Ill-posedness studies in Banach spaces

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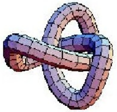

Vortrag zum Minisymposium am 11. Februar 2026 für Herrn Professor

ANDREAS NEUBAUER

des Instituts für Industriemathematik der J. Kepler Universität Linz,

anlässlich seiner bevorstehenden Pensionierung

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This excerpt of talk also presents joint work with:

Jens Flemming (Dresden)

Stefan Kindermann (Linz)

Outline

1. New facets of ill-posedness classification in Banach spaces

2. The fascination of Mazur-type operators

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1. New facets of ill-posedness classification in Banach spaces
2. The fascination of Mazur-type operators

Ill-posedness classification in Banach spaces

Let X and Y be infinite dimensional real **Banach spaces**.

We consider **operator equations** modelling inverse problems,

and focus here on **linear inverse problems**

$$Ax = y \qquad (x \in X,\ y \in Y) \qquad (*)$$

with **bounded linear forward operators** $A \in \mathcal{L}(X, Y)$.

For well-posedness, it is crucial to meet the stability condition.
Naive distinction: $(*)$ is well-posed if and only if $\mathcal{R}(A)$ closed.
Not naive for Hilbert spaces: $(*)$ is well-posed iff $A^\dagger$ is bounded.
Also not naive for injective A mappings in Banach spaces.

Ill-posedness characterization in literature: rough selection

▷ G. Wahba: Ill-posed problems: Numerical and statistical methods for mildly, moderately and severely ill-posed problems with noisy data. Technical Report No. 595. Madison, University of Wisconsin, 1980.

▷ B.H.: *Regularization for Applied Inverse and Ill-Posed Problems*. B. G. Teubner, Leipzig, 1986.

▷ M. Z. Nashed: A new approach to classification and regularization of ill-posed operator equations. In: H. W. Engl and C. W. Groetsch (Eds.), Inverse and Ill-posed Problems (Sankt Wolfgang, 1986), volume 4 of Notes Rep. Math. Sci. Engrg., pp. 53–75. Academic Press, Boston, MA, 1987.

▷ A. K. Louis: *Inverse und schlecht gestellte Probleme*. Teubner, Stuttgart, 1989.

▷ H. W. Engl, M. Hanke and A. Neubauer: *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

▷ O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, F. Lenzen: *Variational Methods in Imaging*. Springer, New York, 2009.

▷ T. Schuster, B. Kaltenbacher, B.H. and K. S. Kazimierski: *Regularization Methods in Banach Spaces*. Walter de Gruyter, Berlin/Boston, 2012.

▷ S. Lu and S. V. Pereverzev: *Regularization Theory for Ill-Posed Problems*. Walter de Gruyter, Berlin/Boston, 2013.

▷ B.H. and R. Plato: On ill-posedness concepts, stable solvability and saturation. *J. Inverse Ill-Posed Probl.* **26** (2018), pp. 287–297.

In all Banach spaces, not isomorphic to a Hilbert space, null-spaces $\mathcal{N}(A)$ can be **uncomplemented.** Such concept already occurs in the seminal paper by Nashed 1987, but

▷ J. FLEMMING: *Variational Source Conditions, Quadratic Inverse Problems, Sparsity Promoting Regularization*. Frontiers in Mathematics. Birkhäuser, Cham, 2018.

first evaluated the impact of uncomplemented null-spaces for ill-posedness in the context of ℓ^1-regularization.

This inspired us to prepare the article

▷ B.H. AND S. KINDERMANN: Classification of ill-posedness for bounded linear operators in Banach spaces. To appear. Preprint: arXiv:2505.12931, May 2025.

S. Kindermann discovered the **hybrid case** in Banach spaces:

Definition (hybrid-type)

We call the operator equation $(*)$ as of hybrid-type if the operator A is strictly singular and its range $\mathcal{R}(A)$ contains an infinite-dimensional closed subspace.

Proposition

For an operator equation $(*)$ of hybrid-type, the operator A is not compact, and its null-space $\mathcal{N}(A)$ is uncomplemented.

