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Convergence rates for the iteratively regularized Gauss–Newton method in Banach spaces*

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Abstract
In this paper we consider the iteratively regularized Gauss–Newton method (IRGNM) in a Banach space setting and prove optimal convergence rates under approximate source conditions. These are related to the classical concept of source conditions that is available only in Hilbert space. We provide results in the framework of general index functions, which include, e.g. Hölder and logarithmic rates. Concerning the regularization parameters in each Newton step as well as the stopping index, we provide both *a priori* and *a posteriori* strategies, the latter being based on the discrepancy principle.

1. Introduction

We are going to consider a nonlinear ill-posed operator equation

$$F(x) = y \tag{1}$$

where the possibly nonlinear operator $F : \mathcal{D}(F) \subseteq X \to Y$ with domain $\mathcal{D}(F)$ maps between real Banach spaces X and Y. For simplicity, let the symbol $\|\cdot\|$ designate the norm for both spaces. Specifically, we assume X to be reflexive and uniformly smooth. For some of our results we will assume that X is q-convex with some $q > 1$.

Since we are interested in the ill-posed situation, i.e. F fails to be continuously invertible, and the data are contaminated with noise, regularization has to be applied (see, e.g., [4, 25], and references therein).

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Throughout this paper we will assume that an exact solution $x^\dagger \in \mathcal{D}(F)$ of (1) exists, i.e. $F(x^\dagger) = y$, and that the (deterministic) noise level δ in an upper estimate

$$\|y - y^\delta\| \leqslant \delta \tag{2}$$

of the difference between exact right-hand side y and noisy data y^δ is known.

Tikhonov-type variational regularization in Banach spaces has been studied recently with error estimates measured by Bregman distances, e.g. in [3] for linear ill-posed problems, and in [9, 12, 13, 19–21, 23] for nonlinear ill-posed problems (1).

Iterative regularization approaches in Hilbert spaces pose an attractive alternative to variational regularization methods. These approaches were comprehensively analyzed in the monographs [1, 17] (see also the references therein). So far, to the authors' best knowledge, iterative solvers for nonlinear ill-posed problems in Banach spaces have only been formulated in [1, section 4.3] and [18]. In [1], the case $X = Y$ was considered and convergence including rates under sufficiently strong source conditions was proven for generalized Gauss–Newton methods. On the other hand, in [18] convergence of the iteratively regularized Gauss–Newton method and the nonlinear Landweber iteration has been proven in the general situation of possibly different Banach spaces X and Y without imposing any source condition. For an analysis of Landweber-type methods in Banach space we refer to [10] and [24].

The aim of this paper is to provide rate results for the iteratively regularized Gauss–Newton method in a complementary situation, i.e. under weaker source conditions than those assumed in [1], and for not necessarily equal preimage and image space. The obtained rates will be called optimal referring to corresponding optimal rate results in Hilbert space settings.

For Hilbert spaces X by spectral theory one can define at a point $x^\dagger$, where F is Gâteaux differentiable with derivative $F'(x^\dagger)$, linear operators $f(F'(x^\dagger)^*F'(x^\dagger)) : X \to X$ for any index function f. We call a function $f : (0, \infty) \to (0, \infty)$ (or its restriction to a right neighborhood of zero) the index function if f is continuous and strictly increasing with $\lim_{t\to 0+} f(t) = 0$. The properties of non-negativity and self-adjointness of the operator $F'(x^\dagger)^*F'(x^\dagger) : X \to X$ carry over to the new operators. This allows expressing the smoothness of the solution $x^\dagger$ to (1) with respect to the linearization $F'(x^\dagger)$ of the forward operator F in that point. Depending on the specific character of such occurring smoothness Hölder source conditions and general source conditions (see below (8) and (12), respectively) leads to corresponding convergence rates for various regularization methods. For Banach spaces, however, we have $F'(x^\dagger)^* : Y^* \to X^*$ and hence $f(F'(x^\dagger)^*F'(x^\dagger))$ is not well defined. Since general source conditions measuring the solution's smoothness are not available, additional ideas and concepts have to be exploited. Originally developed in [11] for linear ill-posed problems, the concept of approximate source conditions can help to bridge this gap also in the nonlinear case (see, e.g., [9]). In this context, the degree of violation of a benchmark source condition is expressed by so-called distance functions $d(R)$.

The iteratively regularized Gauss–Newton method can be generalized to a Banach space setting by calculating iterates $x_{k+1}^\delta = x_{k+1}^\delta(\alpha_k)$ in a variational form as

$$x_{k+1}^\delta(\alpha) \in \operatorname{argmin}_{x\in\mathcal{D}(F)} \left\|T_k\left(x - x_k^\delta\right) + g_k\right\|^r + \alpha\,\|x - x_0\|^p\,, \qquad k = 0, 1, \ldots, \tag{3}$$

where $p, r \in (1, \infty)$, $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence of regularization parameters, x_0 is some *a priori* guess and we abbreviate

$$T_k = F'\left(x_k^\delta\right), \qquad g_k = F\left(x_k^\delta\right) - y^\delta.$$

Under the assumptions on X the functional $x \mapsto \frac{1}{p}\|x\|^p$ is strictly convex and Fréchet-differentiable for all $p > 1$. Hence, the subdifferential $J_p(x) := \partial\left\{\frac{1}{p}\|x\|^p\right\}$ is single valued and the corresponding duality mapping J_p with the gauge function $t \mapsto t^{p-1}$ is continuous and bijective from X to its dual space X^*. This in general nonlinear mapping J_p is characterized by

$$x^* \in J_p(x) \quad \Longleftrightarrow \quad \langle x^*, x\rangle = \|x\|^p \qquad \text{and} \qquad \|x^*\| = \|x\|^{p-1},$$

where $\langle x^*, x\rangle$ with $x \in X$ and $x^* \in X^*$ is the dual pairing of X and X^*. To analyze convergence rates we employ the Bregman distance $\Delta_p(\tilde{x}, x)$ between $\tilde{x} \in X$ and $x \in X$, defined as

$$\Delta_p(\tilde{x}, x) = \frac{1}{p}\|\tilde{x}\|^p - \frac{1}{p}\|x\|^p - \langle J_p(x), \tilde{x} - x\rangle.$$

If X is q-convex, then there is a constant $\underline{c} > 0$ depending on q such that

$$\Delta_q(\tilde{x}, x) \geqslant \underline{c}\|\tilde{x} - x\|^q \quad \text{for all } \tilde{x}, x \in X \tag{4}$$

(see, e.g., [2, lemma 2.7]).

2. Approximate source conditions and variational inequalities

In order to overcome the absence of Hölder and general source conditions we first extend the Hilbert space standard source condition [4, p 277, formula (11.2)] to the Banach space setting as

$$\exists\, w \in Y^* : \quad J_p(x^\dagger - x_0) = F'(x^\dagger)^* w. \tag{5}$$

Under condition (5) we can estimate

$$|\langle J_p(x^\dagger - x_0), x - x^\dagger\rangle| = |\langle w, F'(x^\dagger)(x - x^\dagger)\rangle| \leqslant \|w\|\|F'(x^\dagger)(x - x^\dagger)\|,$$

which implies the variational inequality

$$\exists\, \beta > 0\, \forall\, x \in \mathcal{D}(F) : \quad |\langle J_p(x^\dagger - x_0), x - x^\dagger\rangle| \leqslant \beta\|F'(x^\dagger)(x - x^\dagger)\|, \tag{6}$$

where in contrast to to the ideas of [12] we only use $\|F'(x^\dagger)(x - x^\dagger)\|$ instead of $\|F(x) - F(x^\dagger)\|$ on the right-hand side.

Usually (see [12] and [13]) variational inequalities for proving convergence rates for the Tikhonov-type regularization in Banach spaces have to hold for appropriate $x \in \mathcal{D}(F)$ in an additive form

$$\exists \beta_1, \beta_2 > 0 \;: \quad |\langle J_p(x^\dagger - x_0), x - x^\dagger\rangle| \leqslant \beta_1 \Delta_p(x, x^\dagger) + \beta_2\|F(x) - F(x^\dagger)\|$$

rather than in the product form (6). Note, however, that the additive form under the assumption

$$\exists K > 0\, \forall x \in \mathcal{D}(F) : \quad \|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \leqslant K\Delta_p(x, x^\dagger) \tag{7}$$

immediately follows from the product form by the triangle inequality.

By avoiding $\|F(x) - F(x^\dagger)\|$ on the right-hand side we are up to some extent independent of the tangential cone condition (7). In particular, we will, e.g., prove optimal rates under a mere Lipschitz condition on F' provided (5) holds.

Moreover, for the Banach space setting the form (6) allows us to use as a substitute for the Hölder-type Hilbert space source condition

$$\exists\, w \in X : \quad J_2(x^\dagger - x_0) = x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^{\nu/2} w, \tag{8}$$

for $0 < \nu < 1$, the following variational inequality:

$$\exists\, \beta > 0\, \forall x \in \mathcal{B} : \quad |\langle J_p(x^\dagger - x_0), x - x^\dagger\rangle| \leqslant \beta D_p^{x_0}(x^\dagger, x)^{\frac{1-\nu}{2}}\|F'(x^\dagger)(x - x^\dagger)\|^\nu. \tag{9}$$

