MODULUS OF CONTINUITY FOR CONDITIONALLY STABLE ILL-POSED PROBLEMS IN HILBERT SPACE

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Abstract. One of the fundamental results in the theory of ill-posed inverse problems asserts that these problems can become conditionally well-posed when restricting the domain of the forward operator in an appropriate manner. This leads to the study of certain moduli of continuity for the associated restricted inverse operator. The authors systematically study this modulus of continuity and highlight its intimate connection to error bounds of various regularizing procedures. The contributions of V. K. Ivanov and his concept of quasi-solutions are fundamental for such analysis.

1. Introduction

The study of linear ill-posed operator equations, say

$$Ax = y, \tag{1}$$

is a fundamental issue within the theory of inverse problems, where, within the present context the forward operator $A : X \to Y$ is assumed to be an *injective* linear mapping between infinite-dimensional separable Hilbert spaces X and Y endowed with inner products $\langle \cdot , \cdot \rangle$ and norms $\| \cdot \|$. Ill-posedness means that the range $\mathcal{R}(A)$ of the operator A is not closed, which is equivalent to the fact that the inverse operator $A^{-1} : \mathcal{R}(A) \subset Y \to X$ is unbounded (not continuous), and hence for arbitrarily small $\delta > 0$ it holds true that

$$\sup \{ \|x\|, \quad x \in X : \ \|Ax\| \le \delta\} = \infty. \tag{2}$$

However, when restricting the domain of the forward operator A to certain subsets M of X and consequently the domain of A^{-1} to images of M this no longer needs to be the case. Most famously, by Tikhonov's theorem (see, [35] and also e.g. [1, Lemma 2.2]) the operator A^{-1} restricted to images of compact sets M is continuous and the corresponding function

$$\bar{\omega}(A^{-1}, M, \delta) := \sup \{ \|x_1 - x_2\|, \quad x_1, x_2 \in M, \|Ax_1 - Ax_2\| \le \delta\} \tag{3}$$

tends to zero as $\delta \to 0$. This result is fundamental as it asserts a *conditional stability*, and the study of properties of the function $\bar{\omega}$ from (3) has attracted attention. V. K. Ivanov was one of the first who used this for the analysis of ill-posed problems. Most frequently this function is exploited as it is closely related to error bounds, and we recall this connection. In particular the function from (3) suggests to consider instead of the original equation (1), where we denote by $x^\dagger$ its true solution, the nearby equation

$$y^\delta = Ax + \delta\xi \tag{4}$$

as model with bounded $\|\xi\| \le 1$, where the variable δ from (3) expresses the noise level in (4). Throughout this paper let $y^\delta \in Y$ denote noisy data of y corresponding to (1) with noise level $\delta > 0$ satisfying

$$\|y - y^\delta\| \le \delta. \tag{5}$$

A systematic study of properties of $\bar{\omega}$ is lacking, and the authors try to close this gap. We start this study with some elementary analysis in Section 2. In particular we focus on the related to (3) function

$$\omega\,(M,\delta) := \sup\left\{\|x\|, \quad x \in M,\ \|Ax\| \le \delta\right\}, \tag{6}$$

which will be called *modulus of continuity*, throughout. Then, in Section 3 we provide the few examples of sets M where the modulus of continuity can be computed explicitly. Again, such results are related to the work of V. K. Ivanov, see [17]. As mentioned before, the relation of the moduli of continuity to minimal errors of any reconstruction method based on *noisy data* is essential. We recall this basic relation in Section 4 and accompany this by showing that the bounds are sharp for *ellipsoids* in Hilbert space. We use the modulus of continuity to derive error bounds for various classes of *regularizing* procedures. Such procedures complement the *ansatz* of *quasi-solutions* pioneered by V. K. Ivanov in [14, 15, 16], and try to circumvent specific weakness in some points.

2. Elementary properties

We will restrict ourselves to centrally symmetric and convex sets M, which means that with $x_1, x_2 \in M$ also the elements $-x_2$ and $(x_1 - x_2)/2$ belong to M. Given some constant $R > 0$ we agree to denote the set $RM := \{Rz \in X, z \in M\}$. The following result summarizes the elementary properties of moduli of continuity, useful in case that this is a finite function of δ, which will be used throughout the study.

Theorem 1. *For the moduli of continuity from (6) the following properties hold:*

(a) $\omega\,(M,\delta)$ *is a positive and non-decreasing function for* $\delta > 0$*. If* M *is bounded then it is constant for* $\delta \ge \bar{\delta} := \sup_{x\in M} \|Ax\|$.
(b) If M *is relatively compact then* $\lim_{\delta\to 0} \omega\,(M,\delta) = 0$.
(c) $\omega\,(RM,\delta) = R\,\omega\,(M,\delta/R)$ *for* $R > 0$.
(d) $\omega\,(M,C\delta) \le C\omega\,(M,\delta)$ *for* $C > 1$.
(e) $\omega\,(RM,C\delta) \le \max\{C,R\}\,\omega\,(M,\delta)$ *for* $C,R > 0$.
(f) the decay rate of $\omega\,(M,\delta) \to 0$ *as* $\delta \to 0$ *is at most linear.*

(g) *The modulus of continuity* ω *from (6) is related to the function* $\bar{\omega}$ *from (3) by the inequality chain*

$$\omega\left(M,\delta\right) \le \bar{\omega}(M,\delta) \le 2\omega\left(M,\delta/2\right) = \omega\left(2M,\delta\right), \quad \delta > 0.$$

Proof. Item (a) is obvious. To prove item (b) we assume that there would be a constant $\omega_0 > 0$ with $\omega\left(M,\delta\right) \ge \omega_0 > 0$. Then we can find for each $n \in \mathbb{N}$ some $x_n \in M$ with $\|Ax_n\| \le 1/n$ but $\|x_n\| \ge \omega_0$. Thus $Ax_n \to 0$ for $n \to \infty$, but by Tikhonov's theorem $x_n \to 0$ in contradiction to $\|x_n\| \ge \omega_0$.