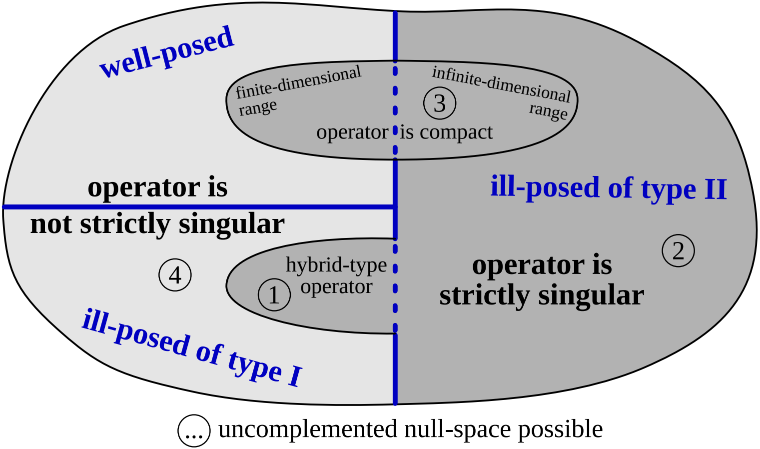
Definition (State of the art for general Banach spaces)

The linear operator equation $(*)$ is called **well-posed** if
the range $\mathcal{R}(A)$ of A is a **closed** subset of Y and
moreover the null-space $\mathcal{N}(A)$ of A is **complemented** in X.
Otherwise the equation $(*)$ is called **ill-posed**.

In the ill-posed case, $(*)$ is called **ill-posed of type I** if
$\mathcal{R}(A)$ contains an **infinite-dimensional closed subspace.**
Otherwise the ill-posed equation $(*)$ is **ill-posed of type II**.

▷ J. FLEMMING AND B.H.: New aspects of ill-posedness classification in Banach spaces. Paper submitted to *Communications in Optimization Theory*, January 2026. Preprint: arXiv:2511.06690, November 2025.

▷ B.H. AND J. FLEMMING: A note on the Goldberg-Thorp example in light of the classification of linear ill-posed problems in Banach spaces. *Journal of Inverse and Ill-Posed Problems* (to appear 2026).



Case distinction for bounded linear operators in Banach spaces

Outline

The fascination of Mazur-type operators

▷ S. BANACH AND S. MAZUR: Zur Theorie der linearen Dimension. *Studia Mathematica* **4** (1933), pp. 100-112.

Definition (Mazur-type operator)

We say that a bounded linear operator $A_1^Y : \ell^1 \to Y$ mapping into a separable infinite-dimensional Banach space Y is of Mazur-type if there is a sequence $(\zeta^{(k)})_{k\in\mathbb{N}}$ in Y which is dense in the unit sphere in Y, with $\zeta^{(k)} \neq \zeta^{(l)}$ for $k \neq l$, and of the form

$$[A_1^Y]\,x = \sum_{k=1}^{\infty} x_k\,\zeta^{(k)} \qquad (x = (x_1, x_2, \ldots) \in \ell^1).$$

▷ S. F. ALBIAC AND N. J. KALTON: *Topics in Banach Space Theory*. Springer, New York, 2006, Section 2.3.

Proposition

For a separable infinite-dimensional Banach space Y, all Mazur-type operators $A_1^Y : \ell^1 \to Y$ are mapping ℓ^1 *onto* Y.
If Y is reflexive, then all Mazur-type operators A_1^Y are strictly singular, hence $(*)$ is of hybrid-type and ill-posed of type I.

The Goldberg-Thorp example: Let A be a continuous linear operator from ℓ^1 onto ℓ^2. Then A is strictly singular, but A^* has a bounded inverse.

▷ S. GOLDBERG AND E THORP: On some open questions concerning strictly singular operators. *Proc. Amer. Math. Soc.* **14** (1963), pp. 334-336.

Such A exist as a consequence of Mazur-type operators.
Are there other such operators A?

For $A := A_1^{\ell^2}$, equ. $(*)$ is of hybrid-type and ill-posed of type I.

The adjoint operator $A^* : \ell^2 \to \ell^\infty$ is injective and of the form

$$A^*\,\eta = (\langle \eta, \zeta^{(1)} \rangle_{\ell^2}, \langle \eta, \zeta^{(2)} \rangle_{\ell^2}, \ldots) \in \ell^\infty.$$

The adjoint equation $A^*\eta = z$ is well-posed.