **77** (2019), pp. 3054–3067.

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Degree of ill-posedness for compositions with non-compact linear operators

We consider for Hilbert spaces X, Y, Z the ill-posed equation

$$Ax = y, \qquad x \in X, \quad y \in Y, \qquad\qquad (*)$$

with **compact composite linear forward** operator

$$A : X \xrightarrow{D} Z \xrightarrow{B} Y,$$

where $A = B \circ D : X \to Y$ is a composition of the **compact** linear operator D with infinite dimensional range $\mathcal{R}(D)$ and the bounded **non-compact** operator B with non-closed range $\mathcal{R}(B) \neq \overline{\mathcal{R}(B)}^Y$.

In the nomenclature of NASHED 1987, the inner problem

$$D\,x \;=\; z\,,$$

is ill-posed of **type II** due to the compactness of D, whereas the outer problem

$$B\,z \;=\; y$$

is ill-posed of **type I**, since B is non-compact.

General open question:

What impact does the non-compact operator B with non-closed range have on the degree of ill-posedness of $(*)$?

Nashed states that

"...an equation involving a bounded non-compact operator with non-closed range is **less ill-posed** than an equation with a compact operator with infinite-dimensional range."

▷ M. Z. NASHED: A new approach to classification and regularization of ill-posed operator equations, In: Inverse and Ill-posed Problems Sankt Wolfgang, 1986 (Eds.: H. W. Engl and C. W. Groetsch), Academic Press, Boston, 1987, pp. 53–75.

Specific open question:

Can the non-compact operator B with non-closed range in $A = B \circ D$ 'destroy' the degree of ill-posedness from D?

In $X = Y = Z = L^2(0,1)$ we have, for the

integration operator $[Jx](s) := \int\limits_0^s x(t)dt$ and classes of

multiplication operators $[Mx](t) := m(t)x(t)$ with multiplier

functions $m \in L^\infty(0,1)$ possessing essential zeros, that

$$\sigma_i(M \circ J) \asymp \sigma_i(J) \asymp i^{-1} \qquad (i \in \mathbb{N}).$$

The non-compact $B = M$ **does not 'destroy'** the singular value decay rate of $D = J$ by the composition $A = M \circ J$.

For that fact we refer to:

▷ M. FREITAG AND B.H.: Analytical and numerical studies on the influence of multiplication operators for the ill-posedness of inverse problems. *J. Inv. Ill-Posed Problems* **13** (2005), pp. 123-148.

▷ B.H. AND L. VON WOLFERSDORF: Some results and a conjecture on the degree of ill-posedness for integration operators with weights. *Inverse Problems* **21** (2005), pp. 427-433.

▷ B.H. AND L. VON WOLFERSDORF: A new result on the singular value asymptotics of integration operators with weights. *Journal of Integral Equations and Applications* **21** (2009), pp. 281-295.

For the compact operator $A = B \circ D : X \to Y$ and $D : X \to Z$ with non-closed ranges $\mathcal{R}(A)$ and $\mathcal{R}(D)$ we have upper bounds for the singular values of A as

$$\sigma_i(A) \le \|B\|_{\mathcal{L}(Z,Y)}\, \sigma_i(D)\,.$$

Lower bounds based on a conditional stability estimate are given as follows:

Theorem 1 (cf. Thm. 2.1 of [HM22])

Suppose that there exists an index function $\Psi : (0,\infty) \to (0,\infty)$ such that for $0 < \delta \le \|A\|_{\mathcal{L}(X,Y)}$ the conditional stability estimate

$$\sup\{\, \|Dx\|_Z : \ \|Ax\|_Y \le \delta,\ \|x\|_X \le 1\} \le \Psi(\delta)$$

holds. Then we have

$$\Psi^{-1}(\sigma_i(D)) \le \sigma_i(A) \qquad (i = 1, 2, \ldots)\,.$$

▷ B.H. and P. Mathé: The degree of ill-posedness of composite linear ill-posed problems with focus on the impact of the non-compact Hausdorff moment operator. *ETNA* **57** (2022), pp. 1–16.

