Variational method for reconstructing the source in elliptic systems from boundary observations

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Abstract: In this paper we investigate the problem of identifying the source term f in the elliptic system

$$-\nabla \cdot \big(Q\nabla\Phi\big) = f \text{ in } \Omega \subset R^d, 1 \le d \le 3,\ Q\nabla\Phi \cdot \vec{n} = j \text{ on } \partial\Omega \text{ and } \Phi = g \text{ on } \partial\Omega$$

from a single noisy measurement couple (j_δ, g_δ) of the Neumann and Dirichlet data (j, g) with noise level $\delta > 0$. In this context, the diffusion matrix Q is given. A variational method of Tikhonov-type regularization with specific misfit term and quadratic stabilizing penalty term is suggested to tackle this inverse problem. The method proves to be a modified variant of the Lavrentiev regularization with implicit forward operator. Using the variational discretization concept, where the PDE is discretized with piecewise linear, continuous finite elements, we show the convergence of regularized finite element approximations. Moreover, we derive an error bound and corresponding convergence rates for discrete regularized solutions under a suitable source condition, which typically occurs in the theory of Lavrentiev regularization. For the numerical solution we propose a conjugate gradient method. To illustrate the theoretical results, a numerical case study is presented which supports our analytical findings.

Key words and phrases: Inverse source problem, Tikhonov regularization, Lavrentiev regularization, finite element method, source condition, convergence rates, ill-posedness, conjugate gradient method, Neumann problem, Dirichlet problem.

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1 Introduction

Let Ω be an open bounded connected domain of $R^d, 1 \le d \le 3$ with boundary $\partial\Omega$. We consider the elliptic system

$$-\nabla \cdot \big(Q\nabla\Phi\big) = f \text{ in } \Omega, \tag{1.1}$$

$$Q\nabla\Phi \cdot \vec{n} = j^\dagger \text{ on } \partial\Omega \text{ and} \tag{1.2}$$

$$\Phi = g^\dagger \text{ on } \partial\Omega, \tag{1.3}$$

where $\vec{n}$ is the unit outward normal on $\partial\Omega$ and the diffusion matrix Q is given. Furthermore, we assume that $Q := (q_{rs})_{1\le r,s\le d} \in L^\infty(\Omega)^{d\times d}$ is symmetric and satisfies the uniformly ellipticity condition

$$Q(x)\xi \cdot \xi = \sum_{1\le r,s\le d} q_{rs}(x)\xi_r\xi_s \ge \underline{q}|\xi|^2 \text{ a.e. in } \Omega \tag{1.4}$$

for all $\xi = (\xi_r)_{1\le r\le d} \in R^d$ with some constant $\underline{q} > 0$.

The system (1.1)–(1.3) is overdetermined, i.e., if the Neumann and Dirichlet boundary conditions $j^\dagger \in H^{-1/2}(\partial\Omega) := H^{1/2}(\partial\Omega)^*$, $g^\dagger \in H^{1/2}(\partial\Omega)$, and the source term $f \in L^2(\Omega)$ are given, then there may be no Φ satisfying this system. In this paper we assume that the system is consistent and our aim is to reconstruct a function $f \in L^2(\Omega)$ and a function $\Phi \in H^1(\Omega)$ in the system (1.1)–(1.3) from a noisy measurement couple $(j_\delta, g_\delta) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ of the exact Neumann and Dirichlet data $\big(j^\dagger, g^\dagger\big)$, where $\delta > 0$ stands for the measurement error, i.e., we assume the noise model

$$\big\|j_\delta - j^\dagger\big\|_{H^{-1/2}(\partial\Omega)} + \big\|g_\delta - g^\dagger\big\|_{H^{1/2}(\partial\Omega)} \le \delta. \tag{1.5}$$

