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Some results and a conjecture on the degree of ill-posedness for integration operators with weights

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Abstract

In this paper, we are looking for answers to the question whether a non-compact linear operator with non-closed range applied to a compact linear operator mapping between Hilbert spaces can, in a specific situation, destroy the degree of ill-posedness determined by the singular value decay rate of the compact operator. We partially generalize a result of Vu Kim Tuan and Gorenflo (1994 *Inverse Problems* **10** 949–55) concerning the non-changing degree of ill-posedness of linear operator equations with fractional integral operators in $L^2(0,1)$ when weight functions appear. For power functions $m(t) = t^\alpha$ ($\alpha > -1$), we prove the asymptotics $\sigma_n(A) \sim \frac{\int_0^1 m(t)\,\mathrm{d}t}{\pi n}$ for the singular values of the composite operator $[Ax](s) = m(s)\int_0^s x(t)\,\mathrm{d}t$ in $L^2(0,1)$. We conjecture this asymptotic behaviour also for exponential functions $m(t) = \exp(-1/t^\alpha)(\alpha > 0)$ that play some role for the local degree of ill-posedness for a nonlinear inverse problem in option pricing in Hein and Hofmann (2003 *Inverse Problems* **19** 1319–38).

1. Introduction

Since Wahba in 1980 distinguished, in her paper [25], between mildly, moderately and severely linear ill-posed problems

$$Ax = y \qquad (x \in X, y \in Y, A \in \mathcal{L}(X,Y)), \tag{1}$$

the discussion of measures for the ill-posedness of inverse problems and its consequences for condition numbers of discretized problems and regularization, partially exploiting Hilbert scale techniques, plays an important role in the theory of inverse problems (see, e.g., [1, 2, 4, 15, 18, 20–23, 26]). If only the smoothing properties of the injective compact linear forward operator A mapping between infinite-dimensional Hilbert spaces are considered, then the decay rate of the positive, non-increasing sequence $\{\sigma_n(A)\}_{n=1}^\infty$ of singular values of A

tending to zero as $n \to \infty$ is a frequently used measure of ill-posedness (see, e.g., [4, p 40], [9, p 31] and [19, p 235]). It defines a finite degree $\mu \in (0, \infty)$ of ill-posedness if $\sigma_n(A) \asymp n^{-\mu}$ is valid[3]. In [14], we suggested considering the more general interval of ill-posedness

$$[\underline{\mu}(A), \overline{\mu}(A)] = \left[\liminf_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n}, \limsup_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n}\right],$$

where $\underline{\mu}$ and $\overline{\mu}$ can also be zero and infinity. As proposed and motivated in [7, 10, 11], such measures are also helpful for evaluating the local behaviour of ill-posedness for nonlinear operator equations

$$F(x) = y \qquad (x \in D(F) \subseteq X, \, y \in Y) \tag{2}$$

with continuous nonlinear forward operator F at a point $x_0 \in D(F)$ by considering a linearized version of (2) as an equation (1) with the Fréchet derivative $A = F'(x_0)$, which is compact whenever F is compact ([3, theorem 4.19]). For non-compact linear operators A, in particular multiplication operators, some ideas concerning ill-posedness measures were presented in [12]. Some interdependences between an ill-posed nonlinear equation (2) and its linearization with respect to the local degree of ill-posedness characterized by $F'(x_0)$ and the degree of nonlinearity of F at x_0 including consequences for regularization were formulated in [13].

In this paper, for $X = Y = L^2(0, 1)$ with norm $\|\cdot\|$, we try to measure the ill-posedness of the class of linear operator equations (1) with an injective integral operator

$$[Ax](s) = m(s) \int_0^s x(t)\, \mathrm{d}t \qquad (0 \leqslant s \leqslant 1), \tag{3}$$

which is a composition $A = M \circ J$ of the simple integration operator

$$[Jx](s) = \int_0^s x(t)\, \mathrm{d}t \qquad (0 \leqslant s \leqslant 1) \tag{4}$$

and an injective multiplication operator

$$[Mh](t) = m(t)h(t) \qquad (0 \leqslant t \leqslant 1). \tag{5}$$

In this paper, we only consider multiplier (weight) functions

$$m \in L^1(0, 1) \qquad \text{with} \quad |m(t)| > 0 \quad \text{a.e. on } [0, 1] \tag{6}$$