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The mystery of the Hausdorff moment operator in a composition with the compact integration operator

We recall the Hausdorff moment operator $B^{(H)} : L^2(0,1) \to \ell^2$

$$[B^{(H)}z]_j := \int_0^1 t^{j-1} z(t)dt \qquad (j = 1, 2, \ldots).$$

to apply it later in a composition $A = B^{(H)} \circ J$ with the compact integration operator
$J : L^2(0,1) \to L^2(0,1)$ with $[Jx](s) := \int_0^s x(t)dt \;\; (0 \le s \le 1)$.

For the subsequent proposition and assertions on $B^{(H)}$ see:

$\triangleright$ D. GERTH, B.H., C. HOFMANN AND S. KINDERMANN: The Hausdorff moment problem in the light of ill-posedness of type I. *EJMCA* **9** (2021), pp. 57–87.

Proposition (cf. Props. 3-5 of [GHHK21])

For the operator $B^{(H)}: L^2(0,1) \to \ell^2$ we have the properties: $B^{(H)}$ is a bounded, injective and **non-compact** linear operator with $\|B^{(H)}\|_{\mathcal{L}(L^2(0,1),\ell^2)} = \sqrt{\pi}$ and **non-closed range** $\mathcal{R}(B^{(H)})$. The adjoint operator $(B^{(H)})^*: \ell^2 \to L^2(0,1)$ attains the form

$$[(B^{(H)})^* y](t) = \sum_{j=1}^{\infty} y_j \, t^{j-1} \qquad (0 \le t \le 1).$$

We have $B^{(H)} = \mathbb{L}\, Q$ with an isometry $Q: L^2(0,1) \to \ell^2$ and a lower triangular operator $\mathbb{L}: \ell^2 \to \ell^2$ being the lower Cholesky factor of the infinite Hilbert matrix $\mathbb{H} = \left(\frac{1}{i+j-1}\right)_{i,j=1}^{\infty}: \ell^2 \to \ell^2$.

This means that $\quad \mathbb{L}\,\mathbb{L}^* = \mathbb{H} = B^{(H)}(B^{(H)})^*$.

Isometry $[Q\,x]_j = \langle x, L_j \rangle_{L^2(0,1)}$ for ONS of Legendre polynomials $\{L_j\}_{j=1}^{\infty}$ with $\mathrm{span}(L_1 ..., L_j) = \mathrm{span}(1, t, ..., t^{j-1})$ (Gram-Schmidt).

Now we consider $(*)$ with the compact composition

$$A = B^{(H)} \circ J : L^2(0,1) \to \ell^2$$

as forward operator.

Proposition (cf. Thm. 3.1 of [HM22])

There is a positive constant C_0 such that

$$\sup\{\, \|Jx\|_{L^2(0,1)} : \|B^{(H)}(Jx)\|_{\ell^2} \le \delta,\ \|x\|_{L^2(0,1)} \le 1\} \le \frac{C_0}{\ln(1/\delta)}\,.$$

This proposition yields with Theorem 1 by setting $X = Z = L^2(0,1)$, $Y = \ell^2$ and $\Psi(\delta) = \frac{C_0}{\ln(1/\delta)}$ the following

Corollary 1

There exists a positive constant $\underline{C}$ such that

$$\exp(-\underline{C}\, i) \le \sigma_i(B^{(H)} \circ J) \quad (i = 1, 2, ...).$$

However, further detailed studies allow us to prove that $B^{(H)}$ **has the power to 'destroy' the ill-posedness degree of J.**

Theorem 2 (cf. Thm. 5.1 of [HM22])

For the composite operator $A = B^{(H)} \circ J$ there exists a positive constant C such that

$$\sigma_i(B^{(H)} \circ J) \leq \frac{C}{i^{3/2}} \quad (i \in \mathbb{N}).$$

Hence, there is also a positive constant K such that that

$$\sigma_i(B^{(H)} \circ J)/\sigma_i(J) \leq \frac{K}{i^{1/2}} \quad (i \in \mathbb{N}).$$