To formulate precisely the problem, we first give some notations. Let us denote by $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ the continuous Dirichlet trace operator with $\gamma^{-1} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ its continuous right inverse operator, i.e., $(\gamma \circ \gamma^{-1})g = g$ for all $g \in H^{1/2}(\partial\Omega)$. We set

$$H^1_\diamond(\Omega) := \left\{ u \in H^1(\Omega) \;\Big|\; \int_{\partial\Omega} \gamma u dx = 0 \right\} \text{ and } H^{1/2}_\diamond(\partial\Omega) := \left\{ g \in H^{1/2}(\partial\Omega) \;\Big|\; \int_{\partial\Omega} g(x)dx = 0 \right\}$$

and denote by C_Ω the positive constant appearing in the Poincaré-Friedrichs inequality (cf. [35])

$$C_\Omega \int_\Omega \varphi^2 \le \int_\Omega |\nabla\varphi|^2 \text{ for all } \varphi \in H^1_\diamond(\Omega). \tag{1.6}$$

Since $H^1_0(\Omega) := \left\{ u \in H^1(\Omega) \mid \gamma u = 0 \right\} \subset H^1_\diamond(\Omega)$, the inequality (1.6) is still valid for all $\varphi \in H^1_0(\Omega)$. Furthermore, by (1.4), the coercivity condition

$$\|\varphi\|^2_{H^1(\Omega)} \le \frac{1 + C_\Omega}{C_\Omega} \int_\Omega |\nabla\varphi|^2 \le \frac{1 + C_\Omega}{C_\Omega \underline{q}} \int_\Omega Q\nabla\varphi \cdot \nabla\varphi \tag{1.7}$$

holds for all $\varphi \in H^1_\diamond(\Omega)$.

Now, for any fixed $(j, g) \in H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ we can simultaneously consider the Neumann problem

$$-\nabla \cdot (Q\nabla u) = f \text{ in } \Omega \text{ and } Q\nabla u \cdot \vec{n} = j \text{ on } \partial\Omega \tag{1.8}$$

as well as the Dirichlet problem

$$-\nabla \cdot (Q\nabla v) = f \text{ in } \Omega \text{ and } v = g \text{ on } \partial\Omega. \tag{1.9}$$

By the aid of (1.7) and the Riesz representation theorem, we conclude that for each $f \in L^2(\Omega)$ there exists a unique weak solution u of the problem (1.8) in the sense that $u \in H^1_\diamond(\Omega)$ and satisfies the identity

$$\int_\Omega Q\nabla u \cdot \nabla\varphi = \langle j, \gamma\varphi \rangle + (f, \varphi) \tag{1.10}$$

for all $\varphi \in H^1_\diamond(\Omega)$, where notation $\langle j, g \rangle$ stands for the value of the function $j \in H^{-1/2}(\partial\Omega)$ at $g \in H^{1/2}(\partial\Omega)$ and the notation (f, φ) is the scalar inner product of f and φ on the space $L^2(\Omega)$. Then we can define the *Neumann operator*

$$\mathcal{N} : L^2(\Omega) \to H^1_\diamond(\Omega) \text{ with } f \mapsto \mathcal{N}_f j,$$

which maps each $f \in L^2(\Omega)$ to the unique weak solution $\mathcal{N}_f j := u$ of the problem (1.8). Similarly, the problem (1.9) also attains a unique weak solution v in the sense that $v \in H^1(\Omega)$, $\gamma v = g$ and the identity

$$\int_\Omega Q\nabla v \cdot \nabla\psi = (f, \psi) \tag{1.11}$$

holds for all $\psi \in H^1_0(\Omega)$. The *Dirichlet operator* is defined as

$$\mathcal{D} : L^2(\Omega) \to H^1_\diamond(\Omega) \text{ with } f \mapsto \mathcal{D}_f g,$$

which maps each $f \in L^2(\Omega)$ to the unique weak solution $\mathcal{D}_f g := v$ of the problem (1.9). Therefore, for any fixed $f \in L^2(\Omega)$ we can define the so-called *Neumann-to-Dirichlet map*

$$\begin{aligned} \Lambda_f : H^{-1/2}(\partial\Omega) &\to H^{1/2}_\diamond(\partial\Omega) \\ j &\mapsto \Lambda_f j := \gamma\mathcal{N}_f j. \end{aligned}$$