We rewrite

$$\begin{aligned}\omega\left(RM,C\delta\right) &= \sup\left\{\|x\|, \quad x \in RM, \|Ax\| \le C\delta\right\} \\ &= R\,\sup\left\{\frac{\|x\|}{R}, \quad \frac{x}{R} \in M, \frac{\|Ax\|}{R} \le \frac{C}{R}\delta\right\} \\ &= R\,\omega\left(M, \frac{C}{R}\delta\right).\end{aligned} \tag{7}$$

For $C = 1$ this yields (c). Clearly, the function

$$\frac{\omega\left(M,\delta\right)}{\delta} = \sup\left\{\|z\|, \quad z \in \frac{1}{\delta}M, \|Az\| \le 1\right\}, \quad \delta > 0,$$

is a non-increasing function as for declining δ the set where we compute the supremum for widens. Thus, for $C > 1$ we have that

$$\frac{\omega\left(M,C\delta\right)}{C\delta} \le \frac{\omega\left(M,\delta\right)}{\delta}, \tag{8}$$

which shows (d). Furthermore, we do a distinction of cases to show (e) from (7):

$$\omega\left(RM,C\delta\right) \le \begin{cases} C\omega\left(M,\delta\right), & C \ge R \text{ by (d)} \\ R\omega\left(M,\delta\right), & C < R \text{ by monotonicity.}\end{cases}$$

Again, if we consider $\bar{\delta}$ as in (a), then it follows from (8) that

$$\bar{C} := \frac{\omega\left(M,\bar{\delta}\right)}{\bar{\delta}} \le \frac{\omega\left(M,\delta\right)}{\delta}, \quad 0 \le \delta < \bar{\delta},$$

which proves (f).

The first inequality in (g) follows from $0 \in M$ for centrally symmetric M. With arbitrary $\varepsilon > 0$ we choose $x_1, x_2 \in M$ with $\|Ax_1 - Ax_2\| \le \delta$ and $\|x_1 - x_2\| \ge \bar{\omega}(M,\delta) - \varepsilon$. As M is centrally symmetric $\tilde{x} := (x_1 - x_2)/2 \in M$ and we conclude $\|A\tilde{x}\| \le \delta/2$. We complete the proof by estimating

$$\omega\left(M,\delta/2\right) \ge \|\tilde{x}\| = \|(x_1 - x_2)/2\| \ge 1/2\,\left(\bar{\omega}(M,\delta) - \varepsilon\right)$$

and letting $\varepsilon \to 0$. $\square$

We close this section with the following result, which exhibits the importance of the discrepancy $\|A\hat{x} - y^\delta\|$ at any point $\hat{x} \in X$ for bounding the modulus of continuity.

Lemma 1. *Let M be centrally symmetric and convex, and suppose that $x \in M$. If $\hat{x} = \hat{x}(y^\delta) \subset M$ is an element derived from data $y^\delta \in Y$ satisfying*

$$\|A\hat{x} - y^\delta\| \le \varepsilon$$

for some $\varepsilon > 0$, then we have

$$\|x - \hat{x}\| \le 2\,\omega\,(M, (\delta + \varepsilon)/2) = \omega\,(2M, \delta + \varepsilon)\,. \tag{9}$$

Proof. By assumption both elements x and $\hat{x}$ belong to the set M. Moreover, using the triangle inequality we have that

$$\|Ax - A\hat{x}\| \le \|Ax - y^\delta\| + \|A\hat{x} - y^\delta\| \le \delta + \varepsilon.$$

Therefore we can bound

$$\|x - \hat{x}\| \le \bar{\omega}(M, \delta + \varepsilon).$$

By Theorem 1(g) we can complete the proof of the lemma. □

3. Explicit computation of moduli of continuity

For several cases of sets M one can actually compute the corresponding moduli $\omega\,(M, \delta)$. This was first done for *source sets* with respect to the operator $H := A^*A$ in [17], and we will slightly extend this analysis here.

On the other hand the modulus of continuity can be evaluated in case that $M \subset X$ is some finite-dimensional subspace. This was done, in slightly different form, in [21]. For the latter analysis it is important to assume that the operator A is compact, i.e., there is a triple $\{\sigma_j\,;\,u_j\,;\,v_j\}_{j=1}^\infty$, the singular system of A, with positive singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_j \ge \sigma_{j+1} \ge \ldots$ tending to zero as $j \to \infty$. Moreover, $\{u_j\}_{j=1}^\infty \subset X$ is a complete orthonormal system in X and $\{v_j\}_{j=1}^\infty \subset Y$ is an orthonormal system in Y such that $Au_j = \sigma_j v_j$ and $A^* v_j = \sigma_j u_j$ for $j = 1, 2, \ldots$.

Both results below are important as they relate the modulus of continuity to intrinsic geometric properties of the sets M.

3.1. Source sets.

Source sets $M \subset X$ are defined using the notion of an *index function.* Complying with [9, 11] we introduce

Definition 1. A real function $\varphi(t)$ $(0 \le t \le \bar{t})$ with $\bar{t} > 0$ is called an index function, if this function is continuous and strictly increasing with $\varphi(0) = 0$.

At some place we shall need the following additional condition.

Definition 2. An index function $\varphi(t)$, $0 \le t \le \bar{t}$, is said to obey a Δ_2-condition, if there is a constant $0 < C_2 < \infty$ such that $\varphi(2t) \le C_2\,\varphi(t)$, $0 < t \le \bar{t}/2$.

By Theorem 1(d) a Δ_2-condition automatically holds for the modulus of continuity with $C_2 = 2$.

Given the operator A from (1) we assign the nonnegative self-adjoint operator $H := A^*A\colon X \to X$. Sets $M \subset X$ are called source sets if there is an index function $\varphi(t),\ 0 \le t \le \|H\|$ such that

$$M := H_\varphi = \{x = \varphi(H)v, \quad \|v\| \le 1\}\,. \tag{10}$$

Remark 1. The above notation is frequently used in the form $H_\varphi(R) = R \cdot H_\varphi$. However, given some pair φ and $R < \infty$ we can switch to the pair $R\varphi$ and 1 to always reduce the consideration to the standardized source sets $H_\varphi := H_\varphi(1)$.

As mentioned above, for source sets H_φ the modulus $\omega\left(H_\varphi, \delta\right)$ was computed in [17], however, in a rudimentary form. Later in [8] this result was revisited within the framework of *variable Hilbert scales* as the families of Hilbert spaces generated from index functions φ are called since then. In its more recent form the result was first stated in [26], and the present approach follows that study.