The non-compact Hausdorff moment operator $B^{(H)}$ is able to **increase** in a composition **the degree of ill-posedness** 1 of J at least by $1/2$. Thus, $\sigma_i(B^{(H)} \circ J) \asymp \sigma_i(J)$ is **violated.**

As a consequence of Corollary 1 and Theorem 2 we have:

Corollary 2

For the compact composite operator $A = B^{(H)} \circ J$ there exist positive constants $\underline{C}$ and $\overline{C}$ such that

$$\exp(-\underline{C}\, i) \le \sigma_i\,(A) \le \frac{\overline{C}}{i^{3/2}} \quad (i = 1, 2, ...).$$

The gap between lower and upper bounds for $\sigma_i(A)$ is too large.

Open question (Hausdorff mystery)

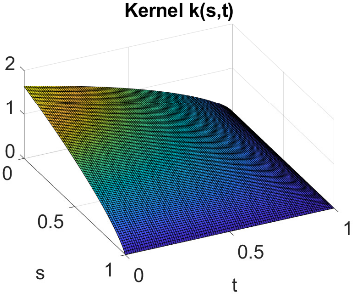
Is the linear operator equation $(*)$ with forward operator $A = B^{(H)} \circ J$ moderately or severely ill-posed?

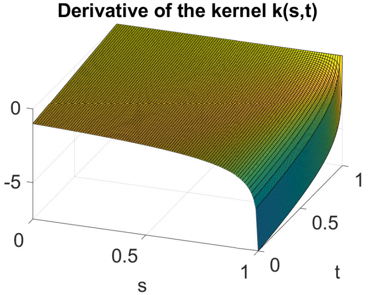
▷ D. GERTH, B.H.: A note on open questions asked to analysis and numerics concerning the Hausdorff moment problem. *EJMCA* **10** (2022), pp. 40–50.

Arguments pro moderate ill-posedness:

For the Hilbert-Schmidt operator $A = B^{(H)} \circ J$ we have

$$[A^* Ax](s) = \int_0^1 k(s,t)\, x(t)\, dt \quad (0 \le s \le 1) \quad \text{with} \quad k(s,t) = \sum_{j=1}^{\infty} \frac{(1-s^j)(1-t^j)}{j^2}\,.$$





Kernel k is smooth, but partial derivative $\frac{\partial k}{\partial s}$ has a pole at $s = 1$.

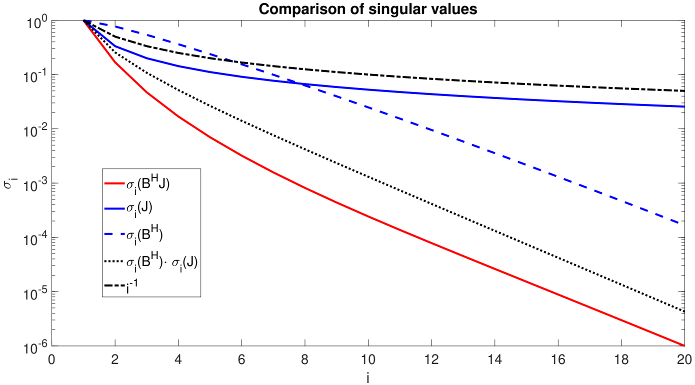
Limited kernel-smoothness seems to mismatch an exponential decay of $\sigma_i(A)$ for $A = B^{(H)} \circ J$.

But the following open question should be answered:

Open question (kernel smoothness and ill-posedness)

Under which conditions can an operator equation $(*)$ with a Hilbert-Schmidt operator A mapping from $L^2(0,1)$ into an arbitrary Hilbert space Y with non-closed range $\mathcal{R}(A)$ be severely (exponentially) ill-posed, provided that the kernel $k \in C([0,1] \times [0,1])$ from $A^*A : L^2(0,1) \to L^2(0,1)$ has limited smoothness, which means that k is not infinitely many continuously differentiable on the whole closed unit square?

Arguments against moderate ill-posednes:



Semi-logarithmic plot of singular values of $n \times n$-matrices with $n = 10^4$ supporting points representing **discretization matrices** of the operators A, $B^{(H)}$ and J.