We mention that since $H^1_0(\Omega) \subset H^1_\diamond(\Omega)$, we from (1.10) have that $\int_\Omega Q\nabla\mathcal{N}_f j \cdot \nabla\psi = (f, \psi)$ for all $\psi \in H^1_0(\Omega)$. In view of (1.11) we therefore conclude

$$\Lambda_f j = g \text{ if and only if } \mathcal{N}_f j = \mathcal{D}_f g,$$

where the identities

$$\mathcal{N}_f j = \mathcal{N}_f 0 + \mathcal{N}_0 j \quad \text{and} \quad \mathcal{D}_f g = \mathcal{D}_f 0 + \mathcal{D}_0 g \tag{1.12}$$

are satisfied, and the operators $f \mapsto \mathcal{N}_f 0$ and $f \mapsto \mathcal{D}_f 0$ are linear and bounded from $L^2(\Omega)$ into itself. Furthermore,

$$\Lambda_f j = \gamma \mathcal{N}_f j = \gamma \mathcal{N}_0 j + \gamma \mathcal{N}_f 0 = \Lambda_0 j + \Lambda_f 0,$$

where $\Lambda_0 j$ is linear, self-adjoint, bounded and invertible, as the diffusion Q is smooth enough (cf. [34]).

As in Electrical Impedance Tomography or Calderón's problem [4, 15, 34] one can pose the question whether the source distribution f inside a physical domain Ω can be determined from an *infinite* number of observations on the boundary $\partial\Omega$, i.e. from the Neumann-to-Dirichlet map Λ_f:

$$f_1, f_2 \in L^2(\Omega) \quad \text{with} \quad \Lambda_{f_1} = \Lambda_{f_2} \quad \Rightarrow \quad f_1 = f_2 \text{ ?}$$

To our best knowledge, the above question is still open so far. In case an observation Λ_δ of Λ_f being available one can use a certain regularization method to approximate the sought source. For example, one can consider for operator norms $\|\cdot\|_*$ a minimizer of the problem

$$\min_{f \in L^2(\Omega)} \|\Lambda_f - \Lambda_\delta\|_*^2 + \rho \|f - f^*\|^2_{L^2(\Omega)}$$

as a reconstruction along the lines of Tikhonov's regularization method, where $\rho > 0$ is the regularization parameter and f^* is an a-priori estimate of the sought source.

However, in practice we have only a *finite* number of observations and the task is to reconstruct the identified source, at least by numerical approximations. Furthermore, for simplicity of exposition we below restrict ourselves to the case of just one observation pair (j_δ, g_δ) being available, while the approach described here can be easily extended to multiple measurements $\left(j^i_\delta, g^i_\delta\right)_{i=1,\dots,I}$. The inverse problem is thus stated as follows.

$$\textit{Given } \left(j^\dagger, g^\dagger\right) \in H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega) \textit{ with } \Lambda_f j^\dagger = g^\dagger, \textit{ find } f \in L^2(\Omega). \tag{$\mathcal{IP}$}$$

In other word, the interested problem is, for given $\left(j^\dagger, g^\dagger\right) \in H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$, to find some $f \in L^2(\Omega)$ and consequently $\Phi \in H^1_\diamond(\Omega)$ such that the system (1.1)–(1.3) is satisfied in the weak sense. Precisely, we define the general solution set

$$\mathcal{I}\left(j^\dagger, g^\dagger\right) := \left\{ f \in L^2(\Omega) \mid \Lambda_f j^\dagger = g^\dagger \right\} = \left\{ f \in L^2(\Omega) \mid \mathcal{N}_f j^\dagger = \mathcal{D}_f g^\dagger \right\} \tag{1.13}$$

of the inverse problem ($\mathcal{IP}$). The source identification problem as described here is well known to be not uniquely determined from boundary observations (see a counterexample in [3]), i.e., the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ fails to be a singleton. Since not the Neumann-to-Dirichlet map is given, but only one pair $\left(j^\dagger, g^\dagger\right)$, the problem is even highly underdetermined. Thus instead we will search for uniquely determined f^*-minimum-norm solutions $f^\dagger$ (cf. Section 4) which are minimizers of the problem

$$\min_{f \in \mathcal{I}(j^\dagger, g^\dagger)} \|f - f^*\|^2_{L^2(\Omega)}. \tag{$\mathcal{IP} - MN$}$$