Here

$$\mathcal{B} = \mathcal{D}(F) \cup \mathcal{B}_\rho(x_0)$$

with $\mathcal{B}_\rho(x_0)$ being a closed ball with radius $\rho > 0$ around x_0, and we use the notation

$$D_p^{x_0}(\tilde{x}, x) := \Delta_p(\tilde{x} - x_0, x - x_0).$$

Precisely, the intermediate source condition (9) can be motivated from the Hilbert space case, since the usual source condition (8) implies (9)

$$\begin{aligned}|\langle J_2(x^\dagger - x_0), x - x^\dagger\rangle| &= |\langle w, (F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}(x - x^\dagger)\rangle| \\ &\leqslant \|w\| \|x - x^\dagger\|^{1-\nu} \|(F'(x^\dagger)^* F'(x^\dagger))^{1/2}(x - x^\dagger)\|^\nu \\ &= \|w\| \|x - x^\dagger\|^{1-\nu} \|F'(x^\dagger)(x - x^\dagger)\|^\nu\end{aligned}$$

by the interpolation inequality taking into account that $D_p^{x_0}(x^\dagger, x) = \|x - x^\dagger\|^2$. More generally, one can consider index functions $f : (0, \infty) \to (0, \infty)$ with

$$\phi := (f^2)^{-1} \text{ being convex} \tag{10}$$

and assume the variational inequality

$$\forall x \in \mathcal{D}(F),\ x \neq x^\dagger : \quad |\langle J_p(x^\dagger - x_0), x - x^\dagger\rangle| \leqslant D_p^{x_0}(x^\dagger, x)^{1/2} f\left(\frac{\|F'(x^\dagger)(x - x^\dagger)\|^2}{D_p^{x_0}(x^\dagger, x)}\right) \tag{11}$$

to hold, which again can be motivated from the Hilbert space case. Namely, if (10) holds, by Jensen's inequality, the general Hilbert space source condition

$$\exists\, w \in X : \quad J_2(x^\dagger - x_0) = x^\dagger - x_0 = f(F'(x^\dagger)^* F'(x^\dagger))w \tag{12}$$

implies

$$\begin{aligned}|\langle J_2(x^\dagger - x_0), x - x^\dagger\rangle| &= |\langle w, f(F'(x^\dagger)^* F'(x^\dagger))(x - x^\dagger)\rangle| \\ &\leqslant \|w\| \|x - x^\dagger\| f\left(\frac{\|F'(x^\dagger)(x - x^\dagger)\|^2}{\|x - x^\dagger\|^2}\right).\end{aligned}$$

This includes, e.g., logarithmic source conditions as appropriate for exponentially ill-posed problems, cf, [14].

Now we will show that variational inequalities like (9) and (11) can also be concluded from the approach of approximate source conditions outlined in [9] for the situation of nonlinear problems and Tikhonov regularization. We refer to (5) as a benchmark source condition, which can be expected to hold only in very specific situations. However, it is always fulfilled in an approximate manner as

$$\exists\, r_R \in X^*, \qquad \exists\, w_R \in Y^*, \qquad \|w_R\|_{Y^*} \leqslant R : \quad J_p(x^\dagger - x_0) = F'(x^\dagger)^* w_R + r_R \tag{13}$$

for all $R \geqslant 0$. Based on this observation, we define a distance function $d(R)$ for all $R \geqslant 0$ measuring the distance of the element $J_p(x^\dagger - x_0)$ with respect to sets in X^* which occur when the operator $F'(x^\dagger)^* : Y^* \to X^*$ is applied to closed balls with radius R in the space Y^*, i.e.

$$d(R) := \inf_{w \in Y^* : \|w\|_{Y^*} \leqslant R} \|J_p(x^\dagger - x_0) - F'(x^\dagger)^* w\|_{X^*}. \tag{14}$$

The distance function is well defined as a non-negative and non-increasing continuous function for all $R \geqslant 0$. Since by Alaoglu's theorem the unit ball in Y^* is weak* compact and the dual norm function is weak* lower semicontinuous, the infimum in (14) is a minimum and assumed in some $w_R \in Y^*$. Under the condition

$$J_p(x^\dagger - x_0) \in \overline{\mathcal{R}(F'(x^\dagger)^*)}^{\|\cdot\|_{X^*}} \setminus \mathcal{R}(F'(x^\dagger)^*) \tag{15}$$

it is evident that $d(R)$ is strictly positive for all $R \geqslant 0$ and tends to zero as $R \to \infty$, cf [9, lemma 4.1 and remark 4.2]. In such a case the decay rate of the distance function

$d(R)$ to zero as $R \to \infty$ measures the degree of violation of $J_p(x^\dagger - x_0)$ with respect to the benchmark source condition (5). As the following proposition will show, this degree of violation determines the function f in variational inequalities like (11).

Proposition 1. *Let X be q-convex. Under conditions (4) and (15) let $\overline{d}$ be a continuous and strictly decreasing majorant of the distance function d from (14) in the sense that the inequality $0 < d(R) \leqslant \overline{d}(R)$ holds for all $R > 0$ and that we have the limit condition $\lim_{R\to\infty} \overline{d}(R) = 0$. Then a variational inequality*

$$|\langle J_q(x^\dagger - x_0), x - x^\dagger\rangle| \leqslant D_q^{x_0}(x^\dagger, x)^{1/q} f\left(\frac{\|F'(x^\dagger)(x - x^\dagger)\|^q}{D_q^{x_0}(x^\dagger, x)}\right) \tag{16}$$

holds with the index function

$$f(t) = 2\,\max\{1, \underline{c}^{-1/q}\}\overline{d}\left(\Psi^{-1}(t)\right) \quad t > 0,$$

$$\textit{with} \quad \Psi(R) = \left(\frac{\overline{d}(R)}{R}\right)^q \quad R > 0, \tag{17}$$

for all $x \in \mathcal{D}(F)$ such that $x - x^\dagger \notin \mathcal{N}(F'(x^\dagger))$.

Proof. Since the infimum in (14) is a minimum, we have for all $R \geqslant 0$ an additive decomposition (13) with $\|r_R\|_{X^*} = d(R)$. Then the following equations and estimates can be stated for $0 < R < \infty$:

$$\begin{aligned}|\langle J_q(x^\dagger - x_0), x - x^\dagger\rangle| &= |\langle F'(x^\dagger)^* w_R + r_R, x - x^\dagger\rangle| \\ &= |\langle w_R, F'(x^\dagger)(x - x^\dagger)\rangle + \langle r_R, x - x^\dagger\rangle| \\ &\leqslant R\|F'(x^\dagger)(x - x^\dagger)\| + d(R)\|x - x^\dagger\|.\end{aligned}$$

Taking into account the q-convexity of X this yields

$$\begin{aligned}|\langle J_q(x^\dagger - x_0), x - x^\dagger\rangle| &\leqslant R\|F'(x^\dagger)(x - x^\dagger)\| + \frac{d(R)}{\underline{c}^{1/q}} D_q^{x_0}(x^\dagger, x)^{1/q} \\ &\leqslant R\|F'(x^\dagger)(x - x^\dagger)\| + \frac{\overline{d}(R)}{\underline{c}^{1/q}} D_q^{x_0}(x^\dagger, x)^{1/q} \\ &\leqslant \max\{1, \underline{c}^{-1/q}\}[R\|F'(x^\dagger)(x - x^\dagger)\| + \overline{d}(R) D_q^{x_0}(x^\dagger, x)^{1/q}].\end{aligned}$$

Since $\Psi(R)$ is strictly decreasing and continuous for $0 < R < \infty$ with limits $\lim_{R\to 0} \Psi(R) = \infty$ and $\lim_{R\to\infty} \Psi(R) = 0$, the equation $\Psi(R) = \left(\frac{\|F'(x^\dagger)(x - x^\dagger)\|}{D_q^{x_0}(x^\dagger, x)^{1/q}}\right)^q$ has a unique solution $R_0 > 0$ for all $x \in \mathcal{D}(F)$ such that $x - x^\dagger \notin \mathcal{N}(F'(x^\dagger))$. For that $R_0 > 0$ the two terms in the last sum above coincide and we obtain the estimate (16). As $\Psi^{-1}(t)$ is strictly decreasing for all $0 < t < \infty$ with limits $\lim_{t\to 0} \Psi^{-1}(t) = \infty$ and $\lim_{t\to\infty} \Psi^{-1}(t) = 0$, under the assumption on $\overline{d}$ stated in the proposition the composite function $\overline{d} \circ \Psi^{-1}$ is an index function. This completes the proof. □

Remark 1. The function f from (17) has the following property: by using the monotonicity inverting substitution $R := \Psi^{-1}(t)$, the quotient function

$$\zeta(t) := \frac{t^{1/q}}{\overline{d}(\Psi^{-1}(t))} = \frac{\Psi(R)^{1/q}}{\overline{d}(R)} = \frac{\overline{d}(R)}{R\overline{d}(R)} = \frac{1}{R}$$

is strictly increasing for $0 < t < \infty$, and tends to zero as $t \to 0$ and $R \to \infty$, respectively. Hence, the quotient $\frac{f(t)}{t^{1/q}}$ is strictly decreasing for all $t > 0$. Moreover, we should note here that for 2-convex Banach spaces X, i.e. for $q = 2$, the variational inequality (16) obtained by proposition 1 attains the form (11) which is required as an assumption in the theorems 1 and 2 below. Furthermore, we have to mention that a function $\overline{f}$ that occurs when Ψ is replaced in (17) by a majorant function $\overline{\Psi}$ (with same monotonicity and limit properties as Ψ) is also an index function and a majorant of f. That fact will be exploited in remark 3.

3. Convergence rates with *a priori* parameter choice

To prove convergence rates we make the following assumption on the nonlinearity of F:

$$\sup_{\substack{v,\tilde{v}\in X,\\ x^\dagger+v\in\mathcal{B}\\ x^\dagger+\tilde{v}\in\mathcal{B}}} \frac{\|(F'(x^\dagger+\tilde{v})-F'(x^\dagger))v\|}{\|F'(x^\dagger)v\|^{\tilde{c}_1}D_p^{x_0}(x^\dagger,v+x^\dagger)^{\tilde{c}_2}\|F'(x^\dagger)\tilde{v}\|^{\tilde{c}_3}D_p^{x_0}(x^\dagger,\tilde{v}+x^\dagger)^{\tilde{c}_4}} \leqslant K \tag{18}$$

with

$$\tilde{c}_1+\tilde{c}_2\frac{2\nu}{\nu+1}\geqslant\frac{1}{2}, \qquad \tilde{c}_3+\tilde{c}_4\frac{2\nu}{\nu+1}\geqslant\frac{1}{2} \tag{19}$$

as well as

$$\begin{aligned}&\tilde{c}_1+\tilde{c}_2 r\geqslant\tfrac{1}{2} \text{ and } \tilde{c}_3+\tilde{c}_4 r\geqslant\tfrac{1}{2}\\ &\text{and}\\ &\big(\big(\tilde{c}_1+\tilde{c}_2 r>\tfrac{1}{2}\wedge\tilde{c}_3+\tilde{c}_4 r>\tfrac{1}{2}\big) \text{ or } K \text{ sufficiently small}\big).\end{aligned}$$

The latter, for $\nu=1$, follows from the usual Lipschitz condition on F' in terms of the Bregman distance in X:

$$\sup_{\substack{v,\tilde{v}\in X,\\ x^\dagger+v\in\mathcal{B}\\ x^\dagger+\tilde{v}\in\mathcal{B}}} \frac{\|(F'(x^\dagger+\tilde{v})-F'(x^\dagger))v\|^2}{D_p^{x_0}(x^\dagger,v+x^\dagger)D_p^{x_0}(x^\dagger,\tilde{v}+x^\dagger)} \leqslant L^2.$$

Note the relation to the concept of degree of nonlinearity, see, e.g, [9], with (18) implying (2.5) in [9, definition 2.5] for $c_1=\tilde{c}_1+\tilde{c}_3$, $c_2=\tilde{c}_2+\tilde{c}_4$. The necessity of using a slightly stronger condition here comes from the need for estimating the difference between the derivatives of F in the proof of theorem 1, see (38) below.