To formulate the result for source sets H_φ we assign to the function φ the related function

$$\Theta(t) := \sqrt{t}\varphi(t), \quad 0 \le t \le \bar{t}. \tag{11}$$

Theorem 2. *If the function* $t \mapsto \varphi^2((\Theta^2)^{-1}(t))$ *is concave, then*

$$\omega^2(H_\varphi, \delta) = s(\delta), \quad 0 < \delta \le \Theta^2(\|H\|), \tag{12}$$

where s is a piecewise linear spline, interpolating

$$s(\Theta(\sigma_j^2)) = \varphi^2(\sigma_j^2), \quad j = 1, 2, \dots. \tag{13}$$

The concavity assumption of Theorem 2 is essential. It ensures that the points $\left(\Theta(\sigma_j^2), \varphi^2(\sigma_j^2)\right),\ j = 1, 2, \dots$ are extremal in the sub-graph of the spline s from above. This, in turn, corresponds to the convexity of the sub-graph, a condition imposed in [17].

One could guess that the modulus of continuity $\omega\left(H_\varphi, \delta\right)$ is always a concave function, but we can show this only for the auxiliary function $\omega^2\left(H_\varphi, \sqrt{\delta}\right)$.

Proposition 1. *For any index function φ the function*

$$\tau(\delta) := \omega^2\left(H_\varphi, \sqrt{\delta}\right), \qquad \delta > 0, \tag{14}$$

is concave.

Proof. For any source element $v = \sum_{j=1}^{\infty} f_j u_j$ with $\|v\| \le 1$ where $f_j = \langle v, u_j\rangle$, denote the Fourier coefficients, we let

$$\beta_j := \varphi^2(\sigma_j^2), \quad \gamma_j := \Theta^2(\sigma_j^2), \quad \kappa_j := f_j^2 \ge 0, \quad j = 1, 2, ...,$$

and rewrite

$$\tau(\delta) = \sup \left\{ \sum_{j=1}^{\infty} \beta_j \kappa_j, \quad \sum_{j=1}^{\infty} \kappa_j \le 1, \quad \sum_{j=1}^{\infty} \gamma_j \kappa_j \le \delta \right\}.$$

Consider arbitrarily chosen $0 < \delta_1 < \delta < \delta_2$ and $\delta = \lambda\delta_1 + (1-\lambda)\delta_2$ for some appropriate $0 < \lambda < 1$. With given $\varepsilon > 0$ we can find convex linear combinations $\{\mu_j\}$ and $\{\nu_j\}$ for which

$$\sum_{j=1}^{\infty} \gamma_j \mu_j \le \delta_1, \quad \sum_{j=1}^{\infty} \beta_j \mu_j \ge \tau(\delta_1) - \varepsilon\,,$$

and

$$\sum_{j=1}^{\infty} \gamma_j \nu_j \le \delta_2, \quad \sum_{j=1}^{\infty} \beta_j \nu_j \ge \tau(\delta_2) - \varepsilon\,.$$

Then each $\kappa_j := \lambda\mu_j + (1-\lambda)\nu_j$ again constitutes a convex linear combination and we have $\sum\limits_{j=1}^{\infty} \gamma_j\kappa_j = \lambda \sum\limits_{j=1}^{\infty} \gamma_j \mu_j + (1-\lambda) \sum\limits_{j=1}^{\infty} \gamma_j \nu_j \le \delta$, hence

$$\begin{aligned}
\tau(\delta) &\ge \sum_{j=1}^{\infty} \beta_j \kappa_j \\
&= \lambda \sum_{j=1}^{\infty} \beta_j \mu_j + (1-\lambda) \sum_{j=1}^{\infty} \beta_j \nu_j \\
&\ge \lambda\tau(\delta_1) + (1-\lambda)\tau(\delta_2) - \varepsilon\,.
\end{aligned}$$

Letting $\varepsilon \to 0$ proves the required concavity assertion. □

Remark 2. The function τ from Proposition 1 is the *smallest concave* index function which interpolates the points $\left(\Theta(\sigma_j^2), \varphi^2(\sigma_j^2)\right)$, $j = 1, 2, \ldots$.

We summarize the results for source sets as follows.

Corollary 1 (see [26, Thm. 1])**.** *Let φ be an index function.*

(i) If $t \to \varphi^2((\Theta^2)^{-1}(t))$ is concave, then

$$\omega\left(H_\varphi, \delta\right) \le \varphi(\Theta^{-1}(\delta)), \quad 0 < \delta \le \bar{\delta}. \tag{15}$$

(ii) If φ obeys a Δ_2-condition and the singular numbers obey $\sigma_{j+1}/\sigma_j \ge \gamma > 0$, then there is a constant $C_\gamma > 0$ such that

$$\omega\left(H_\varphi, \delta\right) \ge C_\gamma^{-1} \varphi(\Theta^{-1}(\delta)), \quad 0 < \delta \le \bar{\delta}. \tag{16}$$

Remark 3. We stress that the bound from (15) can be proved directly and not relying on the representation from Theorem 2 by using some interpolation inequality from [27] which proves useful for the analysis of variable Hilbert scales.

3.2. **Finite dimensional subspaces.** We next discuss the modulus of continuity for finite dimensional subspaces $M := X_n \subset X$ with $\dim(X_n) = n$ spanned by n linear independent elements $e_1, e_2, ..., e_n$ of X, i.e., $X_n = \operatorname{span}\{e_1, e_2, ..., e_n\}$. Although the subspaces $X_n \subset X$ are non-compact, the corresponding modulus of continuity $\omega(X_n, \delta)$ can be studied, and it turns out to be related to the *modulus of injectivity* $j(A, X_n)$, defined next, which is a quantity measuring how well the operator A when restricted to X_n is invertible. Precisely, we let

$$j(A, X_n) := \inf_{0 \neq x \in X_n} \frac{\|Ax\|}{\|x\|}. \tag{17}$$

The following result relates the modulus of continuity to the modulus of injectivity.