Due to Lemma 4.4 below, the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is non-empty, closed and convex, hence $f^\dagger$ is uniquely determined. On the other hand, for all $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ the equation $\mathcal{N}_f j^\dagger = \mathcal{D}_f g^\dagger$ is fulfilled. However, we have to solve this equation with noise data $(j_\delta, g_\delta) \in H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ of $\left(j^\dagger, g^\dagger\right)$ satisfying (1.5). The simplest variety of regularization may be to consider a minimizer of the Tikhonov functional

$$\|\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\|^2_{\mathcal{X}} + \rho \|f - f^*\|^2_{L^2(\Omega)}$$

over $f \in L^2(\Omega)$ as an approximate solution to $f^\dagger$, where $\mathcal{X} = L^2(\Omega)$ or $\mathcal{X} = H^1(\Omega)$.

In present work we adopt the variational approach of Kohn and Vogelius [28, 29, 30] in using cost functions containing the gradient of forward operators to the above mentioned inverse source problem. More precisely, we use the convex function

$$\mathcal{J}_\delta(f) := \int_\Omega Q \nabla \left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) \cdot \nabla \left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) dx, \tag{1.14}$$

(cf. Lemma 2.3) instead of the mapping $f \mapsto \|\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\|^2_{\mathcal{X}}$, together with Tikhonov regularization and consider the *unique* solution $f_{\rho,\delta}$ of the strictly convex minimization problem

$$\min_{f\in L^2(\Omega)} \mathcal{J}_\delta(f) + \rho\|f - f^*\|^2_{L^2(\Omega)} \tag{$\mathcal{P}_{\rho,\delta}$}$$

as the regularized solution. The motivation in using this cost functional $\mathcal{J}_\delta$ as misfit functional is that for all $\xi \in L^2(\Omega)$ the inequality

$$\mathcal{J}_0(\xi) := \int_\Omega Q\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right)\cdot\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right)dx \geq \frac{C_\Omega \underline{q}}{1+C_\Omega}\left\|\mathcal{N}_\xi j^\dagger - \mathcal{D}_\xi g^\dagger\right\|^2_{H^1(\Omega)} \geq 0$$

holds true and $\mathcal{J}_0(f) = 0$ at any $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. The advantage is evident, because the minimizer $f_{\rho,\delta}$ satisfies the equation

$$f - f^* = -\frac{1}{\rho}\left(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta\right) \tag{1.15}$$

(see Theorem 3.2 below). Due to formula (1.15), the approach proves to be a modified variant of the Lavrentiev regularization (see, e.g., [2, 13, 25, 42]) with implicit forward operator. Furthermore, for convenience in numerical analysis with the finite element methods we consider here the Tikhonov regularization. The use of different convex penalty terms, e.g. total variation, may be a work for us in future.

Let $\mathcal{N}^h_f j_\delta$ and $\mathcal{D}^h_f g_\delta$ be corresponding approximations of the solution maps $\mathcal{N}_f j_\delta$ and $\mathcal{D}_f g_\delta$ in the finite dimensional space $\mathcal{V}^h_1$ of piecewise linear, continuous finite elements. We then consider the discrete regularized problem corresponding to $(\mathcal{P}_{\rho,\delta})$, i.e., the following strictly convex minimization problem

$$\min_{f\in L^2(\Omega)} \int_\Omega Q\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)\cdot\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)dx + \rho\|f - f^*\|^2_{L^2(\Omega)}. \tag{$\mathcal{P}^h_{\rho,\delta}$}$$

Using the variational discretization concept introduced in [24], we show in Section 3 that the unique solution $f^h_{\rho,\delta}$ of the problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ automatically belongs to the finite dimensional space $\mathcal{V}^h_1$. Thus, a discretization of the admissible set $L^2(\Omega)$ can be avoided.