An *a priori* choice of α_k and k_* satisfying

$$\alpha_0\leqslant 1, \qquad \alpha_k\to 0 \quad\text{as}\quad k\to\infty, \qquad 1\leqslant\frac{\alpha_k}{\alpha_{k+1}}\leqslant\hat{C} \quad\text{for all}\quad k \tag{21}$$

and

$$k_*(\delta)=\min\Big\{k\in\mathbb{N}\ :\ \alpha_k^{\frac{\nu+1}{r(\nu+1)-2\nu}}\leqslant\tau\delta\Big\}, \quad \text{in the case of (9)} \tag{22}$$

$$k_*(\delta)=\min\big\{k\in\mathbb{N}\ :\ \alpha_k\leqslant\varphi_r(\tau\delta)\big\}, \quad \text{in the case of (11)} \tag{23}$$

with

$$\varphi_r(t)=t^{r-2}\Theta^{-1}(t), \qquad \Theta(\lambda):=f(\lambda)\sqrt{\lambda} \tag{24}$$

yields the following rate result.

Proposition 2. *Assume that a solution $x^\dagger$ to (1) exists, and that F satisfies (18) with (19), (20). Moreover, let $p,r\in(1,\infty)$, let τ be chosen sufficiently large and let x_0 be close enough to $x^\dagger$ so that $D_p^{x_0}(x^\dagger,x_0)$ is sufficiently small. Additionally, assume that $\mathcal{B}_{\bar{\rho}}^{\Delta}(x^\dagger)\subseteq\mathcal{B}$ for some $\bar{\rho}>0$, where $\mathcal{B}_{\bar{\rho}}^{\Delta}(x^\dagger)$ is a ball with respect to the Bregman distance.*

Then for all $k\leqslant k_*(\delta)-1$ with $k_*(\delta)$ according to (22), (23), the iterates $x_{k+1}^\delta:=x_{k+1}^\delta(\alpha_k)$ with α_k according to (21) are well defined.

Proof. The assertion follows from results in [18]. □

Theorem 1. *Let the assumptions of proposition 2 be satisfied.*

(i) Let a variational inequality (9) with β sufficiently small hold.
Then, with the a priori choice (22) we obtain optimal convergence rates

$$D_p^{x_0}(x^\dagger, x_{k_*}) = O\big(\delta^{\frac{2\nu}{\nu+1}}\big), \quad \text{as} \;\; \delta \to 0 \tag{25}$$

as well as in the noise free case $\delta = 0$

$$\begin{aligned} \big\|T\big(x_{k+1}^\delta - x^\dagger\big)\big\| &= O\big(\alpha_k^{\frac{\nu+1}{r(\nu+1)-2\nu}}\big), \\ D_p^{x_0}\big(x^\dagger, x_{k+1}^\delta\big) &= O\big(\alpha_k^{\frac{2\nu}{r(\nu+1)-2\nu}}\big) \end{aligned} \tag{26}$$

for all $k \in \mathbb{N}$.

(ii) Let a variational inequality (11) with

$$t \mapsto \frac{f(t)}{\sqrt{t}} \quad \text{monotonically decreasing}, \tag{27}$$

and

$$\forall 0 < t \leqslant \hat{t}: \quad f(\hat{C}_r t) \leqslant \hat{C}_f f(t) \tag{28}$$

$$\forall 0 < t \leqslant \tilde{t}: \quad f(\tilde{C}_r t) \leqslant \tilde{C}_f f(t) \tag{29}$$

with

$$\begin{aligned} &\hat{C}_r = \big(\hat{C}\hat{C}_f^{2-r}\big)^{2/r}, \qquad \tilde{C}_r = \big(\tilde{C}\tilde{C}_f^{2-r}\big)^{2/r}, \\ &1 \leqslant \hat{C}_\varphi := \big(\hat{C}\hat{C}_f^2\big)^{1/r} < \frac{1}{(2C_\kappa)^{1/r}K}, \qquad 1 \leqslant \tilde{C}_\varphi := \big(\tilde{C}\tilde{C}_f^2\big)^{1/r}, \\ &\tilde{C} = (2M)^{2-r}\hat{C}, \qquad \hat{C}, C_\kappa, M \text{ as in (21), (50), (51)} \\ &\hat{t} = \Theta^{-1}\big(\hat{C}_\varphi \varphi_r^{-1}(\alpha_0)\big)/\hat{C}_r, \qquad \tilde{t} = \Theta^{-1}\big(\tilde{C}_\varphi \varphi_r^{-1}((2M)^{r-2}\alpha_0)\big)/\tilde{C}_r, \end{aligned} \tag{30}$$

hold and assume

$$\tilde{c}_1 = \tilde{c}_3 = \tfrac{1}{2}, \qquad \tilde{c}_2 = \tilde{c}_4 = 0,$$

as well as K sufficiently small in (18).
Then with the a priori choice (23), we obtain optimal convergence rates

$$D_p^{x_0}(x^\dagger, x_{k_*}) = O(f^2(\Theta^{-1}(\delta))) = O\left(\frac{\delta^2}{\Theta^{-1}(\delta)}\right) \quad \text{as} \;\; \delta \to 0 \tag{31}$$

with Θ as in (24), as well as in the noise free case $\delta = 0$

$$\begin{aligned} \big\|T\big(x_{k+1}^\delta - x^\dagger\big)\big\| &= O\left(\varphi_r^{-1}(\alpha_k)\right), \\ D_p^{x_0}\big(x^\dagger, x_{k+1}^\delta\big) &= O\big(f\big(\Theta^{-1}\big(\varphi_r^{-1}(\alpha_k)\big)\big)^2\big) \end{aligned} \tag{32}$$

for all $k \in \mathbb{N}$.

Remark 2. Condition (27) implies for all $C > 0$ the inequality

$$f(\Theta^{-1}(Ct)) \leqslant \max\{\sqrt{C}, 1\} f(\Theta^{-1}(t)) \quad (t \geqslant 0). \tag{33}$$

Because of the monotonicity of the index functions f and Θ^{-1}, we have $f(\Theta^{-1}(Ct)) \leqslant f(\Theta^{-1}(t))$ for $0 < C \leqslant 1$. On the other hand, by substituting $u := \Theta(t)$ we have that $\frac{f(\Theta^{-1}(\tau))}{\sqrt{\tau}} = \frac{f(u)}{\sqrt{\Theta(u)}} = \sqrt{\frac{f(u)}{\sqrt{u}}}$ showing in view of (27) that these quotient functions with positive arguments τ and u, respectively, are both monotonically increasing. Consequently, we have $\frac{f(\Theta^{-1}(Ct))}{\sqrt{Ct}} \leqslant \frac{f(\Theta^{-1}(t))}{\sqrt{t}}$ for $C > 1$. Both facts imply together (33).

Moreover, condition (27) means that the variational inequality condition determined by the index function f is not too strong, i.e. the decay rate of $f(t) \to 0$ as $t \to 0$ is not faster than the corresponding decay rate of $\sqrt{t}$. A sufficient condition for that is the concavity of f^2 which is equivalent to condition (10). From remark 1 we learned that condition (27) is satisfied for the function f from proposition 1 whenever $q = 2$. By the same arguments it follows that this remains true for all $2 \leqslant q < \infty$.

We wish to point out that (18), (19) and (20) get weaker for a larger smoothness index ν, which corresponds to results in Hilbert space (see, e.g., [5]), where—as here—in the case $\nu = 1$ a Lipschitz condition suffices to prove optimal convergence rates. In the case of a general index function f, we have to restrict ourselves to the strongest case in (18), (19) and (20) corresponding to $\nu = 0$.

Note that q-convexity of X is not required for the results of theorem 1. If X is q-convex, then inequality (4) implies

$$\|\tilde{x} - x\|^q = O\left(\delta^{\frac{2\nu}{\nu+1}}\right), \quad \text{as } \delta \to 0$$

in case (i) of theorem 1 and

$$\|\tilde{x} - x\|^q = O(f^2(\Theta^{-1}(\delta))) = O\left(\frac{\delta^2}{\Theta^{-1}(\delta)}\right), \quad \text{as } \delta \to 0$$

in case (ii) of theorem 1.