While numerically the singular values of J decay as suggested by the theory, the **singular values of $\mathbf{A = B^{(H)} \circ J}$ decay exponentially** in the numerical experiments.

Numerics indicates **severe ill-posedness** of $A = B^{(H)} \circ J$, but

▷ D. GERTH: A note on numerical singular values of compositions with non-compact operators. *ETNA* **57** (2022).

yields some arguments for the conjecture that **an exponential decay of matrix singular values is possible even if the singular values of the infinite dimensional operator $A: L^2(0,1) \to \ell^2$ decay slowly.**

We consider the $n \times n$-matrices $\mathbb{J}_n, \mathbb{B}_n^{(H)}$ and $\mathbb{A}_n$ as discretized versions of the operators J, $B^{(H)}$ and $A = B^{(H)} \circ J$, calculated with n supporting points over the interval $[0,1]$ and with n moments of the truncated Hausdorff moment operator.

$\mathbb{H}_n = \left(\frac{1}{i+j-1}\right)_{i,j=1}^{n}$: n-dimensional segment of Hilbert matrix $\mathbb{H}$.

$1 \le \sigma_1(\mathbb{H}_n) \le \pi = \lim_{n\to\infty} \sigma_1(\mathbb{H}_n), \qquad \sigma_n(\mathbb{H}_n) \approx \hat{C}\exp(-3.526n).$

$\mathbb{B}_n^{(H)}$ constructed such that $\sigma_i(\mathbb{B}_n^{(H)}) = (\sigma_i(\mathbb{H}_n))^{1/2} \ (i=1,2,\dots,n).$

Owing to [Beckermann19, formula (4.8)] we have that

$$\sigma_i(\mathbb{B}_n^{(H)}) \le 2\,[\varphi(n)]^{i-1}\,(\sigma_1(\mathbb{H}_n))^{1/2} \quad (i{=}1,2,\ldots,n{-}1), \qquad (+)$$

with factor $\quad 0 < \varphi(n) = \exp\left(-\frac{\pi^2}{2\ln(8n-4)}\right) < 1$

growing very slowly to 1 as $n \to \infty$: $1 - \varphi(n) \sim 1/\ln(n)$.

n	$i=2$	$i=4$	$i=10$	$i=51$
10^2	0.4777	0.1091	0.0013	$9.1932 \cdot 10^{-17}$
10^3	0.5774	0.1926	0.0071	$1.1920 \cdot 10^{-12}$
10^4	0.6459	0.2695	0.0196	$3.2240 \cdot 10^{-10}$
10^6	0.7331	0.3940	0.0612	$1.8129 \cdot 10^{-7}$
10^9	0.8054	0.5224	0.1426	$1.9982 \cdot 10^{-5}$

Values of occurring multiplier $\varphi(n)^{i-1}$ in $(+)$

We conjecture that $(+)$ is approximately an equation if $n \gg i$.

For n fixed: Exponential decay $\sigma_i(\mathbb{B}_n^{(H)}) \sim \exp(-K\,i)$ with $K = K(n) > 0$.

▷ B. Beckermann and A. Townsend: Bounds on the singular values of matrices with displacement structure. *SIAM Review* **61** (2019), pp. 319–344

We have $\sigma_{2i}(\mathbb{A}_n) \le \sigma_i(\mathbb{B}_n^{(H)})\, \sigma_i(\mathbb{J}_n)$.

If n is small or medium in size:
$\sigma_i(\mathbb{A}_n)$ dominated by $\sigma_i(\mathbb{B}_n^{(H)}) \sim \exp(-K(n)\, i)$

If n is very large and $\varphi(n) \approx 1$:
$\sigma_i(\mathbb{A}_n)$ more dominated by $\sigma_i(\mathbb{J}_n) \sim 1/i$.

This yields some rough explanation for the contradiction.

Is numerics reaching its limits here to evaluate the degree of ill-posedness for the infinite dimensional problem?

However, by now there is no final unveiling of this mystery!