such that the composition operator $A = M \circ J$ is compact. Linear operators (3), in particular, occur as Fréchet derivatives $F'(x_0)$ of composite nonlinear operators $F = N \circ J : D(F) \subset L^2(0, 1) \to L^2(0, 1)$ with half-space domains

$$D(F) = \{x \in L^2(0, 1) : x(t) \geqslant c_0 \geqslant 0 \text{ a.e. on } [0, 1]\} \tag{7}$$

defined as

$$[F(x)](s) = [N(Jx)](s) = k(s, (Jx)(s)) \qquad (0 \leqslant s \leqslant 1; x \in D(F)), \tag{8}$$

where the linear integral operator J forms the inner operator and a nonlinear Nemytskii operator N the outer operator. Here, N is generated by the sufficiently smooth function $k(s, v)((s, v) \in [0, 1] \times [0, \infty))$, where the multiplier function m in (3) is of the form

$$m(t) = \frac{\partial k}{\partial v}(t, (Jx_0)(t)). \tag{9}$$

Such Fréchet derivatives $F'(x_0) = M \circ J$ appear, for example, in iterative solutions of corresponding nonlinear inverse problems (2).

[3] Here the notation $a_n \asymp b_n$ for sequences of positive numbers a_n and b_n denotes the existence of positive constants c_1 and c_2 such that $c_1 \leqslant a_n/b_n \leqslant c_2$ for all sufficiently large n. If moreover $\lim_{n\to\infty} a_n/b_n = 1$ we write $a_n \sim b_n$.

From the explicitly given singular values

$$\sigma_n(J) = \frac{1}{\pi\left(n - \frac{1}{2}\right)} \sim \frac{1}{\pi n} \qquad (n = 1, 2, \ldots)$$

of the compact integration operator $J \in \mathcal{L}(L^2(0,1))$ and for multiplier functions m which satisfy

$$0 < c \leqslant |m(t)| \leqslant C \qquad \text{a.e. on } [0,1], \tag{10}$$

we may derive the inequalities $c\sigma_n(J) \leqslant \sigma_n(A) \leqslant C\sigma_n(J)$ and hence the asymptotics

$$\sigma_n(A) \asymp n^{-1} \tag{11}$$

based on spectral equivalence (see, e.g., [9, lemma 2.46]). That means the degree of ill-posedness is $\mu = 1$ for all such multiplier functions. Examples for nonlinear operators (8) leading to situation (10) are presented in [11].

Note that (10) implies a continuous (non-compact) multiplication operator $M \in \mathcal{L}(L^2(0,1))$ and hence the compactness of the linear operator A. Moreover, it is well known that the operator A from (3) is also compact whenever we have

$$|m(t)| \leqslant C t^{\alpha} \qquad \text{a.e. on } [0,1] \tag{12}$$

for a constant $C > 0$ and some exponent $\alpha > -1$ (cf [27]). We have a non-closed range of M, i.e. $\mathrm{Range}(M) \neq \overline{\mathrm{Range}(M)}$, whenever m has essential zeros. Then an additional ill-posedness factor occurs the strength of which should be evaluated. Source conditions $x_0 = A^* w (w \in L^2(0,1), \|w\| \leqslant R)$ that are important as sufficient conditions for convergence rates in regularization would imply

$$\frac{x_0'}{m} \in L^2(0,1) \qquad \text{and} \qquad \left\| \frac{x_0'}{m} \right\| \leqslant R \tag{13}$$

for A from (3). If we assume that $\lim_{t\to 0} m(t) = 0$ and $t = 0$ is the only essential zero of the non-negative multiplier function m, then the strength of the requirement (13) grows when the decay rate of $m(t) \to 0$ as $t \to 0$ grows. However, we conjecture that the asymptotics (11) remains true for all multiplier functions m satisfying (6) and (12) and that consequently the non-compact operator M does not destroy the degree of ill-posedness determined by the compact operator J. We prove this in the form