Proof. To show (i), observe that under the assumption (9) we get, with the notation $T = F'(x^\dagger)$,

$$\begin{aligned}
\left\|x_{k+1}^\delta - x_0\right\|^p - \|x^\dagger - x_0\|^p &= p\Delta_p\left(x^\dagger - x_0, x_{k+1}^\delta - x_0\right) + p\left\langle J_p(x^\dagger - x_0), x_{k+1}^\delta - x^\dagger\right\rangle \\
&\geqslant pD_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) - p\beta D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)^{(1-\nu)/2}\left\|T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^\nu \\
&\geqslant pD_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) \\
&\quad - p\beta\left(\epsilon D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) + C\left(\epsilon, \frac{\nu+1}{2}\right)\left\|T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{2\nu/(\nu+1)}\right)
\end{aligned} \tag{34}$$

with $\epsilon > 0$ to be chosen sufficiently small later on,

$$C(\epsilon, 1) = 1,$$

and

$$\begin{aligned}
C(\epsilon, \mu) &= \max\left\{1, \phi\left(\left(\frac{\epsilon}{1-\mu}\right)^{1/\mu}\right)\right\} \\
&= \max\left\{1, \frac{\mu}{(1-\mu)^{(\mu+1)/\mu}}\epsilon^{-(1-\mu)/\mu}\right\}
\end{aligned}$$

for $\mu \in (0, 1)$, where $\phi(\lambda) = \frac{\lambda^\mu - \epsilon}{\lambda}$ so that

$$\lambda^\mu \leqslant \epsilon + C(\epsilon, \mu)\lambda \qquad \text{for all } \lambda > 0. \tag{35}$$

By minimality in (3) we have for any solution $x^\dagger \in \mathcal{B}_\rho(x_0)$ of (1)

$$\begin{aligned}
&\left\|T_k\left(x_{k+1}^\delta - x_k^\delta\right) + g_k\right\|^r + \alpha_k\left\|x_{k+1}^\delta - x_0\right\|^p \\
&\quad \leqslant \left\|T_k\left(x^\dagger - x_k^\delta\right) + g_k\right\|^r + \alpha_k\|x^\dagger - x_0\|^p.
\end{aligned} \tag{36}$$

Combining (34) and (36) we get by the simple inequality $(a-b)^r + b^r \geqslant \frac{1}{2^{r-1}} a^r$

$$\begin{aligned}\frac{1}{2^{r-1}} \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^r &+ \alpha_k p(1-\beta\epsilon) D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) \\ &\leqslant \left\| T_k\left(x^\dagger - x_k^\delta\right) + g_k\right\|^r + \left(\left\|(T_k - T)\left(x_{k+1}^\delta - x^\dagger\right)\right\| + \left\| T_k\left(x^\dagger - x_k^\delta\right) + g_k\right\|\right)^r \\ &\quad + \alpha_k p\beta C\left(\epsilon, \frac{\nu+1}{2}\right) \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{2\nu/(\nu+1)}.\end{aligned}$$

The terms on the right-hand side can be estimated by means of (18),

$$\begin{aligned}\left\| T_k\left(x^\dagger - x_k^\delta\right) + g_k\right\| &\leqslant \left\|(T_k - T)\left(x_k^\delta - x^\dagger\right)\right\| + \left\| F\left(x_k^\delta\right) - F(x^\dagger) - T\left(x_k^\delta - x^\dagger\right)\right\| + \delta \\ &\leqslant 2K \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{\tilde{c}_1+\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{\tilde{c}_2+\tilde{c}_4} + \delta \end{aligned} \tag{37}$$

$$\left\|(T_k - T)\left(x_{k+1}^\delta - x^\dagger\right)\right\| \leqslant K \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{\tilde{c}_1} \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)^{\tilde{c}_2} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{\tilde{c}_4} \tag{38}$$

which, together with the simple inequality $(a+b)^r \leqslant 2^{r-1}(a^r + b^r)$, yields

$$\begin{aligned}\frac{1}{2^{r-1}} \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^r &+ \alpha_k p(1-\beta\epsilon) D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) \\ &\leqslant (1+2^{r-1})\left(2K \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{\tilde{c}_1+\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{\tilde{c}_2+\tilde{c}_4} + \delta\right)^r \\ &\quad + 2^{r-1}\left(K \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{\tilde{c}_1} D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)^{\tilde{c}_2} \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{\tilde{c}_4}\right)^r \\ &\quad + \alpha_k p\beta C\left(\epsilon, \frac{\nu+1}{2}\right) \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{2\nu/(\nu+1)}.\end{aligned}$$

Applying the estimate

$$a^\zeta b \leqslant \tilde{\epsilon} a + C(\tilde{\epsilon}, 1-\zeta) b^{1/(1-\zeta)} \tag{39}$$

for $\zeta \in (0, 1]$, that follows from (35) with $\lambda := \frac{b^{1/(1-\zeta)}}{a}$ and $\mu = 1-\zeta$ to the last term, and $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$ to the second term on the right-hand side, we get

$$\left(\frac{1}{2^{r-1}} - \tilde{\epsilon}\right) \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^r + \alpha_k p(1-\beta\epsilon) D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) \tag{40}$$

$$\leqslant (1+2^{r-1})\left(2K \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{\tilde{c}_1+\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{\tilde{c}_2+\tilde{c}_4} + \delta\right)^r \tag{41}$$

$$+ \frac{2^{r-1}K^r}{2} \left\| T\left(x_{k+1}^\delta - x^\dagger\right)\right\|^{2r\tilde{c}_1} D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)^{2r\tilde{c}_2} \tag{42}$$

$$+ \frac{2^{r-1}K^r}{2} \left\| T\left(x_k^\delta - x^\dagger\right)\right\|^{2r\tilde{c}_3} D_p^{x_0}\left(x^\dagger, x_k^\delta\right)^{2r\tilde{c}_4} \tag{43}$$

$$+ C\left(\tilde{\epsilon}, \frac{r(\nu+1)-2\nu}{r(\nu+1)}\right) \left(\alpha_k p\beta C\left(\epsilon, \frac{\nu+1}{2}\right)\right)^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}, \tag{44}$$

where we choose $\tilde{\epsilon} < \frac{1}{2^{r-1}}$. Considering (40) and (44) and neglecting the rest (which is just an estimate of the nonlinearity error) for a moment, we expect that (26) can be obtained, which we prove as follows: dividing (40)–(44) by $\alpha_{k+1}^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}$, using (21), (19) and (22), and defining

$$\gamma_k := \max\left\{ \frac{\left\| T\left(x_k^\delta - x^\dagger\right)\right\|^r}{\alpha_k^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}}, \frac{D_p^{x_0}\left(x^\dagger, x_k^\delta\right)}{\alpha_k^{\frac{2\nu}{r(\nu+1)-2\nu}}} \right\},$$

we get the following estimate:

$$\min\left\{\frac{1}{2^{r-1}}-\tilde{\epsilon},\,p(1-\beta\epsilon)\right\}\gamma_{k+1}\leqslant(1+2^{r-1})2^{r-1}(2K)^r\hat{C}^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}\gamma_k^{\tilde{c}_1+\tilde{c}_3+(\tilde{c}_2+\tilde{c}_4)r}$$
$$+\frac{2^{r-1}K^r}{2}\gamma_{k+1}^{2(\tilde{c}_1+r\tilde{c}_2)}+\frac{2^{r-1}K^r}{2}\hat{C}^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}\gamma_k^{2(\tilde{c}_3+r\tilde{c}_4)}$$
$$+C(\tilde{\epsilon},\frac{r(\nu+1)-2\nu}{r(\nu+1)})\left(p\beta C\left(\epsilon,\frac{\nu+1}{2}\right)\right)^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}\hat{C}^{\frac{r(\nu+1)}{r(\nu+1)-2\nu}}+\frac{(1+2^{r-1})2^{r-1}}{\tau^r}.$$

Therewith we get a recursive estimate of the form

$$\big(1-A\gamma_{k+1}^{2(\tilde{c}_1+r\tilde{c}_2)-1}\big)\gamma_{k+1}\leqslant B\big(\gamma_k^{\tilde{c}_1+\tilde{c}_3+(\tilde{c}_2+\tilde{c}_4)r-1}+\gamma_k^{2(\tilde{c}_3+r\tilde{c}_4)-1}\big)\gamma_k+c,\tag{45}$$

where c can be made small by making β small and τ large.

From this we can now derive an induction step of the form

$$\gamma_k\leqslant\bar{\gamma}\;\Rightarrow\;\gamma_{k+1}\leqslant\bar{\gamma}\tag{46}$$

as follows: using (20) and the fact that A and B will be small if K is small, we can first of all conclude that for $\bar{\gamma},\bar{\zeta}$ sufficiently small, the function

$$\begin{aligned} h(\gamma):\quad (0,\bar{\gamma}) &\to (0,\bar{\zeta})\\ \gamma &\mapsto (1-A\gamma^{2(\tilde{c}_1+r\tilde{c}_2)-1})\gamma\end{aligned}$$

is strictly monotonically increasing and invertible with

$$h^{-1}(\zeta)\leqslant 2\zeta.$$

By using the induction hypothesis $\gamma_k\leqslant\bar{\gamma}$ with a possibly reduced value of $\bar{\gamma}$, we can achieve that the right-hand side of (45) is smaller than $\bar{\zeta}$ so that by applying h^{-1} to both sides of (45), we can conclude

$$\begin{aligned}\gamma_{k+1}&\leqslant 2B\big(\gamma_k^{(\tilde{c}_1+\tilde{c}_3)+(\tilde{c}_2+\tilde{c}_4)r-1}+\gamma_k^{2(\tilde{c}_3+\tilde{c}_4r)-1}\big)\,\gamma_k+2c\\ &\leqslant 2B(\bar{\gamma}^{(\tilde{c}_1+\tilde{c}_3)+(\tilde{c}_2+\tilde{c}_4)r-1}+\bar{\gamma}^{2(\tilde{c}_3+\tilde{c}_4r)-1})\bar{\gamma}+\tfrac{1}{2}\bar{\gamma},\end{aligned}\tag{47}$$

where we use the fact that we can make β small and τ large so that $c<\frac{\bar{\gamma}}{4}$. Now we use (20) again to achieve

$$2B(\bar{\gamma}^{(\tilde{c}_1+\tilde{c}_3)+(\tilde{c}_2+\tilde{c}_4)r-1}+\bar{\gamma}^{2(\tilde{c}_3+r\tilde{c}_4)-1})\leqslant\tfrac{1}{2}$$

by possibly decreasing $\bar{\gamma}$. Inserting this into (47) yields $\gamma_{k+1}\leqslant\bar{\gamma}$.

