New results for variational regularization with oversmoothing penalty term in Banach spaces

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Abstract

In this article on variational regularization for ill-posed nonlinear problems, we are once again discussing the consequences of an oversmoothing penalty term. This means in our model that the searched-for solution of the considered nonlinear operator equation does not belong to the domain of definition of the penalty functional. In the past years, such variational regularization has been investigated comprehensively in Hilbert scales, but rarely in a Banach space setting. Our present results try to establish a theoretical justification of oversmoothing regularization in Banach scales. This new study includes convergence rates results for a priori and a posteriori choices of the regularization parameter, both for Hölder-type smoothness and low order-type smoothness. An illustrative example is intended to indicate the specificity of occurring non-reflexive Banach spaces.

Keywords: Nonlinear ill-posed problem, variational regularization, oversmoothing penalty, convergence rates results, a priori parameter choice strategy, discrepancy principle, logarithmic source conditions

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1 Introduction

The goal of this paper is the theoretical justification of variational regularization with oversmoothing penalties for nonlinear ill-posed problems in Banach scales. Precisely, we consider operator equations of the form

$$F(u) = f^{\dagger}, \tag{1}$$

where $F : \mathcal{X} \supset \mathcal{D}(F) \to \mathcal{Y}$ is a nonlinear operator between infinite-dimensional Banach spaces \mathcal{X} and \mathcal{Y} with norms $\|\cdot\|$. We suppose that the right-hand side $f^{\dagger} \in \mathcal{Y}$ is approximately given as $f^{\delta} \in \mathcal{Y}$ satisfying the deterministic noise model

$$\|f^{\delta} - f^{\dagger}\| \le \delta, \tag{2}$$

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with the noise level $\delta \geq 0$. Throughout the paper, it is assumed that the considered equation (1) has a solution $u^{\dagger} \in \mathcal{D}(F)$ and is, at least at u^{\dagger} , locally ill-posed in the sense of [15, Def. 1.1] and [13, Def. 3]. Consequently, a variant of regularization is required for finding stable approximations to the solution $u^{\dagger} \in \mathcal{D}(F)$ of equation (1), and we exploit in this context a variant of variational regularization with regularization parameter $\alpha > 0$, where the regularized solutions u^{δ}_{α} are minimizers of the extremal problem

$$T^{\delta}_{\alpha}(u) := \|F(u) - f^{\delta}\|^{r} + \alpha \|u - \overline{u}\|_{1}^{r} \to \min, \quad \text{subject to} \quad u \in \mathcal{D}(F), \quad (3)$$

with some exponent r > 0 being fixed. Here, $\|\cdot\|_1$ is a norm on a densely defined subspace \mathcal{X}_1 of \mathcal{X} , which is stronger than the original norm $\|\cdot\|$ in \mathcal{X} . Precisely, we define the stronger norm $\|\cdot\|_1$ by $\|u\|_1 = \|G^{-1}u\|, u \in \mathcal{R}(G)$, where the generator $G : \mathcal{X} \to \mathcal{X}$ with range $\mathcal{R}(G)$ is a bounded linear operator, which is one-to-one and has an unbounded inverse G^{-1} . Further conditions on the operator G are given in Section 2 below. Moreover, the element $\overline{u} \in \mathcal{X}_1 \cap \mathcal{D}(F)$, occurring in the penalty term of the Tikhonov functional T^{δ}_{α} , denotes an initial guess. Note that we restrict our consideration in this study to identical exponents r for the misfit term and the penalty functional in order to avoid unnecessary technical complications.

In the present work, we discuss the nonlinear Tikhonov-type regularization (3) with focus on an oversmoothing penalty term. This means in our model that we have $u^{\dagger} \notin \mathcal{X}_1$, or in other words $||u^{\dagger}||_1 = +\infty$, which is an expression of 'non-smoothness' of the solution u^{\dagger} with respect to the reference Banach space \mathcal{X}_1 . Variational regularization of the form (3) with r = 2 and oversmoothing penalty for nonlinear ill-posed operator equations (1) has been investigated comprehensively in the past four years in Hilbert scales, and we refer to [11, 14] as well as further to the papers [5, 8, 9, 12, 17]. For related results on linear problems, see, e.g., [21]. Our present study continues and extends, along the lines of [14], the investigations on nonlinear problems to Banach scales. This new study includes fundamental error estimates yielding convergence and convergence rates results for a priori and a posteriori choices of the regularization parameter, both for Hölder-type smoothness and low order smoothness. The necessary tools for low order smoothness in the Banach space setting are provided. In addition, a relaxed nonlinearity and smoothing condition on the operator F is considered that turns out to be useful for maximum norms.

Banach space results for the discrepancy principle in a pure equation form have already been proven for the oversmoothing case in the recent paper [3]. In parallel, such results have been developed for oversmoothing subcases to variants of ℓ^1 -regularization and sparsity promoting wavelet regularization in [19, Sec. 5] and [20].

The outline of the remainder is as follows: in Section 2 we summarize prerequisites and assumptions for the main results in the sense of error estimates and convergence rates for the regularized solutions. On the one hand, error estimates for a priori choices of the regularization parameter are presented in Section 3. On the other hand, Section 4 presents results and consequences for using a discrepancy principle. An illustrative example in Section 5 is intended to indicate the specificity of occurring non-reflexive Banach spaces. A numerical case study will be presented in Section 6 that illustrates the theoretical results. Technical details, constructions and verifications for proving the main results of the paper are given in the concluding Section 7.

2 Prerequisites and assumptions

In this section, we introduce a scale of Banach spaces generated by an operator of positive type. Moreover, we define the logarithm of a positive operator and formulate the basic assumptions for our study. The concluding subsection is devoted to well-posedness and stability assertions for the variant of variational regularization under consideration in this paper.

2.1 Non-negative type operators, fractional powers, and regularization operators

Let \mathcal{X} with norm $\|\cdot\|$ be a Banach space and $\mathcal{L}(\mathcal{X})$ with norm $\|\cdot\|_{\mathcal{L}(\mathcal{X})}$ the associated space of bounded linear operators mapping in \mathcal{X} . Furthermore, let the injective operator $G \in \mathcal{L}(\mathcal{X})$ with range $\mathcal{R}(G)$ and unbounded inverse G^{-1} be of non-negative type, i.e.,

$$G + \beta I : \mathcal{X} \to \mathcal{X}$$
 one-to-one and onto, $\| (G + \beta I)^{-1} \|_{\mathcal{L}(\mathcal{X})} \le \frac{\kappa_*}{\beta}, \quad \beta > 0,$ (4)

for some finite constant $\kappa_* > 0$. Fractional powers of non-negative type operators may be defined as follows [1, 2]:

(a) For $0 , the fractional power <math>G^p : \mathcal{X} \to \mathcal{X}$ is defined by

$$G^{p}u := \frac{\sin \pi p}{\pi} \int_{0}^{\infty} s^{p-1} (G+sI)^{-1} Gu \, ds \quad \text{for } u \in \mathcal{X}.$$
(5)

This defines a bounded linear operator on \mathcal{X} .

(b) For arbitrary values $p \ge 1$, the bounded linear operator $G^p : \mathcal{X} \to \mathcal{X}$ is defined by

$$G^p := G^{p - \lfloor p \rfloor} G^{\lfloor p \rfloor}$$

We moreover use the notation $G^0 = I$.

In what follows, we shall need the interpolation inequality for fractional powers of operators, see, e.g., [18] or [7, Proposition 6.6.4]: for each pair of real numbers 0 , there exists some finite constant <math>c = c(p,q) > 0 such that

$$||G^{p}u|| \le c ||G^{q}u||^{p/q} ||u||^{1-p/q} \quad \text{for } u \in \mathcal{X}.$$
(6)

For $0 , the value of the constant can be chosen as <math>c = 2(\kappa_* + 1)$, cf., e.g., [24, Corollary 1.1.19]. Under the stated assumptions on G, for each p > 0, the fractional power G^p is one-to-one, and we use the notation $G^{-p} = (G^p)^{-1}$. We do not need that the operator G has dense range in \mathcal{X} .