Proposition 2. *Let $X_n \subset X$ be any n-dimensional subspace. Then*

$$\omega(X_n, \delta) = \frac{\delta}{j(A, X_n)}. \tag{18}$$

The modulus of continuity is smallest if $X_n := U_n = \{u_1, \dots, u_n\}$ is the n-dimensional subspace corresponding to the n largest singular numbers of the operator A. In this case

$$\omega(U_n, \delta) = \frac{\delta}{\sigma_n}.$$

Proof. The modulus of continuity is given as a solution of a constrained optimization problem and we turn to unconstrained optimization using a Lagrange multiplier, say λ. Precisely, let P_n be the orthogonal projection onto X_n. Then

$$\omega(X_n, \delta) = \|P_n \bar{x}\|,$$

where $\bar{x} \in X$ maximizes

$$F(x) := \|P_n x\|^2 - \lambda \left(\|AP_n x\|^2 - \delta^2\right).$$

This yields $P_n \bar{x} = \lambda A P_n \bar{x}$ with $\|AP_n \bar{x}\|^2 = \delta^2$, such that $\|P_n \bar{x}\| = |\lambda| \, \delta$. We observe that the latter norm is maximized for

$$\lambda_{\max} := \sup_{0 \neq x \in X_n} \frac{\|x\|}{\|Ax\|} = 1/j(A, X_n),$$

which completes the proof of (18). The remaining assertion follows from the well-known fact that the moduli of injectivity are maximal for the eigenspaces as used, in which case these are equal to the n-th Bernstein numbers, which in Hilbert space coincide with the n-th singular numbers. We refer to [25] for more discussion. □

Remark 4. It is easy to see, and explains the name *modulus of injectivity*, that

$$j(A, X_n) = \sup \{\rho, \quad \rho\|x\| \leq \|Ax\| \text{ for all } x \in X_n\}.$$

If we introduce for given X_n vectors $\underline{\lambda} = \sum_{j=1}^{n} \lambda_j e_j$ in this subspace as well as the positive definite and symmetric $n \times n$-matrices $\underline{E}_1 = (\langle e_i, e_j \rangle)_{j=1,2,\ldots,n}^{i=1,2,\ldots n}$ and $\underline{E}_2 = (\langle Ae_i, Ae_j \rangle)_{j=1,2,\ldots,n}^{i=1,2,\ldots n}$, then following the paper [21] we obtain that

$$j(A, X_n) = \inf_{\underline{\lambda}:\, \underline{\lambda}^T \underline{E}_1 \underline{\lambda} = 1} \sqrt{\underline{\lambda}^T \underline{E}_2 \underline{\lambda}} = \xi_{min},$$

where $\xi_{min} = \xi_n$ is the smallest of the n eigenvalues $\xi_1 \geq \xi_2 \geq \ldots \geq \xi_n > 0$ of the generalized eigenvalue problem

$$\underline{E}_2\, \underline{\lambda} = \xi\, \underline{E}_1 \underline{\lambda}.$$

We notice that $\xi_{min} \leq \sigma_n$ (see, e.g., [31]). Hence, the modulus of injectivity is always equal or smaller than the n-th singular number σ_n of the compact operator A. For $e_j = u_j$, $j = 1, 2, \ldots, n$, taken from the singular system of the compact operator A we have a unity matrix $\underline{E}_1$, since the u_j are orthonormal, and a diagonal matrix $\underline{E}_2 = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2\}$ yielding $j(A, X_n) = \sigma_n$.

The representation from (18) clearly exhibits the fact that such finite-dimensional constrains make the problem well-posed with *conditioning* depending on the quality of the chosen subspace X_n through its *modulus of injectivity*. We refer to [38] (see also [10]) for the fact that the condition numbers of occurring $n \times n$-systems for a Galerkin discretization approach to (1) behave equal or worse than $\mathcal{O}(1/\sigma_n)$ as $n \to \infty$. If, for example, (1) is moderately ill-posed in the sense of $\sigma_n \sim n^{-\mu}$ with some exponent $1 \leq \mu < \infty$, then the condition numbers grow with order n^{μ} at least.

4. Impact of the modulus of continuity on error bounds

Moduli of continuity are important as these are related to minimal errors for any methods of reconstruction $\hat{x} = \hat{x}(y^\delta)$ of $x^\dagger$, which shall be the true solution of (1) from noisy data y^δ when $\|y - y^\delta\| \leq \delta$. If the mapping $y^\delta \mapsto \hat{x}$ is linear, then we call this a *linear reconstruction*. As reconstruction error at instance $x \in X$ we let

$$e(\hat{x}, x, \delta) := \sup \left\{ \|\hat{x}(y^\delta) - x\|, \quad \text{for } y^\delta \text{ with } \|Ax - y^\delta\| \leq \delta \right\}. \tag{19}$$

Consequently, we introduce as

$$e(\hat{x}, M, \delta) := \sup_{x \in M} e(\hat{x}, x, \delta), \quad \delta > 0, \tag{20}$$

the *uniform error* on the set M of the reconstruction method $\hat{x}$ at noise level δ. Thus, the minimal error for any method is given as

$$e(M, \delta) := \inf_{\hat{x}: Y \mapsto X} e(\hat{x}, M, \delta). \tag{21}$$

Error analysis results using the modulus of continuity (6), partially in terms of the notation introduced above, were published by several authors, e.g., in the monographs [3, 6, 18, 22, 33, 37] and in the papers [5, 17, 21, 34].

The basic result which relates the modulus of continuity to minimal errors is as follows, and we refer to the monographs [18, 37] for proof and further details.

Proposition 3. *For any* centrally symmetric and convex *set* $M \subset X$ *and any* $\delta > 0$ *it holds true that*

$$\omega(M, \delta) \le e(M, \delta) \le \omega(M, 2\delta). \tag{22}$$

Thus the moduli of continuity allow us to find benchmarks for errors of reconstruction methods $\hat{x} = \hat{x}(y^\delta)$ based on data y^δ.

The interesting result which we are going to establish next states that the above lower bound is sharp for the collection of *ellipsoids* in Hilbert space. Such sets are images of balls under linear bounded mappings. Without loss of generality such mappings may be chosen non-negative and self-adjoint, acting within the space X. Thus, to any injective, positive and self-adjoint operator $G : X \to X$ we assign the ellipsoid

$$M(G) := \{x \in X : \quad x = Gw,\ w \in X,\ \|w\| \le 1\}. \tag{23}$$

Clearly, source sets H_φ are ellipsoids, precisely it holds $H_\varphi = M(\varphi(A^*A))$. For its description and properties we recall the following *minimax result* from [30, Thm. 2.1].

Lemma 2. *If* $M(G)$ *is as in (23) then*

$$\omega(M(G), \delta) = \min_{0 \le \lambda \le 1} \sup \left\{ \|Gv\|, \quad \lambda \|v\|^2 + \frac{1-\lambda}{\delta^2} \|AGv\|^2 \le 1 \right\}. \tag{24}$$

Remark 5. Notice that the right hand side need not be finite unless some restriction on the operator G is imposed. So far we have seen that finiteness can be ensured for source sets with $G := A^*A$, finite-dimensional subspaces X_n with $G := P_n$ the orthogonal projection onto this and compact sets, i.e., when G is any compact operator.