Applying (46) as an induction step we can conclude that

$$\gamma_k\leqslant\bar{\gamma}\qquad\text{for all }\;k\leqslant k_*$$

and therewith, by possibly decreasing $\bar{\gamma}$ to below $\bar{\rho}^2$,

$$D_p^{x_0}\big(x^\dagger,x_k^\delta\big)\leqslant\gamma_k\alpha_k^{\frac{2\nu}{r(\nu+1)-2\nu}}\leqslant\bar{\gamma}\leqslant\bar{\rho}^2\qquad\text{for all }\;k\leqslant k_*$$

provided γ_0 and $D_p^{x_0}(x^\dagger,x_0)$ are sufficiently small. By the assumption $\mathcal{B}_{\bar{\rho}}^{\Delta}(x^\dagger)\subseteq\mathcal{B}$, this yields well definedness of the iterates. Moreover,

$$D_p^{x_0}\big(x^\dagger,x_{k_*}^\delta\big)\leqslant\bar{\gamma}\alpha_{k_*}^{\frac{2\nu}{r(\nu+1)-2\nu}}\leqslant\bar{\gamma}(\tau\delta)^{\frac{2\nu}{\nu+1}}.$$

In the general case (ii) i.e. with the variational inequality (11), we have to apply somewhat different techniques as compared to the special case (9). We get, in place of (34), the estimate

$$\begin{aligned}&\big\|x_{k+1}^\delta-x_0\big\|^p-\|x^\dagger-x_0\|^p\\ &\quad=p\Delta_p\big(x^\dagger-x_0,x_{k+1}^\delta-x_0\big)+p\big\langle J_p(x^\dagger-x_0),x_{k+1}^\delta-x^\dagger\big\rangle\\ &\quad\geqslant pD_p^{x_0}\big(x^\dagger,x_{k+1}^\delta\big)-pD_p^{x_0}\big(x^\dagger,x_{k+1}^\delta\big)^{1/2}f\left(\frac{\|F'(x^\dagger)(x_{k+1}^\delta-x^\dagger)\|^2}{D_p^{x_0}\big(x^\dagger,x_{k+1}^\delta\big)}\right),\end{aligned}\tag{48}$$

which together with (36)–(38) implies

$$\frac{1}{2^{r-1}}\left\|T\left(x_{k+1}^\delta-x^\dagger\right)\right\|^r+\alpha_k p D_p^{x_0}\left(x^\dagger,x_{k+1}^\delta\right)\leqslant\left(1+2^{r-1}\right)\left(2K\left\|T\left(x_k^\delta-x^\dagger\right)\right\|+\delta\right)^r$$
$$+\frac{2^{r-1}K^r}{2}\left\|T\left(x_{k+1}^\delta-x^\dagger\right)\right\|^r+\frac{2^{r-1}K^r}{2}\left\|T\left(x_k^\delta-x^\dagger\right)\right\|^r$$
$$+\alpha_k p D_p^{x_0}\left(x^\dagger,x_{k+1}^\delta\right)^{1/2} f\left(\frac{\left\|T\left(x_{k+1}^\delta-x^\dagger\right)\right\|^2}{D_p^{x_0}\left(x^\dagger,x_{k+1}^\delta\right)}\right)$$

in place of (40)–(44), which by moving the second term on the right-hand side to the left-hand side, using $K^r<\frac{2}{2^{r-1}}$ and (23), yields an inequality of the form

$$\mathrm{t}_{k+1}^r+\alpha_k\mathrm{d}_{k+1}^2\leqslant\kappa\mathrm{t}_k^r+m\left(\varphi_r^{-1}(\alpha_k)\right)^r+M\alpha_k\mathrm{d}_{k+1}f\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right)\tag{49}$$

for all $k\leqslant k_*-1$, where we use the abbreviations

$$\begin{aligned}\mathrm{d}_k&=D_p^{x_0}(x^\dagger,x_k^\delta)^{1/2},\\ \mathrm{t}_k&=\left\|T\left(x_k^\delta-x^\dagger\right)\right\|,\qquad\qquad(50)\\ \kappa&=\frac{2^r(1+2^{r-1})2^{r-1}+2^{r-1}/2}{\tilde c}K^r=C_\kappa K^r,\\ m&=\frac{(1+2^{r-1})2^{r-1}}{\tau^r\tilde c},\\ M&=\frac{p}{\tilde c},\qquad\qquad(51)\\ \tilde c&=\min\left\{\frac{1}{2^{r-1}}-\frac{2^{r-1}K^r}{2},p\right\}.\end{aligned}$$

Now we prove by induction that for all $k\leqslant k_*$ (or in the case $\delta=0$ for all $k\in\mathbb{N}$)

$$\mathrm{d}_k\leqslant C_1 f\left(\Theta^{-1}\left(\varphi_r^{-1}(\alpha_k)\right)\right)\tag{52}$$
$$\mathrm{t}_k\leqslant C_2\varphi_r^{-1}(\alpha_k)\tag{53}$$

where C_2 is sufficiently large so that (cf (30))

$$\hat C_\varphi\leqslant\left(2\left(\kappa+m/C_2^r\right)\right)^{-1/r},\qquad\tilde C_\varphi\leqslant\frac{C_2}{2M}\tag{54}$$

and $C_1:=\sqrt{\frac{C_2^r}{\min\{1,\hat C\}}}$ so that

$$C_1^2\hat C\geqslant C_2^r,\qquad C_1^2\geqslant C_2^r.\tag{55}$$

For this purpose, observe that (49) together with the induction hypothesis implies

$$\mathrm{t}_{k+1}^r+\alpha_k\mathrm{d}_{k+1}^2\leqslant\left(\kappa C_2^r+m\right)\left(\varphi_r^{-1}(\alpha_k)\right)^r+M\alpha_k\mathrm{d}_{k+1}f\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right).\tag{56}$$

We distinguish between two cases:

if $\left(\kappa C_2^r+m\right)\left(\varphi_r^{-1}(\alpha_k)\right)^r\leqslant M\alpha_k\mathrm{d}_{k+1}f\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right)$, we get from (56)

$$\mathrm{t}_{k+1}^r+\alpha_k\mathrm{d}_{k+1}^2\leqslant 2M\alpha_k\mathrm{d}_{k+1}f\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right).\tag{57}$$

Since in the case $\mathrm{d}_{k+1} = 0$ (and therewith $\mathrm{t}_{k+1} = 0$) and in the case $\mathrm{t}_{k+1} = 0$ $\big($and therewith $\mathrm{d}_{k+1} = 0$ by $\mathrm{d}_{k+1}^2 \leqslant 2M\mathrm{d}_{k+1} f\big(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\big)\big)$, the assertions (52) and (53) trivially hold for k replaced by $k+1$, we may assume w.l.o.g. that $\mathrm{d}_{k+1} \neq 0$ and $\mathrm{t}_{k+1} \neq 0$. Multiplying (57) with t_{k+1} and dividing by d_{k+1}^2 we get

$$\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\mathrm{t}_{k+1}^{r-1} + \alpha_k \mathrm{t}_{k+1} \leqslant 2M\alpha_k\Theta\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right),$$

which implies

$$\Phi\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right)\mathrm{t}_{k+1}^{r-1} \leqslant 2M\alpha_k$$

with, according to (27), the monotonically increasing function

$$\Phi : u \mapsto \frac{\sqrt{u}}{f(u)} = \frac{u}{\Theta(u)}$$

and

$$\mathrm{t}_{k+1} \leqslant 2M\Theta\left(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\right) \qquad \text{i.e. } \Theta^{-1}\left(\frac{\mathrm{t}_{k+1}}{2M}\right) \leqslant \frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}, \tag{58}$$

consequently

$$\Phi\left(\Theta^{-1}\left(\frac{\mathrm{t}_{k+1}}{2M}\right)\right)\mathrm{t}_{k+1}^{r-1} \leqslant 2M\alpha_k.$$

Since $\Phi\left(\Theta^{-1}\big(\frac{t}{C}\big)\right)t^{r-1} = C\Theta^{-1}\big(\frac{t}{C}\big)t^{r-2} = \varphi_r\big(\frac{t}{C}\big)C^{r-1}$, this implies

$$\mathrm{t}_{k+1} \leqslant 2M\varphi_r^{-1}((2M)^{2-r}\alpha_k) \tag{59}$$

from which by (58) we get

$$\begin{aligned} \mathrm{d}_{k+1}^2 &\leqslant \frac{\mathrm{t}_{k+1}^2}{\Theta^{-1}\big(\frac{\mathrm{t}_{k+1}}{2M}\big)} = (2M)^2\left(f\left(\Theta^{-1}\left(\frac{\mathrm{t}_{k+1}}{2M}\right)\right)\right)^2 \\ &\leqslant (2M)^2\left(f(\Theta^{-1}(\varphi_r^{-1}((2M)^{2-r}\alpha_k))\right)^2. \end{aligned} \tag{60}$$

Otherwise, if $\big(\kappa C_2^r + m\big)\big(\varphi_r^{-1}(\alpha_k)\big)^r \geqslant M\alpha_k\mathrm{d}_{k+1} f\big(\frac{\mathrm{t}_{k+1}^2}{\mathrm{d}_{k+1}^2}\big)$, we get from (56)

$$\mathrm{t}_{k+1}^r + \alpha_k\mathrm{d}_{k+1}^2 \leqslant 2\big(\kappa C_2^r + m\big)\big(\varphi_r^{-1}(\alpha_k)\big)^r. \tag{61}$$

From (59)–(61), using the identity

$$f(\Theta^{-1}\big(\underbrace{\varphi_r^{-1}(\alpha)}_{=:z}\big) = \frac{z}{\sqrt{\Theta^{-1}(z)}} = z^{r/2}\frac{1}{\sqrt{z^{r-2}\Theta^{-1}(z)}} = \frac{1}{\sqrt{\varphi_r(z)}}z^{r/2} = \frac{1}{\sqrt{\alpha}}\left(\varphi_r^{-1}(\alpha)\right)^{r/2}$$

and (21), we see that in order to complete the induction proof of (52), (53), it suffices to show

$$\varphi_r^{-1}(\alpha) \leqslant \hat{C}_\varphi\varphi_r^{-1}(\alpha/\hat{C}) \quad \forall 0 < \alpha \leqslant \alpha_0, \tag{62}$$

$$\varphi_r^{-1}(\alpha) \leqslant \tilde{C}_\varphi\varphi_r^{-1}(\alpha/\tilde{C}) \quad \forall 0 < \alpha \leqslant (2M)^{r-2}\alpha_0, \tag{63}$$

and use (54), (55). By the definition of φ_r, (62) can be concluded from (28) as follows: with $\hat{C}_\varphi = \sqrt{\hat{C}_r\hat{C}_f}$, $\hat{C}_r = \hat{C}\hat{C}_\varphi^{2-r}$ (cf (30)), $\lambda = \hat{C}_\varphi\varphi_r^{-1}(\alpha/\hat{C})$, $t = \Theta^{-1}(\lambda)/\hat{C}_r$, we have for any