The scale of normed spaces $\{\mathcal{X}_{\tau}\}_{\tau \in \mathbb{R}}$, generated by G, is given by the formulas

$$\mathcal{X}_{\tau} = \mathcal{R}(G^{\tau}) \text{ for } \tau > 0, \qquad \mathcal{X}_{\tau} = \mathcal{X} \text{ for } \tau \le 0,$$
$$\|u\|_{\tau} := \|G^{-\tau}u\| \text{ for } \tau \in \mathbb{R}, \ u \in \mathcal{X}_{\tau}. \tag{7}$$

For $\tau < 0$, topological completion of the spaces $\mathcal{X}_{\tau} = \mathcal{X}$ with respect to the norm $\|\cdot\|_{\tau}$ is not needed in our setting. We note that $(G^p)_{p\geq 0}$ defines a C_0 -semigroup on

 $\overline{\mathcal{R}(G)}$, which in particular means that $G^p u \to u$ for $p \downarrow 0$ is valid for any $u \in \overline{\mathcal{R}(G)}$ (cf. [7, Proposition 3.1.15]). Finally, we note that

$$\mathcal{R}(G^{\tau_2}) \subset \mathcal{R}(G^{\tau_1}) \subset \overline{\mathcal{R}(G)} \quad \text{for all } 0 < \tau_1 < \tau_2 < \infty.$$
(8)

2.2 The logarithm $\log G$

For the consideration of low order smoothness, we need to introduce the logarithm of G. For selfadjoint operators in Hilbert spaces this can be done by spectral analysis, and we refer in this context for example to [16]. In Banach spaces, $\log G$ may be defined as the infinitesimal generator of the C_0 -semigroup $(G^p)_{p>0}$ considered on $\overline{\mathcal{R}(G)}$:

$$(\log G)u = \lim_{p \downarrow 0} \frac{1}{p} (G^p u - u), \quad u \in \mathcal{D}(\log G),$$

where

$$\mathcal{D}(\log G) = \{ u \in \mathcal{X} : \lim_{p \downarrow 0} \frac{1}{p} (G^p u - u) \text{ exists} \}$$

cf., e.g., [22] or [7, Proposition 3.5.3]. Low order smoothness of an element $u \in \mathcal{X}$ by definition then means $u \in \mathcal{D}(\log G)$. Note that we obviously have $\mathcal{D}(\log G) \subset \overline{\mathcal{R}(G)}$. In addition, $\mathcal{R}(G^p) \subset \mathcal{D}(\log G)$ is valid for arbitrarily small p > 0, which follows from [22, Satz 1]. Summarizing the above notes, we have a chain of subsets of \mathcal{X} as

$$\mathcal{R}(G^p) \subset \mathcal{D}(\log G) \subset \overline{\mathcal{R}(G)} \quad \text{for all } p > 0.$$
(9)

This means that also in the Banach space setting, any Hölder-type smoothness is stronger than low order smoothness.

2.3 Main assumptions

In the following assumption, we briefly summarize the structural properties of the space \mathcal{X} , of the operator F and of its domain $\mathcal{D}(F)$, in particular with respect to the solution u^{\dagger} of the operator equation. Moreover, we make one more assumption concerning G in addition to the requirements on the operator G stated above.

- **Assumption 1.** (a) The infinite-dimensional Banach space \mathcal{X} has a separable predual space \mathcal{Z} with $\mathcal{Z}^* = \mathcal{X}$ such that a weak*-convergence denoted as \rightarrow^* takes place in \mathcal{X} .
- (b) The operator F : X ⊃ D(F) → Y is weak*-to-weak sequentially continuous, i.e., for elements u_n, u₀ ∈ D(F), weak*-convergence u_n →* u₀ in X implies weak convergence F(u_n) → F(u₀) in Y.
- (c) The domain of definition $\mathcal{D}(F) \subset \mathcal{X}$ is a sequentially weak*-closed subset of \mathcal{X} .
- (d) Let $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{X}_1 \neq \emptyset$.
- (e) Let the solution u[†] ∈ D(F) to equation (1) with right-hand side f[†] be an interior point of the domain D(F).

- (f) Let the data f^δ ∈ 𝔅 satisfy the noise model (2), and let the initial guess u satisfy u ∈ 𝔅₁ ∩ 𝔅(F).
- (g) Let a > 0, and let $0 < c_a \le C_a$ and $c_0, c_1 > 0$ be finite constants such that the following holds:
 - For each $u \in \mathcal{D}$ satisfying $||u u^{\dagger}||_{-a} \leq c_0$, we have

$$||F(u) - f^{\dagger}|| \le C_a ||u - u^{\dagger}||_{-a}.$$
 (10)

• For each $u \in \mathcal{D}$ satisfying $||F(u) - f^{\dagger}|| \leq c_1$, we have

$$c_a \| u - u^{\dagger} \|_{-a} \le \| F(u) - f^{\dagger} \|.$$
(11)

(h) The operator $G : \mathcal{X} \to \mathcal{X}$ defined above possesses a pre-adjoint operator $\tilde{G} : \mathcal{Z} \to \mathcal{Z}$ such that $\tilde{G}^* = G$ holds true.

Remark 2. From the inequality (11) of item (g) in Assumption 1, we have for $u^{\dagger} \in \mathcal{X}_1$ that u^{\dagger} is the uniquely determined solution to equation (1) in the set \mathcal{D} . For $u^{\dagger} \notin \mathcal{X}_1$, there is no solution at all to (1) in \mathcal{D} . But in both cases, alternative solutions $u^* \notin \mathcal{X}_1$ with $u^* \in \mathcal{D}(F)$ and $F(u^*) = f^{\dagger}$ cannot be excluded in general. Note that the concept of penalty-minimizing solutions, which is usual in theory of Tikhonov regularization, does not make sense in the case of oversmoothing penalties.

2.4 Existence and stability of regularized solutions

The following two propositions on existence and stability can be immediately taken from [29, § 4.1.1], and we refer in this context also to [10] and [28, § 3.2].

Proposition 3 (well-posedness, cf. Prop. 4.1 of [29]). For all $\alpha > 0$ and $f^{\delta} \in \mathcal{Y}$, there exists a regularized solution $u_{\alpha}^{\delta} \in \mathcal{D}$, minimizing the Tikhonov functional $T_{\alpha}^{\delta}(u)$ in (3) over all $u \in \mathcal{D}(F)$.

Proposition 4 (stability, cf. Prop. 4.2 of [29]). For all $\alpha > 0$, the minimizers $u_{\alpha}^{\delta} \in \mathcal{D}$ of the extremal problem (3) are stable with respect to the data f^{δ} . More precisely, for a data sequence $\{f_n\}$ converging to f^{δ} with respect to the norm topology in \mathcal{Y} , i.e. $\lim_{n\to\infty} ||f_n - f^{\delta}|| = 0$, every associated sequence $\{u_n\}$ of minimizers to the extremal problem

 $||F(u) - f_n||^r + \alpha ||u - \overline{u}||_1^r \to \min, \text{ subject to } u \in \mathcal{D}(F),$

has a subsequence $\{u_{n_k}\}$, which converges in the weak*-topology of \mathcal{X} , and the weak*limit \tilde{u} of each such subsequence is a minimizer u_{α}^{δ} of (3).

In order to prove the applicability of both propositions to our situation, we have to state that the relevant items of Assumptions 3.11 and 3.22 in [29] can be met as a consequence of our Assumption 1 by taking into account Remark 4.9 in [29], where the transfer from the weak-situation to the weak*-situation is explained. In particular,

since G is bounded, we have $||u||_1 \ge \tilde{c}||u||$ with some constant $\tilde{c} > 0$ for all $u \in \mathcal{X}_1$. Then the penalty functional $\Omega : \mathcal{X} \to [0, \infty]$ of the Tikhonov functional T^{δ}_{α} , defined as

$$\Omega(u) := \begin{cases} \|u - \overline{u}\|_{1}^{r} = \|G^{-1}(u - \overline{u})\|^{r} & \text{for } u \in \mathcal{X}_{1} \\ +\infty & \text{for } u \in \mathcal{X} \setminus \mathcal{X}_{1} \end{cases}$$

possesses the required stabilizing property (cf. item (c) of Assumption 3.22 in [29]) as a consequence of the sequential Banach–Alaoglu theorem, which implies that the sublevel sets

$$\{u \in \mathcal{X} : \Omega(u) \le c\} \subset \{u \in \mathcal{X} : \|u - \overline{u}\|^r \le \frac{c}{\tilde{c}^r}\}$$

are weak* sequentially compact in \mathcal{X} for all $c \geq 0$. Moreover, Ω is sequentially weak* lower semi-continuous (cf. item (b) of Assumption 3.22 in [29]), because the existence of a pre-adjoint operator $\tilde{G} : \mathcal{Z} \to \mathcal{Z}$ to G in the sense of item (h) of Assumption 1 ensures that $G : \mathcal{X} \to \mathcal{X}$ is weak*-to-weak* sequentially continuous. This together with the weak* lower semi-continuity of the norm functional in \mathcal{X} yields the sequentially weak* lower semi-continuity of the penalty functional Ω .

3 Error estimate and a priori parameter choices

We start with an error estimate result that provides the basis for the analysis of the regularizing properties, including convergence rates under a priori parameter choices. In what follows, we use the notation

$$\kappa := \frac{1}{r(a+1)}.\tag{12}$$

Theorem 5. Let Assumption 1 be satisfied. Then there exist finite positive constants K_1, α_0 and δ_0 such that for $0 < \alpha \le \alpha_0$ and $0 < \delta \le \delta_0$, an error estimate for the regularized solutions as

$$\|u_{\alpha}^{\delta} - u^{\dagger}\| \le f_1(\alpha) + K_1 \frac{\delta}{\alpha^{\kappa a}}$$
(13)

holds, where $f_1(\alpha)$ for $0 < \alpha \le \alpha_0$ is some bounded function satisfying:

- (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then $f_1(\alpha) \to 0$ as $\alpha \to 0$.
- (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then <math>f_1(\alpha) = \mathcal{O}(\alpha^{\kappa p})$ as $\alpha \to 0$.
- (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then $f_1(\alpha) = \mathcal{O}((\log \frac{1}{\alpha})^{-1})$ as $\alpha \to 0$.