The following fundamental theorem was inspired by Theorem 3.1 of the seminal contribution [30]. Given the mapping G it will use the specific form

$$J_\alpha(v) := \|AGv - y^\delta\|^2 + \alpha \|v\|^2, \quad v \in X \tag{25}$$

of *Tikhonov functional* with regularization parameter $\alpha > 0$. Since

$$S_\alpha := (AG)^* (AG) + \alpha I \colon X \to X. \tag{26}$$

is a boundedly invertible positive self-adjoint operator for all $\alpha > 0$, by differentiation we see that a minimizer v_α^δ of (25) obeying $S_\alpha v_\alpha^\delta = G^* A^* y^\delta$ is uniquely determined. Therefore we let $x_\alpha^\delta = G v_\alpha^\delta$, hence the linear reconstruction $y^\delta \mapsto \hat{x} := x_\alpha^\delta$ is given as

$$x_\alpha^\delta := G \left((AG)^* (AG) + \alpha I \right)^{-1} G^* A^* y^\delta, \tag{27}$$

which is well defined for all $\alpha > 0$.

Theorem 3. *Suppose that $M(G)$ is as in (23) and that $x^\dagger \in M(G)$. Then for each $\delta > 0$ there is a linear reconstruction $\hat{x}_0 = \hat{x}_0(y^\delta)$ with*

$$e(\hat{x}_0, M(G), \delta) = \omega\,(M(G), \delta)\,. \tag{28}$$

Proof. If $\omega\,(M(G),\delta) = \infty$ there is nothing to prove. Otherwise, let $\lambda_0 = \lambda_0(\delta)$ be the parameter for which the minimum in (24) is attained. We assign $\alpha_0 := \delta^2\lambda_0/(1-\lambda_0)$ and the specification of the convex functional (25) as

$$J_0(v) := J_{\alpha_0}(v), \quad v \in X. \tag{29}$$

Let for the true solution hold $x^\dagger = Gv^\dagger$ with $v^\dagger \in X$ and $\|v^\dagger\| \le 1$. Then we have

$$J_0(v^\dagger) = \|AGv^\dagger - y^\delta\|^2 + \alpha_0\|v^\dagger\|^2 \le \delta^2 + \alpha_0 = \frac{\delta^2}{1-\lambda_0}. \tag{30}$$

With the operator S_{α_0} from (26) we observe by straightforward calculations that

$$J_0(v^\dagger) - J_0(v^\delta_{\alpha_0}) = \langle S_{\alpha_0}(v^\dagger - v^\delta_{\alpha_0}), (v^\dagger - v^\delta_{\alpha_0})\rangle - 2\langle (v^\dagger - v^\delta_{\alpha_0}), S_{\alpha_0}v^\delta_{\alpha_0} - G^*A^*y^\delta\rangle.$$

Since by construction $S_{\alpha_0}v^\delta_{\alpha_0} = G^*A^*y^\delta$ we arrive at

$$J_0(v^\dagger) - J_0(v^\delta_{\alpha_0}) = \langle S_{\alpha_0}(v^\dagger - v^\delta_{\alpha_0}), (v^\dagger - v^\delta_{\alpha_0})\rangle.$$

Therefore, and since $J_0(v^\delta_{\alpha_0}) \ge 0$ we conclude, by specifying $\hat{x}_0 := x^\delta_{\alpha_0}$ in (27), and using (30) that

$$\begin{aligned}
e(\hat{x}_0, M, \delta) &= \sup\left\{\|x^\dagger - Gv^\delta_{\alpha_0}\|, \quad x^\dagger = Gv^\dagger,\ \|v^\dagger\| \le 1,\ \|AGv^\dagger - y^\delta\| \le \delta\right\} \\
&\le \sup\left\{\|G(v^\dagger - v^\delta_{\alpha_0})\|, \quad J_0(v^\dagger) \le \frac{\delta^2}{1-\lambda_0}\right\} \\
&= \sup\left\{\|G(v^\dagger - v^\delta_{\alpha_0})\|, \quad \langle S_{\alpha_0}(v^\dagger - v^\delta_{\alpha_0}), (v^\dagger - v^\delta_{\alpha_0})\rangle \le \frac{\delta^2}{1-\lambda_0} - J_0(v^\delta_{\alpha_0})\right\} \\
&= \sup\left\{\|G(v^\dagger - v^\delta_{\alpha_0})\|, \quad \langle S_{\alpha_0}(v^\dagger - v^\delta_{\alpha_0}), (v^\dagger - v^\delta_{\alpha_0})\rangle \le \frac{\delta^2}{1-\lambda_0}\right\}.
\end{aligned}$$

Now we rename $h := v^\dagger - v^\delta_{\alpha_0}$ and use that

$$\langle S_{\alpha_0}h, h\rangle = \|AGh\|^2 + \alpha_0\|h\|^2, \quad h \in X.$$

This leads to

$$\begin{aligned}
e(\hat{x}_0, M, \delta) &\le \sup\left\{\|Gh\|, \quad \langle S_{\alpha_0}h, h\rangle \le \frac{\delta^2}{1-\lambda_0}\right\} \\
&= \sup\left\{\|Gh\|, \quad \frac{1-\lambda_0}{\delta^2}\|AGh\|^2 + \lambda_0\|h\|^2 \le 1\right\}.
\end{aligned}$$

By the choice of λ_0 as minimizer in (24) we obtain from Lemma 2 that $e(\hat{x}_0, M(G), \delta) \le \omega\,(M(G), \delta)$, and together with Proposition 3 this completes the proof of the theorem. $\square$

The proof of Theorem 3 reveals the following interesting corollary which was first presented in [3, Thm. 3.4], we also refer to the more recent [5, Thm. 2.1].