$\alpha \in (0, \alpha_0]$:

$$
\begin{aligned}
f(\hat{C}_r t) \leqslant \hat{C}_f f(t) &\Leftrightarrow \underbrace{\Theta(\hat{C}_r t)}_{\lambda} \leqslant \underbrace{\sqrt{\hat{C}_r \hat{C}_f}}_{=\hat{C}_\varphi} \Theta(t) \\
&\Leftrightarrow \Theta^{-1}(\lambda/\hat{C}_\varphi) \leqslant t = \frac{1}{\hat{C}_r}\Theta^{-1}(\lambda) \\
&\Leftrightarrow (\lambda/\hat{C}_\varphi)^{r-2}\Theta^{-1}(\lambda/\hat{C}_\varphi) \leqslant \frac{1}{\hat{C}_r \hat{C}_\varphi^{r-2}} \lambda^{r-2}\Theta^{-1}(\lambda) \\
&\Leftrightarrow \underbrace{\hat{C}_r \hat{C}_\varphi^{r-2}}_{=\hat{C}} \underbrace{\varphi_r(\lambda/\hat{C}_\varphi)}_{=\alpha/\hat{C}} \leqslant \varphi_r(\lambda) \\
&\Leftrightarrow \varphi_r^{-1}(\alpha) \leqslant \lambda = \hat{C}_\varphi \varphi_r^{-1}(\alpha/\hat{C}),
\end{aligned}
$$

where we have used the fact that the functions φ_r, Θ as well as their inverses are strictly monotonically increasing. Analogously, (63) follows from (29). Therewith, the induction proof of (52), (53) is finished.

The estimates (52), (53) immediately yield (32).

Inserting (23) into (52) for $k = k_*$ directly yields with (33)

$$
\begin{aligned}
\mathrm{d}_{k_*} &\leqslant C_1 f\big(\Theta^{-1}\big(\varphi_r^{-1}(\alpha_{k_*})\big) \leqslant C_1 f(\Theta^{-1}(\tau\delta)\big) \leqslant C_1 \max\{\sqrt{\tau}, 1\} f(\Theta^{-1}(\delta)) \\
&= C_1 \max\{\sqrt{\tau}, 1\} \frac{\Theta(\Theta^{-1}(\delta))}{\sqrt{\Theta^{-1}(\delta)}} = C_1 \max\{\sqrt{\tau}, 1\} \frac{\delta}{\sqrt{\Theta^{-1}(\delta)}}.
\end{aligned}
$$

This provides us with the convergence rate assertion (31) and completes the proof of (ii). □

Corollary 1. *Let the assumptions of propositions 2 and 1 with*

$$q = p = 2,$$

and

$$\forall R \geqslant \hat{R}: \quad \overline{d}\big(\hat{C}_f \hat{C}_r^{-1/p} R\big) \leqslant \hat{C}_f \overline{d}(R), \tag{64}$$

$$\forall R \geqslant \tilde{R}: \quad \overline{d}\big(\tilde{C}_f \tilde{C}_r^{-1/p} R\big) \leqslant \tilde{C}_f \overline{d}(R), \tag{65}$$

with (30) hold, where

$$\hat{R} = \Psi^{-1}\big(\Theta^{-1}\big(\hat{C}_\varphi \varphi_r^{-1}(\alpha_0)\big)/\hat{C}_r\big), \qquad \tilde{R} = \Psi^{-1}\big(\Theta^{-1}\big(\tilde{C}_\varphi \varphi_r^{-1}((2M)^{r-2}\alpha_0)\big)/\tilde{C}_r\big).$$

Moreover, assume that

$$\tilde{c}_1 = \tilde{c}_3 = \tfrac{1}{2}, \qquad \tilde{c}_2 = \tilde{c}_4 = 0,$$

and K is sufficiently small in (18).

Then, with the a priori choice (23), we obtain convergence rates (31), (32) with f as in (17).

Proof. The assertion follows by a combination of part (ii) of theorem 1, proposition 1 and the fact that (28), (29) can be concluded from (64), (65): with $R = \Psi^{-1}(t)$ we get for any $t \in (0, \hat{t}]$:

$$
\begin{aligned}
\frac{f(\hat{C}_r t)}{f(t)} &= \frac{\overline{d}(\Psi^{-1}(\hat{C}_r t))}{\overline{d}(R)} = \left(\frac{(\overline{d}(\Psi^{-1}(\hat{C}_r t))/\Psi^{-1}(\hat{C}_r t))^p}{(\overline{d}(R)/R)^p}\right)^{1/p} \frac{\Psi^{-1}(\hat{C}_r t)}{R} \\
&= \left(\frac{\hat{C}_r t}{t}\right)^{1/p} \frac{\Psi^{-1}(\hat{C}_r t)}{\Psi^{-1}(t)} = \hat{C}_r^{1/p} \frac{\Psi^{-1}(\hat{C}_r t)}{\Psi^{-1}(t)} \leqslant \hat{C}_f,
\end{aligned}
$$

since we have the equivalences

$$\begin{aligned}\Psi^{-1}(\hat{C}_r t) \leqslant \hat{C}_f \hat{C}_r^{-1/p}\Psi^{-1}(t) &\Leftrightarrow \hat{C}_r t \geqslant \Psi\big(\hat{C}_f \hat{C}_r^{-1/p}\Psi^{-1}(t)\big)\\ &\Leftrightarrow \hat{C}_r \Psi(R) \geqslant \Psi\big(\hat{C}_f \hat{C}_r^{-1/p} R\big)\\ &\Leftrightarrow \hat{C}_r(\overline{d}(R)/R)^p \geqslant \big(\overline{d}\big(\hat{C}_f \hat{C}_r^{-1/p} R\big)/\big(\hat{C}_f \hat{C}_r^{-1/p} R\big)\big)^p\\ &\Leftrightarrow \hat{C}_f^p \overline{d}(R)^p \geqslant \overline{d}\big(\hat{C}_f \hat{C}_r^{-1/p} R\big)^p,\end{aligned}$$

where we have used the fact that Ψ^{-1} is strictly decreasing. Analogously we get (29). Note that (27) is automatically satisfied for f defined by (17), see remark 1. Moreover, in the case $x_{k+1}^\delta - x^\dagger \in \mathcal{N}(F'(x^\dagger))$ that is not covered by proposition 1, we can conclude from proposition 2.1 in [17] and $\tilde{c}_1 = \tilde{c}_3 = \frac{1}{2}$, $\tilde{c}_2 = \tilde{c}_4 = 0$ as well as K sufficiently small in (18) that x_{k+1}^δ solves (1). □

4. Convergence rates with *a posteriori* parameter choice

If the exponent ν in the source condition is not known, we require a nonlinearity assumption that corresponds to the strongest case $\nu = 0$ in (18)–(20), namely the tangential cone condition

$$\|F(x) - F(\bar{x}) - F'(x)(x - \bar{x})\| \leqslant c_{tc}\|F(x) - F(\bar{x})\| \quad \forall x, \bar{x} \in \mathcal{B} \tag{66}$$

for some $0 < c_{tc} < 1$, $\rho > 0$. Note that (18) for $\nu = 0$ with K sufficiently small becomes (66) at $x = x^\dagger$ with $c_{tc} = \frac{K}{1-K}$.

Therewith, we can prove convergence rates with *a posteriori* choices of the regularization parameters α_k

$$\underline{\sigma}\|g_k\| \leqslant \big\|T_k\big(x_{k+1}^\delta(\alpha_k) - x_k^\delta\big) + g_k\big\| \leqslant \overline{\sigma}\|g_k\| \tag{67}$$

(cf [6]), and of the stopping index k_* by the discrepancy principle:

$$k_*(\delta) = \min\big\{k \in \mathbb{N} : \big\|F\big(x_k^\delta\big) - y^\delta\big\| \leqslant \tau\delta\big\}. \tag{68}$$

Proposition 3. *Assume that a solution $x^\dagger$ to (1) exists, that F is weakly sequentially closed (see, e.g., (11), (12) in [18]), and satisfies (64) with c_{tc} sufficiently small*

$$c_{tc} < \underline{\sigma} < \overline{\sigma} < 1.$$

Moreover, let τ be chosen sufficiently large so that

$$c_{tc} + \frac{1 + c_{tc}}{\tau} \leqslant \underline{\sigma} \text{ and } c_{tc} < \frac{1 - \overline{\sigma}}{2}, \tag{69}$$

and let x_0 be close enough to $x^\dagger$ so that $D_p^{x_0}(x^\dagger, x_0)$ is sufficiently small. Additionally, assume that either

(a) $F'(x) : X \to Y$ is weakly closed for all $x \in \mathcal{D}(F)$ and Y reflexive
or
(b) $\mathcal{D}(F)$ is weakly closed

and

$$\delta < \frac{\|F(x_0) - y^\delta\|}{\tau}.$$

Then for all $k \leqslant k_*(\delta) - 1$ with $k_*(\delta)$ according to (68), the iterates

$$x_{k+1}^\delta := \begin{cases} x_{k+1}^\delta(\alpha_k), \text{ with } \alpha_k \text{ as in (67)} & \text{if } \big\|T_k\big(x_0 - x_k^\delta\big) + g_k\big\| \geqslant \overline{\sigma}\|g_k\| \\ x_0 & \text{else}\end{cases}$$

are well defined.