$$\sigma_n(A) \sim \frac{\int_0^1 m(t)\,\mathrm{d}t}{\pi n} \sim \left(\int_0^1 m(t)\,\mathrm{d}t\right) \sigma_n(J) \qquad \text{as} \quad n \to \infty \tag{14}$$

for the family of power functions

$$m(t) = t^{\alpha} \qquad (0 < t \leqslant 1) \quad \text{and} \quad \alpha > -1 \tag{15}$$

in the subsequent paragraph. Sophisticated numerical experiments in [6] suggest assuming that formula (14) also holds for the family

$$m(t) = \exp\left(-\frac{1}{t^{\alpha}}\right) \qquad (0 < t \leqslant 1) \quad \text{and} \quad \alpha > 0, \tag{16}$$

even though such exponential multiplier functions (16) decrease faster to zero as $t \to 0$ than the power functions (15) and hence seem to have a higher potential for ill-posedness. Multiplier functions

$$K_1 \exp\left(-\frac{c_1}{t}\right) \leqslant m(t) \leqslant K_2 \exp\left(-\frac{c_2}{\sqrt{t}}\right) \qquad (0 < t \leqslant 1;\, c_1, c_2, K_1, K_2 > 0)$$

occurring in a nonlinear inverse problem of option pricing are bounded below and above by exponential functions and were recently discussed in [8, p 1333]. This problem, which aims to determine a time-dependent volatility function, leads to an operator A from (3), where A as the Fréchet derivative of the associated forward operator F is of the form (8) and the function k generating the Nemytskii operator N represents the Black–Scholes function U_{BS} analysed in detail in the paper [8].

For the class of composite linear compact operators $A_\gamma = M \circ J_\gamma$ mapping in $L^2(0, 1)$ and defined by

$$[A_\gamma x](s) = s^{-\beta} \int_0^s \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} x(t)\,\mathrm{d}t \qquad (0 < s \leqslant 1;\ \gamma > \beta > 0), \tag{17}$$

where J_γ are fractional integral operators of order $\gamma > 0$ and the multiplier functions m are power functions with a weak pole, similar considerations as in this paper were performed by Vu Kim Tuan and Gorenflo in 1994 (see [24]). Using Gegenbauer polynomials they proved the asymptotics

$$\sigma_n(A_\gamma) \asymp n^{-\gamma} \tag{18}$$

for $0 \leqslant \beta < \frac{\gamma}{2}$ and conjectured that (18) also remains true for $\frac{\gamma}{2} \leqslant \beta < \gamma$. To our knowledge, results on the degree of ill-posedness of the class (17) do not seem to be published in the inverse problems literature for multiplier functions m with zeros ($\beta < 0$). Here we focus on the situation $\gamma = 1$ and present some results and a conjecture for that subcase.

2. Statement on the singular value asymptotics

In the following we prove the asymptotics (14) for the compact operator A from (3) with power functions (15).

Theorem 2.1. *For the non-increasing sequence $\{\sigma_n(A)\}_{n=1}^\infty$ of singular values of the compact linear operator A in $L^2(0, 1)$ defined by*

$$[Ax](s) = s^\alpha \int_0^s x(t)\,\mathrm{d}t \qquad (0 < s \leqslant 1) \tag{19}$$

with exponent $\alpha > -1$, we have the asymptotics

$$\sigma_n(A) \sim \frac{1}{(\alpha+1)\pi n} = \left(\int_0^1 m(t)\,\mathrm{d}t\right) \frac{1}{\pi n} \qquad as \quad n \to \infty. \tag{20}$$

Proof. Let $\sigma = \sigma(A) > 0$ be a singular value of A from (19). Then $\lambda = \frac{1}{\sigma^2} > 0$ satisfies the eigenvalue equation $u - \lambda A^*Au = 0$ for some non-trivial function $u \in L^2(0, 1)$. Taking into account the explicitly given structures of $[A^*y](t) = \int_t^1 m(s)y(s)\,\mathrm{d}s$ and $[A^*Ax](t) = \int_t^1 m^2(s)\big(\int_0^s x(\tau)\,\mathrm{d}\tau\big)\,\mathrm{d}s$, for $m(t) = t^\alpha$ ($\alpha > -1$) the eigenvalue equation can be rewritten as