Theorem 5, the proof of which can be found in [26], allows us to derive regularizing properties of variational regularization with oversmoothing penalty and to obtain convergence and rates results for appropriate a priori parameter choices that culminate in Theorem 6. For evaluating the strength of smoothness for the three different occurring situations in Theorem 5 (no explicit smoothness, Hölder smoothness and low order smoothness) we recall the chain (9) of range conditions.

The following theorem is a direct consequence of Theorem 5, because its proof is immediately based on the error estimate (13) with the respective properties of the function $f_1(\alpha)$.

Theorem 6. Let Assumption 1 be satisfied.

• (No explicit smoothness) Let $u^{\dagger} \in \overline{\mathcal{R}(G)}$. Then for any a priori parameter choice $\alpha_* = \alpha(\delta)$ satisfying $\alpha_* \to 0$ and $\frac{\delta}{\alpha_*^{\alpha_*}} \to 0$ as $\delta \to 0$, we have

$$\|\,u_{\alpha_*}^\delta-u^\dagger\,\|\to 0 \quad \text{ as } \delta\to 0$$

• (Hölder smoothness) Let $u^{\dagger} \in \mathcal{X}_p$ for some $0 . Then for any a priori parameter choice satisfying <math>\alpha_* = \alpha(\delta) \sim \delta^{1/(\kappa(p+a))}$, we have

$$\|u_{\alpha_{\alpha}}^{\delta} - u^{\dagger}\| = \mathcal{O}(\delta^{p/(p+a)}) \quad as \ \delta \to 0$$

• (Low order smoothness) Let $u^{\dagger} \in \mathcal{D}(\log G)$. Then for any a priori parameter choice satisfying $\alpha_* = \alpha(\delta) \sim \delta$, we have

$$||u_{\alpha_*}^{\delta} - u^{\dagger}|| = \mathcal{O}((\log \frac{1}{\delta})^{-1}) \quad as \ \delta \to 0.$$

4 Results for a discrepancy principle and consequences

For the specification of a suitable discrepancy principle, the behaviour of the misfit functional $\alpha \mapsto ||F(u_{\alpha}^{\delta}) - f^{\delta}||$ needs to be understood, for $\delta > 0$ fixed. The basic properties are summarized in the following proposition. As a preparation, we introduce the following parameter:

$$e_r = \begin{cases} 1, & \text{if } r \ge 1, \\ 2^{-1+1/r} & \text{otherwise.} \end{cases}$$
(14)

Proposition 7. Let Assumption 1 be satisfied. Then for $\delta > 0$ fixed, the function $\alpha \mapsto ||F(u_{\alpha}^{\delta}) - f^{\delta}||$ is non-decreasing, with

$$\lim_{\alpha \to 0} \|F(u_{\alpha}^{\delta}) - f^{\delta}\| \le e_r \delta, \quad \lim_{\alpha \to \infty} \|F(u_{\alpha}^{\delta}) - f^{\delta}\| = \|F(\overline{u}) - f^{\delta}\|.$$

In addition, we have $\lim_{\alpha \to \infty} ||u_{\alpha}^{\delta} - \overline{u}|| = 0.$

Proof. This follows along the lines of the proof of [14, Proposition 4.5]. Details are thus omitted here. \Box

Algorithm 8 (Discrepancy principle). Let $b > e_r$ and c > 1 be finite constants. (a) If $||F(\overline{u}) - f^{\delta}|| \le b\delta$ holds, then choose $\alpha_* = \infty$, i.e., $u_{\infty}^{\delta} := \overline{u} \in \mathcal{D}$.

(b) Otherwise, choose a finite parameter $\alpha =: \alpha_* > 0$ such that

$$\|F(u_{\alpha_*}^{\delta}) - f^{\delta}\| \le b\delta \le \|F(u_{\gamma_{\delta}}^{\delta}) - f^{\delta}\| \quad \text{for some } \alpha_* \le \gamma_{\delta} \le c\alpha_*, \tag{15}$$

where c > 1 denotes some finite constant.

Remark 9. (a) Practically, a parameter α_* satisfying condition (15) can be determined, e.g., by a sequential discrepancy principle. For more details, see, e.g., Section 6 and [14].

(b) It follows from Proposition 7 that Algorithm 8 is feasible. We note that for $\delta > 0$ fixed, the function $\alpha \mapsto ||F(u_{\alpha}^{\delta}) - f^{\delta}||$ may be discontinuous. For this reason, we do not consider other versions of the discrepancy principle, e.g., $||F(u_{\alpha_*}^{\delta}) - f^{\delta}|| = b\delta$ or $b_1\delta \le ||F(u_{\alpha_*}^{\delta}) - f^{\delta}|| \le b_2\delta$.

We next present the main result of this paper.

Theorem 10. Let Assumption 1 be satisfied, and let the regularization parameter $\alpha_* = \alpha(\delta, f^{\delta})$ be chosen according to the discrepancy principle.

• (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then we have

$$\|u_{\alpha_*}^{\delta} - u^{\dagger}\| \to 0 \quad as \ \delta \to 0.$$

• (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some 0 , then we have

$$||u_{\alpha_*}^{\delta} - u^{\dagger}|| = \mathcal{O}(\delta^{p/(p+a)}) \quad as \ \delta \to 0.$$

• (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then

$$\|u_{\alpha_*}^{\delta} - u^{\dagger}\| = \mathcal{O}((\log \frac{1}{\delta})^{-1}) \quad as \ \delta \to 0.$$

The proof of Theorem 10 is given in Section 7.5.

5 An illustrative example

The following example with specific Banach spaces and nonlinear forward operator is intended to illustrate the assumptions stated above and to indicate the specificity of occurring non-reflexive Banach spaces. This example should show that the general mathematical framework developed in this paper is applicable. The considered basis space is the non-reflexive and non-separable space $\mathcal{X} = L^{\infty}(0, 1)$ with the essential supremum norm $\|\cdot\| = \|\cdot\|_{\infty}$ possessing a separable pre-dual space $\mathcal{Z} = L^1(0, 1)$. The generator *G* of the scale of Banach spaces is given by

$$[Gu](x) = \int_0^x u(\xi) \, d\xi \qquad (0 \le x \le 1, \quad u \in L^\infty(0, 1)). \tag{16}$$

Below we give some properties of G:

- The operator $G: L^{\infty}(0,1) \to L^{\infty}(0,1)$ is of non-negative type with constant $\kappa_* = 2$, see, e.g., [24].
- G is a compact operator, which possesses a compact pre-adjoint operator \tilde{G} : $L^1(0,1) \rightarrow L^1(0,1)$, which is characterized by

$$[\tilde{G}v](x) = \int_x^1 v(\xi) \, d\xi \qquad (0 \le x \le 1, \quad v \in L^1(0,1)).$$

• G has a trivial nullspace and a non-dense range

$$\mathcal{R}(G) = W_0^{1,\infty}(0,1) := \{ u \in W^{1,\infty}(0,1) : u(0) = 0 \},\$$

with

$$\overline{\mathcal{R}(G)} = C_0[0,1] := \{ u \in C[0,1] : u(0) = 0 \}.$$

As a consequence of the last item we have that $\mathcal{X}_1 = W_0^{1,\infty}(0,1)$ with $||u||_1 := ||u'||_{\infty}$. With $\mathcal{X} = \mathcal{Y} = L^{\infty}(0,1)$, the nonlinear forward operator of this example is F:

with $X = \mathcal{Y} = L^{-}(0, 1)$, the holimetal forward operator of this example is F. $L^{\infty}(0, 1) \to L^{\infty}(0, 1)$ given by

$$[F(u)](x) = \exp((Gu)(x)) \qquad (0 \le x \le 1, \quad u \in L^{\infty}(0,1)).$$
(17)

This operator F is weak*-to-weak sequentially continuous, because F is a composition of the continuous outer nonlinear exponential operator and the inner linear integration operator G, both mapping in $L^{\infty}(0,1)$. The operator G transforms weak*-convergent sequences in $L^{\infty}(0,1)$ to norm-convergent sequences in this space, because G is compact and has a pre-adjoint operator (cf. [4, Lemma 2.5]).

Moreover, the operator F is Fréchet differentiable on its domain of definition $\mathcal{D}(F) = L^{\infty}(0,1)$, with $[F'(u)]h = [F(u)] \cdot Gh$. Now consider some function $u^{\dagger} \in L^{\infty}(0,1)$ which is assumed to be fixed throughout this section. We then have

$$c_1 \le Fu^{\dagger} \le c_2 \text{ on } [0,1], \text{ with } c_1 := \exp(-\|Gu^{\dagger}\|_{\infty}) > 0, \ c_2 := \exp(\|Gu^{\dagger}\|_{\infty}),$$

so that

$$c_1|Gh| \le |F'(u^{\dagger})h| \le c_2|Gh|$$
 on $[0,1]$ $(h \in L^{\infty}(0,1)).$ (18)

For any $u \in L^{\infty}(0,1)$, we denote by $\Delta = \Delta(u)$ and $\theta = \theta(u)$ the following functions:

$$\Delta := Fu - Fu^{\dagger} \in L^{\infty}(0,1), \qquad \theta := G(u - u^{\dagger}) \in L^{\infty}(0,1).$$

Thus, $\|u - u^{\dagger}\|_{-1} = \|\theta\|_{\infty}$, and we refer to (7) for the definition of $\|\cdot\|_{-1}$.