As $h, \delta \to 0$ and with an appropriate a-priori regularization parameter choice $\rho = \rho(h,\delta)$, we in Section 4 prove that the sequence $\left(f^h_{\rho,\delta}\right)$ converges to $f^\dagger$ in the $L^2(\Omega)$-norm. Furthermore, the corresponding state sequences $\left(\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta\right)$ and $\left(\mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta\right)$ converge in the $H^1(\Omega)$-norm to $\Phi^\dagger = \Phi^\dagger(f^\dagger, j^\dagger, g^\dagger)$ solving (1.1)–(1.3).

Section 5 is devoted to convergence rates. In this section we also show that if $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ and there is a function $w \in L^2(\Omega)$ such that

$$f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger \tag{1.16}$$

then $f = f^\dagger$, i.e., f is the *unique* f^*-minimum-norm solution of the identification problem. Condition (1.16) appears to be a *source condition* which is typical for Lavrentiev regularization and allows for convergence rates. Precisely, for the known matrix $Q \in C^{0,1}(\overline{\Omega})^{d\times d}$ and the exact data $\left(j^\dagger, g^\dagger\right) \in H^{1/2}(\partial\Omega) \times H^{3/2}(\partial\Omega)$ we derive the convergence rates

$$\left\|\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta - \mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta\right\|^2_{H^1(\Omega)} + \rho\left\|f^h_{\rho,\delta} - f^\dagger\right\|^2_{L^2(\Omega)} = \mathcal{O}\left(\delta^2 + h^2 + h\rho + \delta\rho + \rho^2\right)$$

and

$$\left\|\mathcal{N}^h_{f^h_{\rho,\delta}} j_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} + \left\|\mathcal{D}^h_{f^h_{\rho,\delta}} g_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} = \mathcal{O}\left(\delta^2\rho^{-1} + h^2\rho^{-1} + h + \delta + \rho\right)$$

will be established. Finally, for the numerical solution of the discrete regularized problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ we employ a conjugate gradient algorithm. Numerical results show an efficiency of our theoretical findings.

The source identification problem in PDEs arises in many branches of applied science such as Electro-Encephalo-Graphy, geophysical prospecting and pollutant detection, and attracted great attention from many scientists in the last 30 years or so. For surveys on this subject we may consult in [8, 16, 19, 23, 26, 41] and the references therein. So far, only a limited number of works was investigated the general source identification problem and obtained results concentrated on numerical analysis for the identification problem. In [20, 32, 33] authors have used the dual reciprocity boundary element methods to simulate numerically for the above mentioned identification problem. In case some priori knowledge of the identified source is available, such as a point source, a characteristic function or a harmonic function, numerical methods treating the problem have been obtained in [5, 6, 31, 38]. A survey of the problem of simultaneously identifying the source term and coefficients in elliptic systems from *distributed* observations can be found in [37], where further references can be found.

We conclude this introduction with a remark that so far we have not yet found investigations on discretization analysis for the *general* source identification problem, a fact which motivated the research presented in this paper. Since the main interest is to clearly state our ideas, we only treat the model elliptic problem (1.1) while the approach described here can be easily extended to more general models, e.g., for the source identification problem in diffusion-reaction equations

$$-\nabla \cdot \left(Q\nabla\Phi\right) + \kappa^2\Phi = f \text{ in } \Omega,\ Q\nabla\Phi\cdot\vec{n} + \sigma\Phi = j^\dagger \text{ on } \partial\Omega \text{ and } \Phi = g^\dagger \text{ on } \partial\Omega \tag{1.17}$$

from a measurement (j_δ, g_δ) of $\left(j^\dagger, g^\dagger\right)$, where Q satisfying the condition (1.4), $0 \neq \kappa = \kappa(x) \in L^\infty(\Omega)$, i.e, the set $\{x \in \Omega | \kappa(x) \neq 0\}$ has positive Lebesgue measure, and $\sigma = \sigma(x) \in L^\infty(\partial\Omega)$ with $\sigma \geq 0$ are given. The variational approach is now formulated as the minimizing problem with the misfit