Corollary 2. *Let for the true solution hold $x^\dagger \in M(G)$ and let $0 < \underline{c} < \overline{c} < \infty$ be positive constants. If we consider as reconstruction elements $\hat{x}(y^\delta)$ the regularized solutions $x^\delta_{\alpha(\delta)} = Gv^\delta_{\alpha(\delta)}$ according to (27) by using an a priori choice of the regularization parameter $\alpha = \alpha(\delta)$ such that $\underline{c}\delta^2 \le \alpha(\delta) \le \overline{c}\delta^2$, then we have the error estimate*

$$\|x^\dagger - x^\delta_{\alpha(\delta)}\| \le \max\left\{2\sqrt{1+1/\underline{c}}, 1+\sqrt{1+\overline{c}}\right\} \omega\left(M(G), \delta\right). \tag{31}$$

Proof. A look into the proof of Theorem 3, and using the fact that $v^\delta_{\alpha(\delta)}$ minimizes the functional (25) for $\alpha = \alpha(\delta)$ reveals that

$$J_{\alpha(\delta)}(v^\delta_{\alpha(\delta)}) \le J_{\alpha(\delta)}(v^\dagger) \le \delta^2 + \alpha(\delta).$$

This implies that

$$\|AGv^\delta_{\alpha(\delta)} - y^\delta\| \le \sqrt{\delta^2 + \alpha(\delta)} \le \delta\sqrt{1+\overline{c}}, \tag{32}$$

and

$$\|v^\delta_{\alpha(\delta)}\| \le \frac{1}{\sqrt{\alpha(\delta)}}\sqrt{\delta^2+\alpha(\delta)} \le \sqrt{1+1/\underline{c}}. \tag{33}$$

By Lemma 1 this allows to bound

$$\|x^\dagger - x^\delta_{\alpha(\delta)}\| \le 2\omega\left(\left(\sqrt{1+1/\underline{c}}\right)M, \left(\frac{1+\sqrt{1+\overline{c}}}{2}\right)\delta\right),$$

from which the proof can be completed, using Theorem 1(e). □

5. Stability of regularization schemes

5.1. **Quasi-solutions.** A very traditional approach for exploiting the effect of conditional stability is the *method of quasi-solutions*, as first suggested by V. K. Ivanov (see the monograph [18] for more details and the survey paper [19] for the history of definitions and approaches by Russian scientists).

For compact sets M the approximate solutions $\hat{x}_{qu} = \hat{x}_{qu}(y^\delta)$ are selected by solving the discrepancy minimization problem

$$\|Ax - y^\delta\| \to \min, \qquad \text{subject to} \quad x \in M. \tag{34}$$

Since M is a compactum, the solution set of (34) is non-empty, but not necessarily a singleton. Measured by appropriate set-distances the solutions of (34) stably depend on the data y^δ. The power of this *ansatz* is as follows.

Proposition 4. *Suppose that $x^\dagger \in M$ and let $\hat{x}_{qu}$ be any solution to (34). Then we have the error estimate*

$$\|x^\dagger - \hat{x}_{qu}\| \le 2\omega\left(M, \delta\right). \tag{35}$$

Proof. By construction of $\hat{x}_{qu}$ it holds $\|A\hat{x}_{qu} - y^\delta\| \le \|Ax^\dagger - y^\delta\| \le \delta$. Thus, Lemma 1 applies with $\varepsilon = \delta$ and yields the conclusion. □

Unfortunately, the explicit prescription of the set M is necessary in order to use this method of quasi-solutions. This makes it difficult to find quasi-solutions of practical relevance for M from (23) or (10). Even if an a priori smoothness $x^\dagger \in \mathcal{R}(G)$ with given G is available, i.e., the smoothness of the true solution is expressed by the fact that $x^\dagger$ belongs to the range of G, the occurring scaling factor R from $x^\dagger \in R \cdot M(G)$ can hardly be estimated in practice. Since quasi-solutions in general lie on the boundary of a set $M := R \cdot M(G)$, the subjective choice of the radius $R > 0$ essentially influences the properties of corresponding quasi-solutions.

5.2. **Linear regularization schemes.** Regularization methods (see [36] and e.g. [2, 3, 7, 32]) have a long history in stabilizing the solution process of ill-posed equations (1). We will shortly present the ideas of a general linear regularization scheme unifying a multitude of ansatzes (see, e.g., [1, p. 74ff.]). We start with a family of piecewise continuous functions

$$g_\alpha(t), \quad 0 < t \le a := \|A^*A\|,$$

depending on a regularization parameter α with $0 < \alpha < \alpha_{sup}$, where α_{sup} may be a finite real number or ∞. This parameter compromises between stabilization and approximation of the original problem. Each function g_α describes a regularization method if the following properties hold:

$$\sup_{0<t\le a} |\, t\, g_\alpha(t)\,| \le \gamma_0, \tag{36}$$

$$\lim_{\alpha\to 0} g_\alpha(t) = \frac{1}{t}, \quad 0 < t \le a, \tag{37}$$

$$\sup_{0<t\le a} \sqrt{t}\,|g_\alpha(t)| \le \frac{\gamma_*}{\sqrt{\alpha}}, \tag{38}$$

with constants $\gamma_0, \gamma_* > 0$.

Two approximate solutions gained by the regularization methods need to be distinguished, namely

$$\hat{x}_\alpha := g_\alpha(A^*A)A^*y = g_\alpha(A^*A)A^*Ax^\dagger, \tag{39}$$

in the noise-free case while in the case of noisy data we let

$$\hat{x}_\alpha^\delta := g_\alpha(A^*A)A^*y^\delta. \tag{40}$$

Note that the intrinsic application of the function g_α is with respect to the self-adjoint operator A^*A, see [20, Remark 2.3].

By (36), (39) and (40) we derive

$$\|\hat{x}_\alpha^\delta - \hat{x}_\alpha\| \le \frac{\gamma_*\delta}{\sqrt{\alpha}}\,. \tag{41}$$

As in [23, 26, 27] we measure the *qualification* of a regularization method by index functions ψ.

Definition 3. An index function $\psi(t),\ 0 \le t \le a$, is said to be a qualification with constant $1 \le \gamma < \infty$ of the regularization method g_α applied to the operator equation (1) if for some $\tilde{\alpha} \in (0, a]$

$$\sup_{0<t\le a} |1 - t g_\alpha(t)|\psi(t) \le \gamma\psi(\alpha), \qquad 0 < \alpha \le \tilde{\alpha}\,. \tag{42}$$

One finds examples for concrete methods described by g_α, their qualifications and the corresponding constants in the formulae (36), (38) and (42), e.g., in [11, chapter 2.1] and [20, Remark 2.8].