Below we show that the basic estimates (10) and (11) are satisfied for that example with a = 1. As a preparation, we note that

$$|\Delta - F'(u^{\dagger})(u - u^{\dagger})| \le |\theta| \, |\Delta| \quad \text{on } [0, 1],$$
(19)

and refer in this context to [9, Sect. 4.4]. In this reference, the same F is analyzed as an operator mapping in $L^2(0, 1)$, where moreover its relation to a parameter estimation problem for an initial value problem of a first order ordinary differential equation is outlined.

(a) We first show that (10) holds. Even more general we show that it holds for any $u \in L^{\infty}(0,1)$ sufficiently close to u^{\dagger} , not only for $u \in \mathcal{X}_1$. From (18) we have that

$$|\Delta - F'(u^{\dagger})(u - u^{\dagger})| \ge |\Delta| - |F'(u^{\dagger})(u - u^{\dagger})| \ge |\Delta| - c_2|\theta|$$
 on [0, 1],

and (19) then implies the estimate

$$|\Delta| - c_2|\theta| \le |\theta| |\Delta| \quad \text{on } [0,1].$$

For any $u \in L^{\infty}(0,1)$ satisfying $\|\theta\|_{\infty} \leq \tau < 1$, we thus have $|\Delta| \leq \tau |\Delta| + c_2 |\theta|$ and therefore $(1-\tau)|\Delta| \leq c_2 |\theta|$ on [0,1]. This finally yields

$$\frac{1-\tau}{c_2} \|\Delta\|_{\infty} \le \|\theta\|_{\infty} \quad \text{for } \|\theta\|_{\infty} \le \tau \qquad (0 < \tau < 1),$$

from which the first required nonlinearity condition (10) follows immediately.

(b) We next show that (11) holds, in fact for any $u \in L^{\infty}(0, 1)$ sufficiently close to u^{\dagger} . From (18) we have

$$|\Delta - F'(u^{\dagger})(u - u^{\dagger})| \ge |F'(u^{\dagger})(u - u^{\dagger})| - |\Delta| \ge c_1 |\theta| - |\Delta| \quad \text{on } [0, 1],$$

and (19) then implies that

$$c_1|\theta| \leq |\Delta| + |\theta| \, |\Delta| \quad \text{on } [0,1]$$

For any $0 < \varepsilon < c_1$ and $u \in L^{\infty}(0, 1)$ satisfying $\|\Delta\|_{\infty} \leq c_1 - \varepsilon$, we thus have $c_1|\theta| \leq |\Delta| + (c_1 - \varepsilon)|\theta|$ and therefore $\varepsilon|\theta| \leq |\Delta|$ on [0, 1]. This provides us with the estimate $\varepsilon \|\theta\|_{\infty} \leq \|\Delta\|_{\infty}$, which is valid for $\|\Delta\|_{\infty} \leq c_1 - \varepsilon$ ($0 < \varepsilon < c_1$). This, however, yields directly the second required nonlinearity condition (11) and completes the list of requirements imposed by Assumption 1.

6 Numerical case studies

In this section, we verify the main result in Theorem 10 for the situation of Hölder smoothness. For this, we recall the example considered in Section 5. In particular, for the spaces $\mathcal{X} = \mathcal{Y} = L^{\infty}(0, 1)$, equipped with the essential supremum norm $\|\cdot\| =$ $\|\cdot\|_{\infty}$, we consider the operator equation (1), where the nonlinear forward operator $F : L^{\infty}(0, 1) \rightarrow L^{\infty}(0, 1)$ is given by (17). In this context, the integration operator G, defined as in (16), generates the space

$$\mathcal{X}_1 = \mathcal{R}(G) = W_0^{1,\infty}(0,1) := \{ u \in W^{1,\infty}(0,1) : u(0) = 0 \},\$$

with norm $||u||_1 := ||u'||_{\infty}$. As verified in the previous section, this example satisfies Assumption 1 with a = 1.

In the numerical experiments presented below, we consider the model equation $F(u) = f^{\dagger}$ with $f^{\dagger}(x) = \exp(x^{p+1}/(p+1)), 0 \le x \le 1$, with two different values p from the interval (0, 1). The solution is obviously given by $u^{\dagger}(x) = x^p$ for $0 \le x \le 1$. It satisfies $u^{\dagger} \in \mathcal{X}_p$ which follows from the fact that the fractional powers G^p coincide with Abel integral operators, and thus $[G^p\Gamma(1+p)](x) = x^p, 0 \le x \le 1$, where Γ denotes Euler's gamma function. For details, we refer to [6, p. 9] and [25]. Note that $u^{\dagger} \notin \mathcal{X}_1$, hence we have an oversmoothing penalty term in the Tikhonov functional $T^{\delta}_{\alpha}(u)$ defined as in (3). To find a regularized solution for u^{\dagger} , we set $\overline{u} = 0$ as an initial guess and r = 1 in the minimization problem (3). We use R programming software [27] for the implementation. The interval [0, 1] is partitioned by using equidistant grid points $0 = x_0 < \ldots < x_N = 1$ with N = 100. To approximate the functions u on the given grid, we exploit linear splines that vanish at x = 0. In what follows, we use the notation

$$|u\| := \max_{i=0,\dots,N} |u_i|$$

δ	$lpha_*$	$\ u_{\alpha_*}^\delta-u^\dagger\ $	$\frac{\ u_{\alpha_*}^{\delta}-u^{\dagger}\ }{\delta^{p/(p+1)}}$
0.0500	$7.813 \cdot 10^{-3}$	0.2118	0.4228
0.0250	$7.813 \cdot 10^{-3}$	0.2124	0.4975
0.0125	$1.953 \cdot 10^{-3}$	0.1929	0.5304
0.0062	$2.441\cdot10^{-4}$	0.1496	0.4827
0.0031	$4.883 \cdot 10^{-4}$	0.1559	0.5902
0.0016	$1.221 \cdot 10^{-4}$	0.1217	0.5405
0.0008	$6.104 \cdot 10^{-5}$	0.1157	0.6033
0.0004	$3.052 \cdot 10^{-5}$	0.0920	0.5625
0.0002	$7.629 \cdot 10^{-6}$	0.0919	0.6597

Table 1: Numerical results of Algorithm 8 for p = 0.3.

for the discrete norm. We simulate perturbed observations f_i^{δ} , i = 0, ..., N, as follows:

$$f_i^{\delta} = \begin{cases} f_i^{\dagger} + \delta \frac{\rho_i}{\|\rho\|}, & i = 1, \dots, N, \\ f_0^{\dagger} & i = 0. \end{cases}$$

In this setting, $f_i^{\dagger} = (F(u^{\dagger}))_i = \exp(x_i^{p+1}/(p+1))$, for i = 0, ..., N, denotes the simulated right-hand side of operator equation (1), and the vector $\rho = (\rho_1, ..., \rho_N)^T$ consists of independent and identically distributed standard Gaussian variables ρ_i , for i = 1, ..., N. The discrepancy principle is implemented sequentially as follows (see Remark 4.8 in [14]):

- Choose initial constants $b > e_r$, $\theta > 1$, and $\alpha^{(0)} > 0$.
- If $||F(u_{\alpha^{(0)}}^{\delta}) f^{\delta}|| \ge b\delta$ holds, proceed with $\alpha^{(k)} = \theta^{-k} \alpha^{(0)}$, for k = 1, 2..., until $||F(u_{\alpha^{(k)}}^{\delta}) - f^{\delta}|| \le b\delta \le ||F(u_{\alpha^{(k-1)}}^{\delta}) - f^{\delta}||$ is satisfied for the first time. On that occasion, set $\alpha_* = \alpha^{(k)}$.
- If $||F(u_{\alpha^{(0)}}^{\delta}) f^{\delta}|| \le b\delta$ holds, proceed with $\alpha^{(k)} = \theta^k \alpha^{(0)}$, for k = 1, 2..., until $||F(u_{\alpha^{(k-1)}}^{\delta}) - f^{\delta}|| \le b\delta \le ||F(u_{\alpha^{(k)}}^{\delta}) - f^{\delta}||$ is satisfied for the first time. Set $\alpha_* = \alpha^{(k-1)}$.

Within the minimization steps, we use the command fminunc included in the package pracma. Table 1 and 2 illustrate the results of Algorithm 8 for p = 0.3 and p = 0.7, respectively, and for decreasing values of δ . The initial values are chosen as b = 2, $\theta = 2$, and $\alpha^{(0)} = 1$. Except for the second column of Table 1, all values are rounded to four decimal places. The second columns of the tables present the values of the regularization parameter α_* chosen by the discrepancy principle. The third columns illustrate the corresponding regularization errors. The last columns confirm the statement of Theorem 10.