$$u(t) - \lambda \int_t^1 s^{2\alpha} \left(\int_0^s u(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s = 0 \qquad (0 < t < 1) \tag{21}$$

implying the first boundary condition

$$u(1) = 0. \tag{22}$$

Differentiation of (21) leads to $u'(t) + \lambda t^{2\alpha} \int_0^t u(\tau)\,\mathrm{d}\tau = 0$ and hence the equation

$$\frac{u'(t)}{t^{2\alpha}} + \lambda \int_0^t u(\tau)\,\mathrm{d}\tau = 0 \qquad (0 < t < 1) \tag{23}$$

and the second boundary condition

$$\lim_{t\to 0}\frac{u'(t)}{t^{2\alpha}} = 0. \tag{24}$$

Finally, differentiating (23) and multiplying by the factor $t^{2\alpha+2}$ yields the second-order ODE

$$t^2u''(t) - 2\alpha t u'(t) + \lambda t^{2\alpha+2}u(t) = 0 \qquad (0 < t < 1). \tag{25}$$

Conversely, from the differential equation (25) and the boundary conditions (22) and (24) the integral equation (21) follows, such that the boundary value problem for equation (25) determines the singular values $\sigma = \frac{1}{\sqrt{\lambda}} > 0$ under consideration with associated eigenfunctions u.

The differential equation (25) possesses an explicit solution for all $\alpha > -1$ (see, e.g., formula (1a) on p 440 in [17])

$$u(t) = t^{\alpha+\frac{1}{2}}\mathcal{Z}_\nu\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right), \qquad \nu = \frac{2\alpha+1}{2\alpha+2}, \tag{26}$$

by exploiting the general Bessel function $\mathcal{Z}_\nu$ of order $\nu \in (-\infty, 1)$. Then the general solution of (25) is of the form

$$u(t) = C_1u_1(t) + C_2u_2(t), \tag{27}$$

where

$$u_1(t) = t^{\alpha+\frac{1}{2}}\mathcal{J}_{-\nu}\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right)$$

for all $\alpha > -1$, i.e. for all $\nu \in (-\infty, 1)$, with Bessel function of the first kind $\mathcal{J}_\nu$ (cf [5, section 7.2]) and

$$u_2(t) = t^{\alpha+\frac{1}{2}}\mathcal{J}_{\nu}\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right) \qquad (\nu \in (-\infty, 1), \nu \neq 0, -1, -2, \ldots),$$

respectively,

$$u_2(t) = t^{\alpha+\frac{1}{2}}\mathcal{Y}_{\nu}\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right) \qquad (\nu = 0, -1, -2, \ldots)$$

with Bessel function of the second kind $\mathcal{Y}_\nu$. The constants C_1 and C_2 are to be selected such that the boundary conditions (22) and (24) are satisfied.

To fit the boundary condition (24) at $t = 0$ we consider

$$\frac{u_1'(t)}{t^{2\alpha}} = \frac{t^{\frac{1}{2}}}{\sigma}\mathcal{J}'_{-\nu}\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right) + \left(\alpha+\frac{1}{2}\right)t^{-\alpha-\frac{1}{2}}\mathcal{J}_{-\nu}\left(\frac{t^{\alpha+1}}{(\alpha+1)\sigma}\right).$$

Taking into account the well-known asymptotics (cf [5, section 7.2]) of the Bessel functions of the first kind and their derivatives for $t \to 0$,

$$\mathcal{J}_{-\nu}(t) \sim \frac{1}{\Gamma(1-\nu)}\left\{\left(\frac{t}{2}\right)^{-\nu} - \frac{1}{1-\nu}\left(\frac{t}{2}\right)^{2-\nu}\right\}$$

and

$$\mathcal{J}'_{-\nu}(t) \sim \frac{1}{2\Gamma(-\nu)}\left\{\left(\frac{t}{2}\right)^{-\nu-1} - \frac{1}{1-\nu}\left(\frac{t}{2}\right)^{1-\nu}\right\},$$

we obtain after some algebra

$$\frac{u_1'(t)}{t^{2\alpha}} = \mathcal{O}(t) \qquad \text{as} \quad t \to 0. \tag{28}$$