In order to obtain convergence rates concerning the total error for general regularization methods we use the obvious error decomposition

$$\|\hat{x}_\alpha^\delta - x^\dagger\| \le \|\hat{x}_\alpha - x^\dagger\| + \|\hat{x}_\alpha^\delta - \hat{x}_\alpha\|, \tag{43}$$

and we shall bound the noiseless term $\|\hat{x}_\alpha - x^\dagger\|$ using the modulus of continuity. Therefore, we assume that $x^\dagger, \hat{x}_\alpha \in M$ for sufficiently small $\alpha > 0$. Then the definition of the function $\bar{\omega}$ from (3) yields that

$$\|\hat{x}_\alpha - x^\dagger\| \le \bar{\omega}(M, \|A\hat{x}_\alpha - Ax^\dagger\|), \tag{44}$$

hence we need to bound the discrepancy in order to apply Theorem 1(g). We obtain

$$\|A\hat{x}_\alpha - Ax^\dagger\| = \|A[I - g_\alpha(A^*A)A^*A]x^\dagger\|. \tag{45}$$

Now we can continue in two ways. First, we shall not use any source condition on the exact solution $x^\dagger$. Let us assign to the modulus of continuity $\omega\left(M, \sqrt{t}\right)$ the function

$$\Psi(t) := \sqrt{t}\,\omega\left(M, \sqrt{t}\right), \quad 0 < t \le \bar{t}. \tag{46}$$

Theorem 4. *Let $M \subset X$ be centrally symmetric and convex. Suppose that the regularization method with generator function g_α has the qualification $\sqrt{t}$, and that both $x^\dagger$ and x_α from (39) for sufficiently small $\alpha > 0$ belong to M. If $\alpha(\delta) = \Psi^{-1}(\delta)$ with Ψ from (46) then*

$$\|\hat{x}_\alpha^\delta - x^\dagger\| = \mathcal{O}\left(\omega\left(M, \sqrt{\Psi^{-1}(\delta)}\right)\right) \qquad \text{as } \delta \to 0\,. \tag{47}$$

Proof. If the regularization method g_α has a qualification $\sqrt{t}$ with constant $1 \le \gamma < \infty$ then, as outlined in [20], it holds

$$\|A\hat{x}_\alpha - Ax^\dagger\| \le \gamma\|x^\dagger\|\sqrt{\alpha}.$$

For centrally symmetric and convex sets M this leads with Theorem 1(g) to

$$\|\hat{x}_\alpha - x^\dagger\| \le \bar{\omega}\left(M, \gamma\|x^\dagger\|\sqrt{\alpha}\right) \le 2\omega\left(M, \frac{\gamma\|x^\dagger\|}{2}\sqrt{\alpha}\right). \tag{48}$$

Thus, the total error of regularization is bounded using the decomposition (43) and (41) as

$$\|\hat{x}_\alpha^\delta - x^\dagger\| \le \frac{\gamma_* \delta}{\sqrt{\alpha}} + \max\left\{2, \gamma\|x^\dagger\|\right\} \omega\left(M, \sqrt{\alpha}\right) .$$

with the help of Theorem 1(e). Under the a priori parameter choice be $\alpha(\delta) \sim \Psi^{-1}(\delta)$ we obtain (47), which completes the proof of the theorem. □

More comprehensive and satisfactory results can be found if $M := R \cdot H_\varphi$, i.e., if we can assume that $x^\dagger$ satisfies a source condition $x^\dagger = \varphi(H)v$ with an index function φ and $\|v\| \le R$, and if the regularization g_α has higher qualification.

Remark 6. It was shown in [24] that for compact operators $H = A^*A$ each element $x^\dagger \in X$ belongs to some source set $R \cdot H_\varphi$ for a certain pair (φ, R).

Theorem 5. *Suppose that $x^\dagger \in H_\varphi$ for some index function φ, and suppose that the regularization method with generator function g_α has the qualification $\Theta(t)$ for the function Θ from (11). If for sufficiently small $\alpha > 0$ it holds true that x_α from (39) belongs to H_φ, and if $\alpha(\delta) = \Theta^{-1}(\delta)$ then*

$$\|\hat{x}_{\alpha(\delta)}^\delta - x^\dagger\| = \mathcal{O}\left(\varphi\left(\Theta^{-1}(\delta)\right)\right) \qquad \textit{as } \delta \to 0, \tag{49}$$

provided that the function $t \mapsto \varphi^2(\left(\Theta^2\right)^{-1}(t))$ is concave.

Proof. Since the regularization method g_α has qualification $\Theta(t)$, one may estimate from (45) that

$$\|A\hat{x}_\alpha - Ax^\dagger\| \le \|A[I - g_\alpha(A^*A)A^*A]\varphi(A^*A)\| \le \gamma\Theta(\alpha).$$

Using (44) in conjunction with Theorem 1(g) and for $\alpha > 0$ small enough we conclude that

$$\|\hat{x}_\alpha - x^\dagger\| \le \bar{\omega}(H_\varphi, \gamma\Theta(\alpha)) \le \max\{2, \gamma\}\omega\left(H_\varphi, \Theta(\alpha)\right). \tag{50}$$

Now, using the concavity assumption we infer from Corollary 1(i) that

$$\omega(H_\varphi, \Theta(\alpha)) \le \varphi(\Theta^{-1}(\Theta(\alpha))) = \varphi(\alpha),$$

thus

$$\|\hat{x}_\alpha^\delta - x^\dagger\| \le \frac{\gamma_* \delta}{\sqrt{\alpha}} + \max\{2, \gamma\}\varphi\left(\alpha\right) .$$

Choosing, in this case, the parameter α from $\alpha(\delta) \sim \Theta^{-1}(\delta)$ we obtain (49), which completes the proof. □

Remark 7. The condition $x_\alpha \in M$ for sufficiently small $\alpha > 0$ in Theorem 5 is not hard to fulfill (see [20, Remark 3.3]). Due to [4, Proposition 2.1] such a condition also holds for ellipsoids $M := M(G)$ from (23) in Theorem 4 provided that the operators G and $H = A^*A$ are linked by a range inclusion $\mathcal{R}(G) \subset \mathcal{R}(\varphi(A^*A))$ or equvialent conditions (see [4, 13, 29].

Remark 8. Similar bounds as in (49) can be obtained differently, if the regularization g_α has qualification φ, only, and we refer to [11] for this. However, here we aim at deriving error bounds by using the modulus of continuity, and thus we are to bound the discrepancy. If we do this under smoothness assumptions given by source sets then higher qualification is required, as this is well-known for the discrepancy principle.

Remark 9. One may wonder whether the rate from (49), which takes into account solution smoothness, is always better than that from (47). This is not always the case, since for logarithmic smoothness of the form $\varphi(t) = \log^{-p}(1/t)$ both formulae from (47) and (49) yield the same rate, as $\delta \to 0$, $\|x^\delta_{\alpha(\delta)} - x^\dagger\| = \mathcal{O}(\log^{-p}(1/\delta))$.