Figure 1 shows the shapes (solid lines) of minimizers u_{α}^{δ} of the Tikhonov functional in the case p = 0.3 for a fixed noise level $\delta = 0.0125$ and a series of regularization

δ	α_*	$\ u_{\alpha_*}^\delta-u^\dagger\ $	$rac{\ u^{\delta}_{lpha_*}-u^{\dagger}\ }{\delta^{p/(p+1)}}$
0.0500	0.0156	0.0916	0.3146
0.0250	0.0156	0.0869	0.3967
0.0125	0.0156	0.0703	0.4273
0.0062	0.0078	0.0444	0.3586
0.0031	0.0078	0.0532	0.5726
0.0016	0.0078	0.0253	0.3619
0.0008	0.0005	0.0246	0.4682
0.0004	0.0010	0.0171	0.4323
0.0002	0.0001	0.0125	0.4209

Table 2: Numerical results of Algorithm 8 for p = 0.7.

parameters $\alpha > 0$ with decreasing values. The relative error $||F(u^{\dagger}) - f^{\delta}||/||F(u^{\dagger})||$ is given by 0.01. Dotted lines represent in all five pictures the graph of the solution $u^{\dagger}(x) = x^{0.3}, 0 \le x \le 1$, to be reconstructed. In the first picture, for the largest α , the regularized solution is too smooth. By reducing the values of α , the recovery gets improved. Precisely, the third picture yields the best approximate solution, which corresponds with α_* from the discrepancy principle. As a consequence of the ill-posedness of the problem, more and more oscillating solutions occur when α further tends to zero.

7 Constructions and verifications

In this section, we verify the main result of the paper. For this purpose, we return to the general setting considered in Section 2, i.e., $G : \mathcal{X} \to \mathcal{X}$ denotes a bounded linear operator which is of non-negative type, one-to-one and has an unbounded inverse, where \mathcal{X} is a Banach space.

7.1 Introduction of auxiliary elements

For the auxiliary elements introduced below, we consider linear bounded regularization operators associated with G,

$$R_{\beta}: \mathcal{X} \to \mathcal{X} \quad \text{for } \beta > 0$$
 (20)

and its companion operators

$$S_{\beta} := I - R_{\beta}G \quad \text{for } \beta > 0.$$
⁽²¹⁾



Figure 1: Behaviour of minimizing functions u_{α}^{δ} for $\delta = 0.0125$ and decreasing values of α .

Throughout this section, we assume that the following conditions are satisfied:

$$\|R_{\beta}\|_{\mathcal{L}(\mathcal{X})} \le \frac{c_*}{\beta} \quad \text{for } \beta > 0, \tag{22}$$

$$\|R_{\beta}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{c_{*}}{\beta} \quad \text{for } \beta > 0,$$

$$|S_{\beta}G^{p}\|_{\mathcal{L}(\mathcal{X})} \leq c_{p}\beta^{p} \quad \text{for } \beta > 0, \qquad (0 \leq p \leq p_{0})$$

$$(22)$$

$$R_{\beta}G = GR_{\beta} \quad \text{for } \beta > 0, \tag{24}$$

where $0 < p_0 < \infty$ is a finite number to be specified later, and c_* and c_p denote finite constants. We assume that c_p is bounded as a function of p.

Example 11. An example is given by Lavrentiev's *m*-times iterated method with an integer $m \ge 1$. Here, for $f \in \mathcal{X}$ and $v_0 = 0 \in \mathcal{X}$, the element $R_\beta f$ is given by

$$(G + \beta I)v_n = \beta v_{n-1} + f$$
 for $n = 1, 2, ..., m$, $R_\beta f := v_m$.

The operator R_{β} can be written in the form

$$R_{\beta} = \beta^{-1} \sum_{j=1}^{m} \beta^j (G + \beta I)^{-j},$$

and the companion operator is given by $S_{\beta} = \beta^m (G + \beta I)^{-m}$. For m = 1, this gives Lavrentiev's classical regularization method, $R_{\beta} = (G + \beta I)^{-1}$. For this method, the conditions (22)–(24) are satisfied with $p_0 = m$. In fact, for integer $0 \le p \le m$, estimate (23) holds with constant $c_p = (\kappa_* + 1)^m$, see [24, Lemma 1.1.8]. From this intermediate result and the interpolation inequality (6), inequality (23) then follows for non-integer values $0 , with constant <math>c_p = 2(\kappa_* + 1)^{m+1}$. Δ

We are now in a position to introduce auxiliary elements which provide an essential tool for the analysis of the regularization properties of Tikhonov regularization considered in our setting. They are defined as follows,

$$\widehat{u}_{\beta} := \overline{u} + R_{\beta}G(u^{\dagger} - \overline{u}) = u^{\dagger} - S_{\beta}(u^{\dagger} - \overline{u}) \quad \text{for } \beta > 0,$$
(25)

where G is the generator of the scale of normed spaces introduced in Section 2.1, and $R_{\beta}, \beta > 0$, is an arbitrary family of regularizing operators as in (20) satisfying the conditions (22)-(24) with saturation

$$p_0 \ge 1 + a,$$

and $S_{\beta}, \beta > 0$, denotes the corresponding companion operators, cf. (21). In addition, the solution u^{\dagger} of the operator equation (1) and the corresponding initial guess \overline{u} are as introduced above. The basic properties of the auxiliary elements (25) are summarized in Lemma 15 below.

We next state another property of regularization operators which is also needed below.

Lemma 12. There exists some finite constant c > 0 such that, for each 0 , wehave

$$||R_{\beta}G^{p}||_{\mathcal{L}(\mathcal{X})} \leq c\beta^{p-1} \text{ for } \beta > 0.$$

Proof. Since $R_{\beta}G^{p} = G^{p}R_{\beta}$, for $\kappa_{1} = 2(\kappa_{*} + 1)$ we have

$$\|R_{\beta}G^{p}w\| = \|G^{p}R_{\beta}w\| \le \kappa_{1}\|GR_{\beta}w\|^{p}\|R_{\beta}w\|^{1-p}$$
$$\le \kappa_{1}(c_{0}+1)^{p}c_{*}^{1-p}\|w\|\beta^{p-1}, \qquad w \in \mathcal{X},$$

where the first inequality follows from the interpolation inequality (6). For the meaning of the constants c_0 and c_* , we refer to (22) and (23), respectively.

7.2 Auxiliary results for $\log G$

Lemma 13. For each $u \in \mathcal{D}(\log G)$ and each $0 \le p < p_0$, we have

$$||S_{\beta}G^{p}u|| = \mathcal{O}(\beta^{p}(\log \frac{1}{\beta})^{-1}) \quad as \ \beta \to 0.$$

Proof. By $(G^q)_{q\geq 0}$ a C_0 -semigroup on $\overline{\mathcal{R}(G)}$ is defined, and thus $||G^q||_{\mathcal{L}(\overline{\mathcal{R}(G)})} \leq Ce^{\omega q}$ for $q \geq 0$, where $\omega > 0$ and C > 0 denote suitable constants, and $|| \cdot ||_{\mathcal{L}(\overline{\mathcal{R}(G)})}$ denotes the norm of operators on $\overline{\mathcal{R}(G)}$. Therefore, each real $\lambda > \omega$ belongs to the resolvent set of the operator $\log G : \overline{\mathcal{R}(G)} \supset \mathcal{D}(\log G) \rightarrow \overline{\mathcal{R}(G)}$, i.e., $(\lambda I - \log G)^{-1} : \overline{\mathcal{R}(G)} \rightarrow \overline{\mathcal{R}(G)}$ exists and defines a bounded operator, cf. [23, Theorem 5.3, Chapter 1]. Since

$$\mathcal{R}((\lambda I - \log G)^{-1}) = \mathcal{D}(\lambda I - \log G) = \mathcal{D}(\log G),$$

we can represent u as

$$u = (\lambda I - \log G)^{-1} w$$

with some $w \in \overline{\mathcal{R}(G)}$. From (cf. [23, proof of Theorem 5.3, Chapter 1])

$$u = (\lambda I - \log G)^{-1} w = \int_0^\infty e^{-\lambda q} G^q w \, dq,$$

we obtain

$$S_{\beta}G^{p}u = \int_{0}^{\infty} e^{-\lambda q} S_{\beta}G^{p+q}w \, dq = y_1 + y_2,$$

with

$$y_1 = \int_0^{p_0 - p} e^{-\lambda q} S_\beta G^{p + q} w \, dq, \qquad y_2 = \int_{p_0 - p}^\infty e^{-\lambda q} S_\beta G^{p + q} w \, dq.$$

Below we provide suitable estimates for y_1 and y_2 . The former term can be estimated as follows for $\beta < 1$:

$$||y_1|| \le c_1 ||w|| \int_0^{p_0 - p} \beta^{p+q} dq = c_1 ||w|| \beta^p \frac{1}{\log \beta} \beta^q |_{q=0}^{q=p_0 - p}$$
$$= c_1 ||w|| \beta^p \frac{1}{|\log \beta|} (1 - \beta^{p_0 - p}) \le c_1 ||w|| \beta^p \frac{1}{|\log \beta|},$$

where c_1 denotes a finite constant. The element y_2 can be written as follows:

$$y_2 = \int_{p_0-p}^{\infty} e^{-\lambda q} S_\beta G^{p_0} G^{q-(p_0-p)} w \, dq,$$

and thus we can estimate as follows:

$$\|y_2\| \le c_2 \|w\| \int_{p_0-p}^{\infty} e^{-\lambda q} \beta^{p_0} e^{\omega(q-(p_0-p))} dq$$

$$\le c_3 \|w\| e^{-\omega(p_0-p)} \beta^{p_0} \int_{p_0-p}^{\infty} e^{-(\lambda-\omega)q} dq = \mathcal{O}(\beta^{p_0}) \text{ as } \beta \to 0,$$

where c_2 and c_3 denote suitable finite constants. This completes the proof.