$$\int_\Omega Q\nabla\left(R_f j_\delta - D_f g_\delta\right)\cdot\nabla\left(R_f j_\delta - D_f g_\delta\right)dx + \int_\Omega \kappa^2\left(R_f j_\delta - D_f g_\delta\right)^2 dx + \int_{\partial\Omega}\sigma\left(R_f j_\delta - D_f g_\delta\right)^2 dx$$

over $f \in L^2(\Omega)$, where R and D are the Robin operator and the Dirichlet operator relating with the equation (1.17), respectively. Recently obtained results concerning the inverse source problem for (1.17) on the identification together with numerical algorithms can be found in, e.g., [1, 3, 9, 10].

Throughout the paper we use the standard notion of Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $W^{k,p}(\Omega)$, etc from, for example, [43]. If not stated otherwise we write $\int_\Omega \cdots$ instead of $\int_\Omega \cdots dx$.

2 Preliminaries

We first mention that the solution $\mathcal{N}_f j$ satisfies the following estimate

$$\begin{aligned}\|\mathcal{N}_f j\|_{H^1(\Omega)} &\leq \frac{1+C_\Omega}{C_\Omega \underline{q}}\left(\|j\|_{H^{-1/2}(\partial\Omega)}\|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)} + \|f\|_{L^2(\Omega)}\right)\\ &\leq C_\mathcal{N}\left(\|j\|_{H^{-1/2}(\partial\Omega)} + \|f\|_{L^2(\Omega)}\right),\end{aligned} \tag{2.1}$$

where $C_\mathcal{N} := \frac{1+C_\Omega}{C_\Omega \underline{q}}\max\left(1, \|\gamma\|_{\mathcal{L}\left(H^1(\Omega),H^{1/2}(\partial\Omega)\right)}\right)$. Next, we can rewrite $\mathcal{D}_f g = v_0 + G$, where $G = \gamma^{-1}g$ and $v_0 \in H_0^1(\Omega)$ is the unique solution to the variational problem $\int_\Omega Q\nabla v_0\cdot\nabla\psi = (f,\psi) - \int_\Omega Q\nabla G\cdot\nabla\psi$ for all $\psi \in H_0^1(\Omega)$. Since $\|G\|_{H^1(\Omega)} \leq \left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)}\|g\|_{H^{1/2}(\partial\Omega)}$, we thus obtain the priori estimate

$$\begin{aligned}\|\mathcal{D}_f g\|_{H^1(\Omega)} &\leq \|v_0\|_{H^1(\Omega)} + \|G\|_{H^1(\Omega)} \leq \frac{1+C_\Omega}{C_\Omega\underline{q}}\left(\|f\|_{L^2(\Omega)} + d\|Q\|_{L^\infty(\Omega)^{d\times d}}\|G\|_{H^1(\Omega)}\right) + \|G\|_{H^1(\Omega)}\\ &\leq \frac{1+C_\Omega}{C_\Omega\underline{q}}\|f\|_{L^2(\Omega)} + \left(d\frac{1+C_\Omega}{C_\Omega\underline{q}}\|Q\|_{L^\infty(\Omega)^{d\times d}} + 1\right)\left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)}\|g\|_{H^{1/2}(\partial\Omega)}\\ &\leq C_\mathcal{D}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right),\end{aligned} \tag{2.2}$$

where $C_\mathcal{D} := \max\left(\frac{1+C_\Omega}{C_\Omega\underline{q}}, \left(d\frac{1+C_\Omega}{C_\Omega\underline{q}}\|Q\|_{L^\infty(\Omega)^{d\times d}} + 1\right)\left\|\gamma^{-1}\right\|_{\mathcal{L}\left(H^{1/2}(\partial\Omega),H^1(\Omega)\right)}\right)$ and $Q := (q_{rs})_{1\leq r,s\leq d} \in L^\infty(\Omega)^{d\times d}$ with $\|Q\|_{L^\infty(\Omega)^{d\times d}} := \max_{1\leq r,s\leq d}\|q_{rs}\|_{L^\infty(\Omega)}$.