Proof. For well definedness and convergence without rates, as well as the fact that the iterates remain in $\mathcal{B}$, see theorem 3 in [18]. Note that conditions (a) or (b) guarantee the existence of α_k according to (67) whenever required for the method. $\square$

Theorem 2. *Let the assumptions of proposition 3 be satisfied.*

(i) Under a variational inequality (9) we obtain optimal convergence rates

$$D_p^{x_0}(x^\dagger, x_{k_*}) = O\left(\delta^{\frac{2\nu}{\nu+1}}\right) \quad as \quad \delta \to 0. \tag{70}$$

(ii) Under a variational inequality (11) we obtain optimal convergence rates

$$D_p^{x_0}(x^\dagger, x_{k_*}) = O(f^2(\Theta^{-1}(\delta))) = O\left(\frac{\delta^2}{\Theta^{-1}(\delta)}\right) \quad as \quad \delta \to 0 \tag{71}$$

with Θ as in (24).

Proof. The stopping index $k_*(\delta)$ according to (68) is finite, since on one hand, the case that $\left\|T_k\left(x_0 - x_k^\delta\right) + g_k\right\| < \overline{\sigma}\,\|g_k\|$ and therewith $x_{k+1}^\delta := x_0$ can happen at most every second step:

$$x_{k+1}^\delta = x_0 \;\Rightarrow\; \left\|T_{k+1}\left(x_0 - x_{k+1}^\delta\right) + g_{k+1}\right\| = \|g_{k+1}\| \geqslant \overline{\sigma}\,\|g_{k+1}\|\,,$$

so α_{k+1} can be chosen as in (67) (with k replaced by $k+1$). On the other hand, in steps where α_k is chosen as in (67), the residual norm decreases by a factor of $\frac{\overline{\sigma}+c_{tc}}{1-c_{tc}}$ which is smaller than 1 by (69):

$$\begin{aligned}\|g_{k+1}\| &= \left\|T_k\left(x_{k+1}^\delta - x_k^\delta\right) + g_k + F\left(x_{k+1}^\delta\right) - F\left(x_k^\delta\right) - T_k\left(x_{k+1}^\delta - x_k^\delta\right)\right\| \\ &\leqslant \overline{\sigma}\,\|g_k\| + c_{tc}\left\|F\left(x_{k+1}^\delta\right) - F\left(x_k^\delta\right)\right\| \\ &\leqslant (\overline{\sigma} + c_{tc})\,\|g_k\| + c_{tc}\,\|g_{k+1}\|\,.\end{aligned}$$

Hence,

$$\|g_k\| \leqslant \left(\frac{\overline{\sigma} + c_{tc}}{1 - c_{tc}}\right)^{[k/2]} \leqslant \tau\delta$$

for k sufficiently large.

Estimates (34), (36), together with (66), (2), (67), (68), yield

$$\underline{\sigma}^r\,\|g_k\|^r + \alpha_k\left\|x_{k+1}^\delta - x_0\right\|^p \leqslant \left(c_{tc} + \frac{1 + c_{tc}}{\tau}\right)^r \|g_k\|^r + \alpha_k\|x^\dagger - x_0\|^p \tag{72}$$

for all $k \leqslant k_*(\delta) - 1$, provided $x_k \in \mathcal{B}_\rho(x_0)$.

Inserting (34) into (72) and taking into account (69), (66), we get

$$(1 - \beta\epsilon)D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right) \leqslant \beta C\left(\epsilon, \frac{\nu+1}{2}\right)\left((1 + c_{tc})\left\|F\left(x_{k+1}^\delta\right) - F(x^\dagger)\right\|\right)^{2\nu/(\nu+1)} \tag{73}$$

in the case α_k is chosen according to (67). Hence, with $\epsilon < \beta^{-1}$, for $k = k_* - 1$ the discrepancy principle (68) yields the optimal rate

$$D_p^{x_0}\left(x^\dagger, x_{k_*}^\delta\right) \leqslant \frac{\beta C\left(\epsilon, \frac{\nu+1}{2}\right)}{1 - \beta\epsilon}((1 + c_{tc})(1 + \tau))^{2\nu/(\nu+1)}\,\delta^{2\nu/(\nu+1)},$$

since by the signal to noise ratio assumption $\delta < \|F(x_0) - y^\delta\|/\tau$ we can exclude the case $x_{k_*}^\delta = x_0$, i.e. the case that α_{k_*-1} is not chosen according to (67).

In the general case (11) we get, in place of (34), (73), the estimates (48) and

$$D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)^{1/2} \leqslant f\left(\frac{(1 + c_{tc})^2\left\|F\left(x_{k+1}^\delta\right) - F(x^\dagger)\right\|^2}{D_p^{x_0}\left(x^\dagger, x_{k+1}^\delta\right)}\right)$$

respectively. Hence, with $k = k_* - 1$, using (66) and (68) we get

$$C\delta = \frac{C\delta}{D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big)^{1/2}} D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big)^{1/2}$$
$$\leqslant \frac{C\delta}{D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big)^{1/2}} f\left(\frac{C^2\delta^2}{D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big)}\right) = \Theta\left(\frac{(C\delta)^2}{D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big)}\right)$$

with $C := (1 + c_{tc})(1 + \tau)$ so taking the inverse of Θ on both sides, we get

$$D_p^{x_0}\big(x^\dagger, x_{k_*}^\delta\big) \leqslant \frac{C^2\delta^2}{\Theta^{-1}(C\delta)} \leqslant C^2 \frac{\delta^2}{\Theta^{-1}(\delta)},$$

since $C > 1$ and Θ^{-1} is strictly monotonically increasing. □

Corollary 2. *Under the assumptions of propositions 3, 1 with*

$$q = p = 2,$$

we obtain convergence rates (71), with f as in (17).

Remark 3. Note that proposition 1 together with corollaries 1, 2 for $p = q = 2$ gives a relation between logarithmic decay of the distance function and logarithmic convergence rates (see, e.g., [14, 15]), which are particularly important for exponentially ill-posed problems. For

$$\overline{d}(R) = \ln(R)^{-N} \quad (R > e),$$

with some $N > 0$, we get $\Psi(R) = \frac{1}{\ln(R)^{2N} R^2} \leqslant \frac{1}{R}$; hence with $\check{C} = 2\max\{1, \underline{c}^{-1/2}\}$, we obtain $f(\lambda) = \check{C}\ln(\Psi^{-1}(\lambda))^{-N} \leqslant \check{C}\ln\left(\frac{1}{\lambda}\right)^{-N}$, so $\Theta(\lambda) = f(\lambda)\sqrt{\lambda} \leqslant \check{C}\ln\left(\frac{1}{\lambda}\right)^{-N}\sqrt{\lambda}$, which implies for the quotient terms occurring in the convergence rates of corollaries 1 and 2

$$\frac{\delta^2}{\Theta^{-1}(\delta)} = \frac{[\Theta(\Theta^{-1}(\delta))]^2}{\Theta^{-1}(\delta)} = [f(\Theta^{-1}(\delta))]^2 \leqslant \check{C}^2 \ln\left(\frac{1}{\Theta^{-1}(\delta)}\right)^{-2N} \leqslant \bar{C}_N \ln\left(\frac{1}{\delta}\right)^{-2N}$$

for some $\bar{C}_N > 0$. Here we have considered only the case of sufficiently large $R > 0$ which corresponds with sufficiently small noise levels $\delta > 0$.

5. Two parameter identification examples

In this section, we consider two model problems that have previously been studied in the Hilbert space setting, e.g, in [5–7, 16, 22], and in the Banach space setting in [18]. Since in both examples, X and Y will be defined by Lebesgue or Sobolev–Slobodeckij spaces, we first of all quote some facts on these spaces, see, e.g., [2, 8, 24, 26].

Lemma 1. *Let $\Omega \subseteq \mathbb{R}^{\dim}$ be a smooth domain.*

(a) $L^P(\Omega)$, $W^{m,P}(\Omega)$ *are* $\begin{cases} 2\text{-convex and } P\text{-smooth} & \text{for } 1 < P \leqslant 2 \\ P\text{-convex and } 2\text{-smooth} & \text{for } 2 \leqslant P < \infty. \end{cases}$

(b) The duality mapping J_p is given by

$$J_p(x) = \|x\|_X^{p-P} |x|^{P-1} \operatorname{sgn}(x) \text{ in } X = L^P(\Omega), \tag{74}$$

$$J_p(x) = \|x\|_X^{p-P} (-\nabla(|\nabla x|^{P-2}\nabla x) + |x|^{P-1}\operatorname{sgn}(x))$$
$$\text{in } X = W^{1,P}(\Omega) \text{ if } \frac{\partial x}{\partial n} = 0 \text{ on } \partial\Omega, \tag{75}$$

$$J_p(x) = \|x\|_X^{p-P}(\Delta(|\Delta x|^{P-2}\Delta x) - \nabla(|\nabla x|^{P-2}\nabla x) + |x|^{P-1}\mathrm{sgn}(x))$$
$$\textit{in } X = W^{2,P}(\Omega) \textit{ if } \frac{\partial x}{\partial n} = \Delta x = 0 \textit{ on } \partial\Omega, \tag{76}$$

provided that $W^{2,p}(\Omega)$ *is equipped with the norm*

$$\|x\|_{W^{2,p}(\Omega)} = \left(\int_\Omega (|\Delta x|^P + |\nabla x|^P + |x|^P)\,\mathrm{d}x\right)^{1/P}.$$

Proof. Referring to e.g. [2, 8, 24, 26] for (a), and (74), we only show (75), here. If $\frac{\partial x}{\partial n} = 0$ on $\partial\Omega$, then with $x^* := J_p(x)$ as claimed in (75), we indeed have

$$\langle x^*, x\rangle_{X^*,X} = \int_\Omega \|x\|_X^{p-P}(-\nabla(|\nabla x|^{P-2}\nabla x) + |x|^{P-1}\mathrm{sgn}(x))x\,\mathrm{d}x$$
$$= \|x\|_X^{p-P}\int_\Omega (|\nabla x|^P + |x|^P)\,\mathrm{d}x = \|x\|_X^p,$$

where we have used integration by parts. Assertion (76) can be shown analogously. □