Lemma 14. For each $u \in \mathcal{D}(\log G)$, we have

$$||R_{\beta}u|| = \mathcal{O}(\frac{1}{\beta \log \frac{1}{\beta}}) \quad as \ \beta \to 0.$$

Proof. Follows similar to Lemma 13, by making use of Lemma 12. Details are thus omitted. \Box

7.3 Some preparations for low order rates

In the analysis of low order rates, the functions

$$\varphi(t) = (\log \frac{1}{t})^{-1}, \quad 0 < t < 1, \tag{26}$$

$$\chi_{\pm 1,q}(t) = t^q (\log \frac{1}{t})^{\mp 1}, \quad 0 < t < 1 \qquad (q > 0), \tag{27}$$

will be needed. Below we state some elementary properties of those functions. Note that $\varphi(t) = \chi_{1,0}(t)$ holds, so (26) is a special case of (27).

- (a) For $q \ge 0$, the function $\chi_{1,q}$ is monotonically increasing on the interval (0,1), with $\chi_{1,q}(t) \to 0$ as $t \to 0$.
- (b) For q > 0, the function $\chi_{-1,q}$ is monotonically increasing on the interval $(0, t_0]$, with $t_0 = t_0(q) < 1$ chosen sufficiently small. We have $\chi_{-1,q}(t) \to 0$ as $t \to 0$.
- (c) For q > 0, the inverse function $\chi_{1,q}^{-1} : (0,1) \to \mathbb{R}$ satisfies

$$\chi_{1,q}^{-1}(s) \sim q^{-1/q} \chi_{-1,1}(s)^{1/q}$$
 as $s \to 0$.

This in particular implies that, for each $0 < s_0 < 1$, we have

$$\chi_{1,q}^{-1}(s) \asymp \chi_{-1,1}(s)^{1/q}, \quad 0 < s \le s_0.$$

(d) For each q > 0, we have $\varphi(\chi_{\pm 1,q}(t)) \sim q\varphi(t)$ as $t \to 0$. Thus, in particular for each fixed t_1 small enough and each e > 0, we have

$$\varphi(\chi_{\pm 1,q}(t)^e) \le c_1 \varphi(t), \quad 0 < t \le t_1,$$

for some appropriate constant c_1 .

(e) For each constant $c_2 > 0$, we have $\varphi(c_2 t) \sim \varphi(t)$ as $t \to 0$, and thus in particular

$$\varphi(c_2 t) \asymp \varphi(t), \quad 0 < t \le t_2,$$

for $t_2 < 1/c_2$ fixed.

Here, for two positive, real-valued functions $f, g: (0, t_0) \to \mathbb{R}$, the notation $f(t) \sim g(t)$ as $t \to 0$ means $f(t)/g(t) \to 1$ as $t \to 0$. In addition, $f(t) \asymp g(t)$ for $t \in I \subset (0, t_0)$ means that there are finite positive constants c_1, c_2 such that $c_1f(t) \le g(t) \le c_2f(t)$ for $t \in I$.

7.4 Properties of auxiliary elements

In this section, we present the basic properties of the auxiliary elements, which may be used to verify our convergence results presented below.

Lemma 15. Consider the auxiliary elements from (25), generated by regularization operators $R_{\beta}, \beta > 0$, with saturation $p_0 \ge 1 + a$. Let the three functions $g_i(\beta)$ (i = 1, 2, 3) be given by the following identities:

$$\|\widehat{u}_{\beta} - u^{\dagger}\| = g_1(\beta), \tag{28}$$

$$\|\widehat{u}_{\beta} - u^{\dagger}\|_{-a} = g_2(\beta)\beta^a, \qquad (29)$$

$$\|\widehat{u}_{\beta} - \overline{u}\|_{1} = g_{3}(\beta)\beta^{-1},$$
(30)

for $\beta > 0$, respectively. Those functions $g_i(\beta)$ (i = 1, 2, 3) are bounded and have the following properties:

- (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then we have $g_i(\beta) \to 0$ as $\beta \to 0$ (i = 1, 2, 3).
- (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then <math>g_i(\beta) = \mathcal{O}(\beta^p)$ as $\beta \to 0$ (i = 1, 2, 3).
- (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then $g_i(\beta) = \mathcal{O}((\log \frac{1}{\beta})^{-1})$ as $\beta \to 0$ (i = 1, 2, 3).

Proof. By definition, those three functions g_1, g_2 and g_3 under consideration can be written as follows:

$$g_1(\beta) = \|S_\beta(u^{\dagger} - \overline{u})\|,$$

$$g_2(\beta) = \beta^{-a} \|G^a S_\beta(u^{\dagger} - \overline{u})\|,$$

$$g_3(\beta) = \beta \|R_\beta(u^{\dagger} - \overline{u})\|,$$

and, according to conditions (22)–(24), thus are bounded.

• The three convergence statements under any missing smoothness assumptions all are verified by making use of the uniform boundedness principle. We give some details for the function g_1 . In fact, (23) applied for p = 1 gives $S_\beta z \to 0$ as $\beta \to 0$ for all z from the range $\mathcal{R}(G)$. The uniform boundedness $||S_\beta||_{\mathcal{L}(\mathcal{X})} \leq c_0$, cf. (23) for p = 0, and the denseness of $\mathcal{R}(G)$ in $\overline{\mathcal{R}(G)}$ then gives $g_1(\beta) = ||S_\beta(u^{\dagger} - \overline{u})|| \to 0$ as $\beta \to 0$. The assertions for g_2 and g_3 follow similarly.

• We consider Hölder smoothness next. Since $u^{\dagger}, \overline{u} \in \mathcal{X}_p$ holds, we have $u^{\dagger} - \overline{u} = G^p w$ for some $w \in \mathcal{X}$. The statements are now easily obtained from (23) and Lemma 12.

• We have $u^{\dagger} - \overline{u} \in \mathcal{D}(\log G)$, and the statements now easily follow from Lemmas 13 and 14.

The preceding lemma allows the construction of smooth approximations in \mathcal{X}_1 to u^{\dagger} , which may be used in the subsequent proofs.

Lemma 16. Under the conditions of Lemma 15, the following holds:

• (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then for some parameter choice $\beta = \beta_{\delta}$ we have

$$\|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\| \to 0, \quad \|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\|_{-a} = \mathcal{O}(\delta), \quad \|\widehat{u}_{\beta_{\delta}} - \overline{u}\|_{1} = o(\delta^{-1/a}), \quad (31)$$

as $\delta \to 0$.

• (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then for some parameter choice <math>\beta = \beta_{\delta}$ we have

$$\|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\| = \mathcal{O}(\delta^{\frac{1-p}{p+a}}), \quad \|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\|_{-a} = \mathcal{O}(\delta), \\\|\widehat{u}_{\beta_{\delta}} - \overline{u}\|_{1} = \mathcal{O}(\delta^{-\frac{1-p}{p+a}}) \quad as \ \delta \to 0.$$
(32)

• (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then for some parameter choice $\beta = \beta_{\delta}$ we have

$$\begin{aligned} \|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\| &= \mathcal{O}((\log \frac{1}{\delta})^{-1}), \quad \|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\|_{-a} = \mathcal{O}(\delta), \\ \|\widehat{u}_{\beta_{\delta}} - \overline{u}\|_{1} &= \mathcal{O}(\delta^{-\frac{1}{a}}(\log \frac{1}{\delta})^{-(1+\frac{1}{a})}) \quad as \ \delta \to 0. \end{aligned}$$
(33)

For each of the three cases, the parameter choice $\beta = \beta_{\delta}$ is specified in the proof.

Proof. We consider the following choices of β_{δ} :

- In case of no explicit smoothness, one may choose $\beta_{\delta} = c \delta^{1/a}$.
- In case of Hölder smoothness, one can choose $\beta_{\delta} = c \delta^{\frac{1}{p+a}}$.
- In case of low order smoothness, we consider $\beta_{\delta} = c(\delta \log \frac{1}{\delta})^{1/a}$ for $0 < \delta < \delta_0$, with δ_0 sufficiently small.