Hence u_1 satisfies (24). Analogously, one can show that u_2 does not fulfil (24). This implies $C_2 = 0$ in formula (27) as a consequence of the boundary condition (24). Without loss of generality, we set $C_1 = 1$ and find from the boundary condition (22) the eigenvalue equation

$$\mathcal{J}_{-\nu}\left(\frac{1}{(\alpha+1)\sigma}\right) = 0 \qquad (\alpha > -1) \tag{29}$$

for determining the eigenvalues $\sigma > 0$ for which $u = u_1$ are the corresponding eigenfunctions. The well-known asymptotic behaviour of the nth zero of Bessel functions $\mathcal{J}_{-\nu}$ (cf [16, VIII (Zeros 1., p 146)]) provides the asymptotics

$$\frac{1}{(\alpha+1)\sigma_n} \sim \left(n - \frac{1}{2}\right)\pi - \frac{\nu\pi}{2} + \frac{\pi}{4} \qquad \text{as} \quad n \to \infty$$

and hence the asymptotics of the sequence $\{\sigma_n\}_{n=1}^{\infty}$ of solutions to (29) tending to zero as $n \to \infty$ can be expressed by

$$\sigma_n \sim [(\alpha+1)\pi n]^{-1} \qquad \text{as} \quad n \to \infty. \tag{30}$$

This yields relation (20) and proves the theorem. Finally, note that from the above derivation it follows that the eigenfunction u_1 is absolutely continuous with integrable u_1' of order $\mathcal{O}(t^{2\alpha+1})$ as $t \to 0$ (see formula (28)). □

Corollary 2.2. *For the singular values of a compact linear operator $A = M \circ J$ defined by formulae (3), (4) and (5) with a multiplier function m satisfying for some constants $-1 < \alpha_1 \leqslant \alpha_2$ and $c, C > 0$ the inequalities*

$$ct^{\alpha_2} \leqslant |m(t)| \leqslant Ct^{\alpha_1} \qquad \textit{a.e.} \quad \textit{on } [0, 1], \tag{31}$$

we have

$$\sigma_n(A) \asymp n^{-1}. \tag{32}$$

Proof. By considering theorem 2.1 and the Poincaré–Fischer extremum principle

$$\sigma_n(A) = \max_{X_n \subset X} \min_{x \in X_n, x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where X_n represents an arbitrary n-dimensional subspace of the Hilbert space X (cf, e.g., [1, lemma 4.18] or [9, lemma 2.44]), the asymptotics (32) is an immediate consequence of the inequalities

$$c\sqrt{\int_0^1 s^{2\alpha_2}[(Jx)(s)]^2\,\mathrm{d}s} \leqslant \|Ax\| \leqslant C\sqrt{\int_0^1 s^{2\alpha_1}[(Jx)(s)]^2\,\mathrm{d}s} \qquad \text{for all} \quad x \in L^2(0, 1)$$

that follow from (31). □

Conjecture 2.3. *We conjecture that for all the compact linear operators A from (3) the asymptotic behaviour*

$$\sigma_n(A) \sim \left(\int_0^1 m(t)\,\mathrm{d}t\right)\sigma_n(J) \qquad \textit{as} \quad n \to \infty \tag{33}$$

remains true whenever the multiplier function m satisfies the inequalities

$$0 < m(t) \leqslant Ct^{\alpha} \qquad \textit{a.e.} \quad \textit{on } [0, 1] \tag{34}$$

for some $\alpha > -1$.

In her diploma thesis [6], Freitag performed numerical case studies concerning the decay rates of singular values $\sigma_n(A)$ of composite linear operators A from (3) based on three different numerical approaches:

(i) Numerical solution of corresponding Sturm–Liouville problems,
(ii) Galerkin approximation of A as proposed in [26],
(iii) Rayleigh–Ritz ansatz for A^*A and solving general eigenvalue problems.

She compared, in particular, the two families (15) and (16). In all studies formula (33) could be confirmed rather convincingly and there were no hints that exponential multiplier functions m lead to a higher degree of ill-posedness than power functions. To our knowledge, a stringent proof of formula (33), however, is still missing for the family (16).

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