As a first example, we consider identification of the space-dependent coefficient c in the elliptic boundary value problem

$$-\Delta u + cu = f \quad \text{in } \Omega \tag{77}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{78}$$

from measurements of u in Ω (note that inhomogeneous Dirichlet boundary conditions can be easily incorporated into the right-hand side f if necessary). Here $\Omega \subseteq \mathbb{R}^{\dim}$, $\dim \in \{1, 2, 3\}$ is assumed to be a smooth bounded domain. The forward operator

$$F : \mathcal{D}(F) \subseteq X \to Y \tag{79}$$

and its derivative as well as the Banach space adjoint can be written as

$$F(c) = A(c)^{-1}f,$$
$$F'(c)h = -A(c)^{-1}(h\cdot F(c)), \qquad F'(c)^*w = -F(c)\cdot(A(c)^{-1}w),$$

with

$$\begin{aligned} A(c) : H^2(\Omega)\cap H_0^1(\Omega) &\to L^2(\Omega)g \\ u &\to -\Delta u + cu. \end{aligned}$$

It was shown in [18] that for

$$X = L^P(\Omega), \qquad Y = L^R(\Omega) \tag{80}$$

with

$$P \in (1,\infty), \qquad P \geqslant \frac{\dim}{2}, \qquad R > \frac{P}{P-1}, \qquad R \geqslant \frac{2\dim P}{\dim P + 2P - 2\dim} \tag{81}$$

the assumptions on F in theorem 1 and with

$$P \in (1,\infty), \qquad R \in [2,\infty], \qquad \frac{2R}{R-2} \leqslant P \tag{82}$$

the assumptions on F in theorem 2 are satisfied. Here, the domain of F is set to

$$\mathcal{D}(F) = \{c \in L^P(\Omega) \mid \exists \hat{c} \in L^\infty(\Omega), \hat{c} \geqslant 0 \text{ a.e.} : \|c - \hat{c}\|_{L^P(\Omega)} \leqslant \tilde{\gamma}\}, \tag{83}$$

where $\tilde{\gamma} < \min\{1/\ \|id\|_{H_0^1(\Omega)\to L^{2P/(P-1)}(\Omega)}\ ,\ 1/\ \|id\|_{W^{2,k}\cap H_0^1(\Omega)\to L^{Pk/(P-k)}(\Omega)}\}$ for some

$$\begin{aligned} & k \in [\tilde{a}, \tilde{b}] \cap (1, \infty) \text{ with} \\ & \tilde{a} = \max\{2\mathrm{dim}/(\mathrm{dim}+2), \mathrm{dim}R/(\mathrm{dim}+2R)\}, \\ & \tilde{b} = \min\{P, 2\mathrm{dim}/\max\{0, \mathrm{dim}-2\}, R, PR/(P+R)\} \\ & \text{ and } (k < P \ \wedge\ R < \infty) \text{ or } k > \mathrm{dim}/2 \end{aligned} \tag{84}$$

in the first case (81), and to

$$\mathcal{D}(F) = \{c \in L^\infty(\Omega) \mid \hat{\gamma} \geqslant c \geqslant 0 \text{ a.e.}\} \tag{85}$$

for some $\hat{\gamma} > 0$ in the second case (82).

Therewith the benchmark source condition (5) is equivalent to

$$w = -\|c^\dagger - c_0\|_{L^P}^{p-P} A(c^\dagger)(|c^\dagger - c_0|^{P-1}\mathrm{sgn}(c^\dagger - c_0)) \ \in Y^* = L^{R/(R-1)}(\Omega). \tag{86}$$

Choosing P as small as possible and R as large as possible corresponds to formulating the inverse problem as weakly ill-posed as possible and therewith obviously also to making the source condition (86) as weak as possible. Note that indeed the noise level is in practice often given in the L^∞ norm. Under conditions (81), we might, e.g., set

$$R = \infty, \qquad P = \mathrm{dim}/2 + \varepsilon, \qquad k := \max\{2\mathrm{dim}/(\mathrm{dim}+2), P\},$$

for $\varepsilon > 0$ arbitrarily small, and under conditions (82)

$$R = \infty, \qquad P = 2.$$

This allows for a relaxation as compared to the Hilbert space case $P = R = 2$.

In the second example we deal with the identification of the space-dependent coefficient a in

$$-\nabla(a\nabla u) = f \quad \text{in } \Omega \tag{87}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{88}$$

from measurements of u, where again $\Omega \subseteq \mathbb{R}^{\mathrm{dim}}$, $\mathrm{dim} \in \{1, 2, 3\}$ is assumed to be a smooth bounded domain. Using the differential operator

$$\begin{aligned} A(a) : H^2(\Omega) \cap H_0^1(\Omega) &\to L^2(\Omega) \\ u &\to -\nabla(a\nabla u) \end{aligned}$$

we can write the forward operator, its derivative, as well as the Banach space adjoint as

$$\begin{aligned} & F(a) = A(a)^{-1} f, \\ & F'(a)h = A(a)^{-1}(\nabla(h\nabla F(a))), \qquad F'(a)^* w = -\nabla F(a) \cdot \nabla(A(a)^{-1} w). \end{aligned}$$

It has been shown in [18] that with

$$\mathcal{D}(F) = \{a \in X | a \geqslant \underline{\alpha}\} \tag{89}$$

with $\underline{\alpha} > 0$,

$$X = W^{1,Q}(\Omega), \qquad Y = L^R(\Omega) \tag{90}$$

under conditions

$$Q > \mathrm{dim}, \qquad Q \in (1, \infty), \qquad Q \geqslant \frac{R}{R-1},$$

$$R \leqslant \frac{2\mathrm{dim}}{\max\{0, \mathrm{dim}-2\}} \quad \text{and} \quad (R < \infty\ \vee\ \mathrm{dim} < 2)\,,$$

the assumptions on F in propositions 2, 3, theorems 1, 2, and corollaries 1, 2 are satisfied.

For this example, the benchmark source condition (5) is equivalent to

$$\exists w \in Y^* = L^{R/(R-1)}(\Omega) : -\nabla F(a^\dagger) \cdot \nabla (A(a^\dagger)^{-1} w) \\ = \|e^0\|_{W^{1,Q}}^{p-Q} (-\nabla(|\nabla e^0|^{Q-2} \nabla e^0) + |e^0|^{Q-1} \mathrm{sgn}(e^0)), \tag{91}$$

as well as

$$\frac{\partial e^0}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{92}$$

for

$$e^0 = a^\dagger - a_0,$$

which amounts to a transport equation for $A(a^\dagger)^{-1} w$. In the 1D case $\Omega = (0, L)$, condition (91) becomes

$$w = -\|e^0\|_{W^{1,Q}}^{p-Q} A(a^\dagger) \left(\int_0^{\cdot} \frac{-\left(|e_x^0|^{Q-2} e_x^0\right)_x + |e^0|^{Q-1} \mathrm{sgn}(e^0)}{F(a^\dagger)_x} \,\mathrm{d}x \right) \\ \in Y^* = L^{R/(R-1)}(\Omega); \tag{93}$$

hence, the benchmark source condition is satisfied if $F(a^\dagger)_x$ is bounded away from zero as well as

$$e_x^0(0) = e_x^0(L) = 0 \quad \text{and} \quad e^0 \in W^{3,R/(R-1)}(\Omega). \tag{94}$$

Here we may, e.g. for arbitrarily small $\varepsilon > 0$, set

$$R = \infty, \qquad Q = 1 + \varepsilon \quad \text{if dim} = 1, \tag{95}$$

$$R = \frac{1}{\tilde{\varepsilon}}, \qquad Q = 2 + \varepsilon \quad \text{if dim} = 2, \tag{96}$$

with $\tilde{\varepsilon} \in (0, 1 - 1/(2 + \varepsilon)]$ arbitrarily small

$$R = 6, \qquad Q = 3 + \varepsilon \quad \text{if dim} = 3. \tag{97}$$

In the case dim $= 1$, (94) can be directly compared to the Hilbert space situation $Q = R = 2$, see, e.g., [5], and with (95) yields an obvious relaxation. Note that in the higher dimensional case, the Hilbert space setting requires a higher order Sobolev space, namely $H^s(\Omega)$ with $s \geqslant 1 + \mathrm{dim}/2 - \mathrm{dim}/Q$ so that $H^s(\Omega)$ is continuously embedded in $W^{1,Q}(\Omega)$. The Hilbert space benchmark source condition with $s = 2$ therefore becomes

$$-\nabla F(a^\dagger) \cdot \nabla (A(a^\dagger)^{-1} w) = (\Delta^2 e^0 - \Delta e^0 + e^0) \quad \text{and} \quad \frac{\partial e^0}{\partial n} = \Delta e^0 = 0 \quad \text{on } \partial\Omega,$$

(where we have used (76) with $p = P = 2$), which is obviously stronger than (91), (92) with (96) or (97), since it requires more knowledge on the boundary values of $a^\dagger$ as well as a higher order of differentiability.

Implementation of the IRGNM in Banach space requires numerical solution of the minimization problem (3) with a linear operator T_k in each step. If we do so, e.g., by one of the gradient-type methods devised in [2], we have to apply T_k as well as its Banach space adjoint (which amounts to solving a linear PDE in our parameter identification examples) and the duality mappings $J_p, J_r, J^*_{p/(p-1)} = J_p^{-1}$ in each inner iteration. While J_p, J_r only involve multiplication (and in example (87), (88) by (75) also differentiation), application of J_p^{-1} in example (87), (88) amounts to solving a PDE with the differential operator given by (75), which is even nonlinear unless $P = p = 2$.

6. Conclusions and remarks

In this paper, we provide convergence rate results for the IRGNM under approximate source conditions with general index functions including Hölder and logarithmic rates. Both *a priori* and *a posteriori* parameter choice strategies are studied.

Possible future research will be on the case of enhanced source conditions corresponding to $\nu \in (1, 2]$ (cf [19, 20] for Tikhonov regularization in Banach space). Moreover, different regularization terms in place of $\|x - x_0\|^p$ are of interest. Especially sparsity enhancing terms like the L^1 norm are not covered by the theory of this paper, since $L^1(\Omega)$ is not a uniformly convex space. For this purpose, new ideas will have to be developed and first of all well definedeness and convergence without rates will have to be proven (see [18] for the case of uniformly convex spaces). Like, e.g., in [1] and [17], one might also think of using a general regularization method (in place of Tikhonov) in each Newton step (e.g. the Landweber iteration from [24]).

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