Here, c > 0 denotes an arbitrary constant factor. The first two statements follow as an easy consequence of Lemma 15. The statement on the low order case is also a consequence of Lemma 15. In this case, however, below we consider some details. For this purpose, we will make use of the notation $\varphi(t) = (\log \frac{1}{t})^{-1}$ introduced in Section 7.3. We first note that for some constant $c_1 > 0$, we have

$$\varphi(\beta_{\delta}) \le c_1 \varphi(\delta), \quad 0 < \delta \le \delta_0, \tag{34}$$

which in fact follows easily from the two estimates given in items (d) and (e) introduced in Section 7.3. The three given estimates for the low order case are now consequences of Lemma 15 and estimate (34). For $\|\hat{u}_{\beta\delta} - u^{\dagger}\|$ this is immediate, and in addition, we also obtain the following:

$$\begin{aligned} \|\widehat{u}_{\beta\delta} - u^{\dagger}\|_{-a} &\leq c_2\varphi(\beta\delta)\beta_{\delta}^a \leq c_3\varphi(\delta)(\varphi(\delta)^{-1}\delta)^{a/a} = c_3\delta, \\ \|\widehat{u}_{\beta\delta} - \overline{u}\|_1 \leq c_4\varphi(\beta\delta)\beta_{\delta}^{-1} \leq c_5\varphi(\delta)(\varphi(\delta)\delta^{-1})^{1/a} = c_5\varphi(\delta)^{1+\frac{1}{a}}\delta^{-1/a}. \end{aligned}$$

where c_2, \ldots, c_5 denote appropriately chosen constants. This completes the proof of the lemma.

7.5 **Proof of Theorem 10**

This section is devoted to the proof of our main result, Theorem 10. As a basic ingredient, we need to provide reasonable estimates of the two terms $||u_{\alpha_*}^{\delta} - u^{\dagger}||_{-a}$ and $||u_{\alpha_*}^{\delta} - \overline{u}||_1$. We start with the estimation of the former one.

Lemma 17. Let Assumption 1 be satisfied. We then have

$$\|u_{\alpha_{n}}^{\delta} - u^{\dagger}\|_{-a} = \mathcal{O}(\delta) \quad as \ \delta \to 0$$

Proof. From the choice of α_* and estimate (11), it follows that

$$c_{a} \| u_{\alpha_{*}}^{\delta} - u^{\dagger} \|_{-a} \le \| F(u_{\alpha_{*}}^{\delta}) - f^{\dagger} \| \le \| F(u_{\alpha_{*}}^{\delta}) - f^{\delta} \| + \delta \le (b+1)\delta$$
(35)

for $\delta > 0$ small enough. Note that the upper bound presented at the end of (35) guarantees that estimate (11) is applicable with $u = u_{\alpha_*}^{\delta}$ for δ small enough. This concludes the proof.

Our next goal is to provide appropriate estimates for $||u_{\alpha_*}^{\delta} - \overline{u}||_1$. This requires some preparations. For this purpose, we recall the definition from (12), this is $\kappa = \frac{1}{r(a+1)}$.

Lemma 18. Let Assumption 1 be satisfied. There exists some $\alpha_0 > 0$ such that for $0 < \alpha \le \alpha_0$ and each $\delta > 0$, we have

$$\max\{\|F(u_{\alpha}^{\delta}) - f^{\delta}\|, \, \alpha^{1/r} \|u_{\alpha}^{\delta} - \overline{u}\|_{1}\} \le \psi(\alpha)\alpha^{\kappa a} + e_{r}\delta,$$

where the constant e_r is given by (14). In addition, $\psi(\alpha)$ is a bounded function which satisfies the following:

- (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then $\psi(\alpha) \to 0$ as $\alpha \to 0$.
- (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then <math>\psi(\alpha) = \mathcal{O}(\alpha^{\kappa p})$ as $\alpha \to 0$.
- (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then $\psi(\alpha) = \mathcal{O}((\log \frac{1}{\alpha})^{-1})$ as $\alpha \to 0$.

Proof. Let $u^{\dagger} \in \overline{\mathcal{R}(G)}$. For auxiliary elements of the form (25), with saturation $p_0 \ge 1 + a$, we choose

$$\beta = \beta(\alpha) = \alpha^{\kappa}.$$
(36)

For $\alpha > 0$ small enough, say $0 < \alpha \le \alpha_0$, we have $\hat{u}_\beta \in \mathcal{D}$ because of Lemma 15 and since moreover u^{\dagger} is assumed to be an interior point of $\mathcal{D}(F)$. We thus have

$$(\|F(u_{\alpha}^{\delta}) - f^{\delta}\|^{r} + \alpha \|u_{\alpha}^{\delta} - \overline{u}\|_{1}^{r})^{1/r} \leq (\|F(\widehat{u}_{\beta}) - f^{\delta}\|^{r} + \alpha \|\widehat{u}_{\beta} - \overline{u}\|_{1}^{r})^{1/r}$$

$$\leq e_{r}(\|F(\widehat{u}_{\beta}) - f^{\delta}\| + \alpha^{1/r} \|\widehat{u}_{\beta} - \overline{u}\|_{1})$$

$$\leq e_{r}(\|F(\widehat{u}_{\beta}) - f^{\dagger}\| + \alpha^{1/r} \|\widehat{u}_{\beta} - \overline{u}\|_{1} + \delta).$$
(37)

The first term on the right-hand side of the latter estimate can be written as

$$\|F(\widehat{u}_{\beta}) - f^{\dagger}\| \le C_a \|\widehat{u}_{\beta} - u^{\dagger}\|_{-a} = C_a g_2(\beta) \beta^a = C_a g_2(\alpha^{\kappa}) \alpha^{\kappa a}.$$
 (38)

The estimate in (38) is a consequence of estimate (10), which is applicable with $u = \hat{u}_{\beta}$ for α small enough, and without loss of generality we may assume that small enough means $\alpha \leq \alpha_0$ by choosing α_0 sufficiently small in the beginning. The first identity in (38) follows from representation (29) in Lemma 15.

The second term on the right-hand side of the estimate (37) can be represented as follows:

$$\alpha^{1/r} \| \widehat{u}_{\beta} - \overline{u} \|_1 = \alpha^{1/r} g_3(\beta) \beta^{-1} = g_3(\alpha^{\kappa}) \alpha^{\kappa a},$$

based on (30) of Lemma 15. As a consequence, the estimate of Lemma 18 holds, if the function ψ is chosen as

$$\psi(\alpha) := e_r(C_a g_2(\alpha^{\kappa}) + g_3(\alpha^{\kappa})) \quad \text{for } \alpha \le \alpha_0 \,.$$

The asymptotic behavior of the function ψ stated in the lemma is an immediate consequence of Lemma 15. This completes the proof of the lemma.

As a consequence of the preceding lemma, we can derive reasonable lower bounds for the regularizing parameter α_* obtained by the discrepancy principle, which actually affects the stability of the method.

Corollary 19. Let Assumption 1 be satisfied. Let the parameter $\alpha = \alpha_*$ be chosen according to the discrepancy principle.

- (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then $\alpha_*^{-\kappa a} = o(\delta^{-1})$ as $\delta \to 0$.
- (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then <math>\alpha_*^{-\kappa(p+a)} = \mathcal{O}(\delta^{-1})$ as $\delta \to 0$.
- (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then $\alpha_*^{-\kappa a} = \mathcal{O}(\delta^{-1}(\log \frac{1}{\delta})^{-1})$ as $\delta \to 0$.

Proof. We first note that parameters α_* which stay away from zero can easily be treated in each of the three cases. Note also that this is a related to a degenerated case, and it includes the case $\alpha_* = \infty$.

In the following, we thus may assume that $\alpha_* \leq \alpha_0/c$ and thus $\gamma_{\delta} \leq \alpha_0$ hold, where α_0 is given by Lemma 18, and the constant c and the parameter γ_{δ} are introduced by the discrepancy principle (15). Lemma 18 then implies $b\delta \leq ||F(u_{\gamma_{\delta}}^{\delta}) - f^{\delta}|| \leq \psi(\gamma_{\delta})\gamma_{\delta}^{\kappa a} + e_r\delta$ and thus

$$(b - e_r)\delta \le \psi(\gamma_\delta)\gamma_\delta^{\kappa a}.$$
(39)

The statements of the corollary for the two cases "no explicit smoothness" and "Hölder smoothness" now easily follow from the properties on the function ψ presented in Lemma 18, respectively. Low order smoothness is considered next. In this case, without loss of generality we may assume that $\alpha_0 < 1$. Estimate (39) then means

$$c_1\delta \leq (\log \frac{1}{\gamma_\delta})^{-1}\gamma_\delta^{\kappa a} = \chi_{1,\kappa a}(\gamma_\delta),$$

where $c_1 > 0$ denotes a constant, and the notation from Section 7.3 is used again. From item (a) of that section, we now easily obtain $\chi_{1,\kappa a}^{-1}(c_1\delta) \leq \gamma_{\delta}$, with $\delta > 0$ small enough. This provides the basis for the following estimates, which also utilize items (c) and (e) from Section 7.3:

$$\begin{aligned} &c\alpha_* \ge \gamma_\delta \ge \chi_{1,\kappa a}^{-1} (c_1 \delta) \ge c_2 \chi_{-1,1} (c_1 \delta)^{1/(\kappa a)} \\ &= c_3 (\delta \log \frac{1}{c_1 \delta})^{1/(\kappa a)} \ge c_4 (\delta \log \frac{1}{\delta})^{1/(\kappa a)}, \end{aligned}$$

where c_2, c_3 and c_4 denote appropriately chosen finite constants, and δ is again sufficiently small. A simple rearrangement yields the statement on low order smoothness.

Below, we present suitable estimates for $||u_{\alpha_*}^{\delta} - \overline{u}||_1$.