5.3. **Discretization.** As is well-known (see Section 3.2) ill-posed equations (1) become well-posed after discretization, i.e., if the solutions are searched in a finite-dimensional subspace $X_n \subset X$, say with $\dim(X_n) = n$. In this case quasi-solutions $x_n^\delta \in X_n$ are the uniquely determined minimizers of

$$\|Ax - y^\delta\| \to \min, \qquad \text{subject to} \quad x \in X_n. \tag{51}$$

This procedure is called *one-sided discretization*, and the resulting solutions x_n^δ are obtained as

$$x_n^\delta := P_n \left(P_n A^* A P_n\right)^{-1} P_n A^* y^\delta \tag{52}$$

by using the orthogonal projection P_n onto X_n. This results in the following error bound for the one-sided projection method from (51). We recall the definition of the modulus of injectivity $j(A, X_n)$ from (17).

Lemma 3. *Let $x^\dagger$ be the solution to (1) and x_n^δ be the minimizer of (51), i.e., it is given by (52). Then*

$$\|x^\dagger - x_n^\delta\| \le \operatorname{dist}(x^\dagger, X_n) + \frac{\delta + \|A(I - P_n)x^\dagger\|}{j(A, X_n)}. \tag{53}$$

Proof. We first bound

$$\|x^\dagger - x_n^\delta\| \le \|x^\dagger - P_n x^\dagger\| + \|P_n x^\dagger - x_n^\delta\| = \operatorname{dist}(x^\dagger, X_n) + \|P_n x^\dagger - x_n^\delta\|.$$

The second summand on the right can be bounded using the modulus of continuity, since both $P_n x^\dagger, x_n^\delta \in X_n$. To this end, we need to bound $\|AP_n x^\dagger - Ax_n^\delta\|$. Let us temporarily abbreviate $B_n := AP_n$. Then we bound

$$\begin{aligned}\|AP_n x^\dagger - Ax_n^\delta\| &= \|B_n x^\dagger - B_n \left(B_n^* B_n\right)^{-1} B_n^* \left(Ax^\dagger + \delta\xi\right)\| \\ &\le \|B_n x^\dagger - B_n \left(B_n^* B_n\right)^{-1} B_n^* A x^\dagger\| + \delta \|B_n \left(B_n^* B_n\right)^{-1} B_n^* \xi\| \\ &\le \|A(I - P_n)x^\dagger\| + \delta.\end{aligned}$$

Thus we can use Lemma 1 to obtain that

$$\|P_n x^\dagger - x_n^\delta\| \le \omega\left(X_n, \delta + \|A(I - P_n)x^\dagger\|\right).$$

The proof can be completed by using the representation from Proposition 2. □

This lemma can be used in several ways to establish error bounds from one-sided discretization uniformly for sets M of a priori information. Notice that

$$\|A(I-P_n)x^\dagger\| \leq \|A(I-P_n)\|\|(I-P_n)x^\dagger\|,$$

since $I-P_n$ is an orthogonal projection. Let us denote

$$\eta_n := \|A(I-P_n)\|, \tag{54}$$

which is known to be larger or equal to σ_{n+1}. First, similar to one of the approaches in Section 5.2 implying Theorem 4 we can ignore the smoothness of $x^\dagger$ which yields the following

Corollary 3. *Let $x^\dagger$ be the solution to (1), η_n as in (54), and x_n^δ as in (52). Moreover suppose that there is a constant $0 < C < \infty$ such that*

$$\eta_n \leq C\, j(A, X_n), \quad n \in \mathbb{N}. \tag{55}$$

If $\lim_{n\to\infty} \operatorname{dist}(x^\dagger, X_n) = 0$ and the discretization level $n_* = n_*(\delta)$ is chosen such that

$$n_* \to \infty \quad \textit{and} \quad \delta/\sigma_{n_*+1} \to 0, \qquad \textit{as } \delta \to 0, \tag{56}$$

then $\|x^\dagger - x_{n_*(\delta)}\| \to 0$.

Proof. The proof is based on (53). If $n_* \to \infty$, then the first summand converges to zero. Next, since $j(A, X_n) \geq \eta_n/C \geq \sigma_{n+1}/C$ the quotient $\delta/j(A, X_{n_*})$ tends to zero, if this was true for δ/σ_{n_*+1}. Finally, under (55) we can deduce convergence

$$\frac{\|A(I-P_{n_*})x^\dagger\|}{j(A, X_{n_*})} \leq \frac{\eta_{n_*}}{j(A, X_{n_*})}\|(I-P_{n_*})x^\dagger\| \to 0 \quad \text{as } n_* \to \infty,$$

which completes the proof. □

Remark 10. The validity of bounds as required in (55) is beyond the scope of this study. It is closely related with inequalities of *Bernstein* and *Jackson* type. Such requirements have to be made when studying projection methods for solving ill-posed problems, and we refer to the recent [12, 28, 25]. We just mention that the inequality (55) is fulfilled whenever the spaces X_n are the singular spaces U_n as discussed in Proposition 2.

On the other hand, if the set M is a source set H_φ as in (10), then one can take this into account to obtain an improved error bound.

Corollary 4. *Let $x^\dagger \in H_\varphi$ be the solution to (1) and x_n^δ be given by (52). Suppose that there are constants $0 < c \leq 1 \leq C < \infty$ such that*

$$\|A(I-P_n)\varphi(A^*A)\| \leq C\Theta(\sigma_{n+1}^2), \quad n \in \mathbb{N}, \tag{57}$$

and

$$j(A, X_n) \geq c\sigma_n, \quad n \in \mathbb{N}. \tag{58}$$

If the discretization level $n_* = n_*(\delta)$ is chosen such that

$$\Theta(\sigma_{n_*}^2) = \delta, \tag{59}$$

and if the function $t \mapsto \varphi^2(\left(\Theta^2\right)^{-1}(t))$ is concave then there is a constant $\tilde{C}$ for which

$$\|x^\dagger - x_{n_*}^\delta\| \leq \tilde{C}\varphi(\Theta^{-1}(\delta)), \quad \delta \to 0.$$

Remark 11. Again, the requirements from (57) and (58) are not by chance, and these are fulfilled for the singular systems U_n from Proposition 2. The concavity assumption is used to conclude that (57) also implies that $\|(I - P_n)\varphi(A^*A)\| \leq C\varphi(\sigma_{n+1}^2)$, by interpolation.

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