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Corollary 20. Let Assumption 1 be satisfied. Let the parameter $\alpha = \alpha_*$ be chosen according to the discrepancy principle. Then the following holds:

- (No explicit smoothness) If $u^{\dagger} \in \overline{\mathcal{R}(G)}$, then $\|u_{\alpha_*}^{\delta} \overline{u}\|_1 = o(\delta^{-1/a})$ as $\delta \to 0$.
- (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some $0 , then <math>\|u_{\alpha_*}^{\delta} \overline{u}\|_1 = \mathcal{O}(\delta^{-\frac{1-p}{p+a}})$ as $\delta \to 0$.
- (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then $\|u_{\alpha_*}^{\delta} \overline{u}\|_1 = \mathcal{O}(\delta^{-\frac{1}{a}}(\log \frac{1}{\delta})^{-(1+\frac{1}{a})})$ as $\delta \to 0$.

Proof. For parameters α_* staying away from the origin, say $\alpha_* \ge \alpha_1 > 0$, the statements of the corollary follow immediately, since $||u_{\alpha_*}^{\delta} - \overline{u}||_1$ stays bounded then, as can be seen from the following computations:

$$\begin{aligned} \alpha_1^{1/r} \| u_{\alpha_*}^{\delta} - \overline{u} \|_1 &\leq \alpha_*^{1/r} \| u_{\alpha_*}^{\delta} - \overline{u} \|_1 \leq T_{\alpha}^{\delta} (u_{\alpha_*}^{\delta})^{1/r} \leq T_{\alpha}^{\delta} (\overline{u})^{1/r} \\ &= \| F(\overline{u}) - f^{\delta} \| \leq \| F(\overline{u}) - f^{\dagger} \| + \delta. \end{aligned}$$

Therefore, in the following we may assume $\alpha_* \leq \alpha_0$, where α_0 is given by Lemma 18. The same lemma then implies $\alpha_*^{1/r} \| u_{\alpha_*}^{\delta} - \overline{u} \|_1 \leq \psi(\alpha_*) \alpha_*^{\kappa a} + e_r \delta$ and thus

$$\|u_{\alpha_*}^{\delta} - \overline{u}\|_1 \le \psi(\alpha_*) \, \alpha_*^{-\kappa} + e_r \frac{\delta}{\alpha_*^{1/r}},\tag{40}$$

where we make use of the identity $\kappa a - \frac{1}{r} = -\kappa$. The statements of the corollary now follow by considering the two terms on the right-hand side of (40) separately, respectively. For the two cases "no explicit smoothness" and "Hölder smoothness", this follows from the corresponding estimates from Lemma 18 and Corollary 19. The proof is straightforward, and details thus are omitted here.

Below we present some details for the low order smoothness case. In this case, the estimate of the function ψ given by Lemma 18 yields

$$\|u_{\alpha_{*}}^{\delta} - \overline{u}\|_{1} \le c_{1} (\log \frac{1}{\alpha_{*}})^{-1} \alpha_{*}^{-\kappa} + e_{r} \frac{\delta}{\alpha_{*}^{1/r}}, \tag{41}$$

for some suitable finite constant c_1 . We now can proceed by utilizing the lower estimate of α_* given by Corollary 19, i. e.,

$$c_2(\delta \log \frac{1}{\delta})^{1/(\kappa a)} \le \alpha_* \quad \text{for } 0 < \delta \le \delta_1, \tag{42}$$

with some constant c_2 , and δ_1 is chosen small enough. From (42), the second term on the right-hand side of (41) can be suitable estimated in a straightforward manner, and we omit the details. We next consider the first term on the right-hand side of (41). Without loss of generality, in the following we may assume that $\alpha_0 < 1$ considered in the beginning of the proof is chosen so small such that the function $(\chi_{-1,\kappa}(\alpha))^{-1} = (\log \frac{1}{\alpha})^{-1} \alpha^{-\kappa}$ is monotonically decreasing for $0 < \alpha \le \alpha_0$, cf. item (b) in Section 7.3. From (41), (42) and items (b) in Section 7.3, we then obtain

$$\left(\log \frac{1}{\alpha_*}\right)^{-1} \alpha_*^{-\kappa} \le c_3 \left(\delta \log \frac{1}{\delta}\right)^{-1/a} \sigma, \quad \sigma := \left(-\log(c_2(\delta \log \delta)^{1/(\kappa a)})\right)^{-1},$$

for some constant c_3 . From items (d) and (e) in Section 7.3, it follows that

$$\sigma = \varphi(c_2 \chi_{-1,1}(\delta)^{1/(\kappa a)}) \le c_4 \varphi(\delta) = c_4 (\log \frac{1}{\delta})^{-1},$$

for some constant c_4 . The statement in the third item of the corollary now follows. \Box

We are now in a position to present a proof of the main result of this paper.

Proof of Theorem 10. We start with an elementary error estimate $||u_{\alpha_*}^{\delta} - u^{\dagger}||$ utilizing the auxiliaries,

$$\|u_{\alpha_*}^{\delta} - u^{\dagger}\| \le \|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\| + \|\widehat{u}_{\beta_{\delta}} - u^{\dagger}\|,$$
(43)

where β_{δ} is given by Lemma 16. The error of the auxiliaries on the right-hand side of (43) can be properly estimated using Lemma 16. Below we consider the term $|| u_{\alpha}^{\delta} - \hat{u}_{\beta} ||$ in more detail. From the interpolation inequality (6), it follows

$$\| u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}} \| \le c_1 \| u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}} \|_{-a}^{1/(a+1)} \| u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}} \|_1^{a/(a+1)},$$
(44)

where c_1 denotes some finite constant not depending on δ . The first term on the righthand side of estimate (44) can be estimated by using Lemmas 16 and 17. Precisely, we find the estimates

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta\delta}\|_{-a} \le \|u_{\alpha_*}^{\delta} - u^{\dagger}\|_{-a} + \|\widehat{u}_{\beta\delta} - u^{\dagger}\|_{-a} = \mathcal{O}(\delta) \quad \text{as} \ \delta \to 0,$$

so that estimate (44) simplifies to

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\| \le c_2 \delta^{1/(a+1)} \|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\|_1^{a/(a+1)},$$
(45)

where c_2 denotes some finite constant independent of δ . The last factor on the righthand side of (45) is estimated next, and for this purpose, we make use of the following elementary estimate,

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta\delta}\|_1 \le \|u_{\alpha_*}^{\delta} - \overline{u}\|_1 + \|\widehat{u}_{\beta\delta} - \overline{u}\|_1.$$

$$(46)$$

We now proceed with the estimation of the right-hand side of (46) by distinguishing our different smoothness assumptions.

(a) For $u^{\dagger} \in \overline{\mathcal{R}(G)}$ (no explicit smoothness), from estimate (46), Lemma 16 and Corollary 20 we obtain

$$\| u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}} \|_1 \le o(\delta^{-1/a}) + o(\delta^{-1/a}) = o(\delta^{-1/a}),$$

and estimate (45) then gives

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\| \leq c_2 \delta^{\frac{1}{a+1}} o(\delta^{-\frac{1}{a+1}}) \to 0 \quad \text{as} \ \delta \to 0.$$

This result in combination with estimate (43) and Lemma 16 yields $||u_{\alpha_*}^{\delta} - u^{\dagger}|| \to 0$ as $\delta \to 0$. This is the first statement of Theorem 10.

(b) (Hölder smoothness) If $u^{\dagger} \in \mathcal{X}_p$ for some 0 , then from estimate (46), Lemma 16 and Corollary 20 we obtain

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\|_1 \le \mathcal{O}(\delta^{-\frac{1-p}{p+a}}) + \mathcal{O}(\delta^{-\frac{1-p}{p+a}}) = \mathcal{O}(\delta^{-\frac{1-p}{p+a}}),$$

and estimate (45) then gives

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\| \le c_2 \delta^{\frac{1}{a+1}} \mathcal{O}(\delta^{-\frac{1-p}{p+a}}) = \mathcal{O}(\delta^{\frac{p}{p+a}}) \quad \text{as } \delta \to 0.$$

This estimate combined with estimate (43) and Lemma 16 yields $||u_{\alpha_*}^{\delta} - u^{\dagger}|| = O(\delta^{\frac{p}{p+\alpha}})$ as $\delta \to 0$. This is the second statement of Theorem 10.

(c) (Low order smoothness) If $u^{\dagger} \in \mathcal{D}(\log G)$, then from estimate (46), Lemma 16 and Corollary 20 we obtain

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\|_1 = \mathcal{O}(\delta^{-\frac{1}{a}}(\log \frac{1}{\delta})^{-(1+\frac{1}{a})}),$$

and estimate (45) then gives

$$\|u_{\alpha_*}^{\delta} - \widehat{u}_{\beta_{\delta}}\| \le c_2 \delta^{\frac{1}{a+1}} \mathcal{O}(\delta^{-\frac{1}{a+1}} (\log \frac{1}{\delta})^{-1}) = \mathcal{O}((\log \frac{1}{\delta})^{-1}) \quad \text{as } \delta \to 0.$$

This estimate in combination with (43) and Lemma 16 yields $||u_{\alpha_*}^{\delta} - u^{\dagger}|| = \mathcal{O}((\log \frac{1}{\delta})^{-1})$ as $\delta \to 0$. This is the third and final statement of Theorem 10.

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