Now we summarize some useful properties of the Neumann and Dirichlet operators. The proof of the following result is based on standard arguments and therefore omitted.

Lemma 2.1. *(i) The Neumann operator $\mathcal{N} : L^2(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable. For each $f \in L^2(\Omega)$ the action of the Fréchet derivative in the direction $\xi \in L^2(\Omega)$ denoted by $\eta_{\mathcal{N}} := \mathcal{N}'_f j(\xi) := \mathcal{N}'(f)\xi$ is the unique weak solution in $H^1_\diamond(\Omega)$ of the homogeneous Neumann problem*

$$-\nabla \cdot (Q\nabla\eta_{\mathcal{N}}) = \xi \text{ in } \Omega \text{ and } Q\nabla\eta_{\mathcal{N}} \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

in the sense that it solves the equation $\int_\Omega Q\nabla\eta_{\mathcal{N}} \cdot \nabla\varphi = (\xi, \varphi)$ for all $\varphi \in H^1_\diamond(\Omega)$. Furthermore, the estimate $\|\eta_{\mathcal{N}}\|_{H^1(\Omega)} \le \frac{1+C_\Omega}{C_\Omega \underline{q}} \|\xi\|_{L^2(\Omega)}$ is fulfilled.

(ii) The Dirichlet operator $\mathcal{D} : L^2(\Omega) \to H^1(\Omega)$ is continuously Fréchet differentiable. For each $f \in L^2(\Omega)$ the action of the Fréchet derivative in the direction $\xi \in L^2(\Omega)$ denoted by $\eta_{\mathcal{D}} := \mathcal{D}'_f g(\xi) := \mathcal{D}'(f)\xi$ is the unique weak solution in $H^1_0(\Omega)$ of the homogeneous Dirichlet problem

$$-\nabla \cdot (Q\nabla\eta_{\mathcal{D}}) = \xi \text{ in } \Omega \text{ and } \eta_{\mathcal{D}} = 0 \text{ on } \partial\Omega$$

in the sense that it solves the equation $\int_\Omega Q\nabla\eta_{\mathcal{D}} \cdot \nabla\psi = (\xi, \psi)$ for all $\psi \in H^1_0(\Omega)$. Furthermore, the estimate $\|\eta_{\mathcal{D}}\|_{H^1(\Omega)} \le \frac{1+C_\Omega}{C_\Omega \underline{q}} \|\xi\|_{L^2(\Omega)}$ holds true.

Lemma 2.2. *(i) The map $\mathbf{T} : L^2(\Omega) \to L^2(\Omega)$ defined by*

$$\mathbf{T}(f) := \mathcal{N}_f 0 - \mathcal{D}_f 0$$

is linear, bounded and self-adjoint, i.e., $(\mathbf{T}(f), w) = (f, \mathbf{T}(w))$ for all $f, w \in L^2(\Omega)$.

(ii) For any fixed $(j, g) \in H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ the map $\mathbf{L} : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\mathbf{L}(f) := \mathcal{N}_f j - \mathcal{D}_f g = \mathbf{T}(f) + \mathcal{N}_0 j - \mathcal{D}_0 g$$

is affine linear, continuous and monotone, i.e., $(\mathbf{L}(f) - \mathbf{L}(w), f - w) \ge 0$ for all $f, w \in L^2(\Omega)$.

Proof. (i) It follows from (1.10) that

$$(\mathbf{T}(f), w) = \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla(\mathcal{N}_f 0 - \mathcal{D}_f 0) = \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{N}_f 0 - \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{D}_f 0, \tag{2.3}$$

and similarly,

$$(f, \mathbf{T}(w)) = \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{N}_w 0 - \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{D}_w 0. \tag{2.4}$$

Using (1.10)–(1.11) again, we get

$$\begin{aligned} \int_\Omega Q\nabla\mathcal{N}_w 0 \cdot \nabla\mathcal{D}_f 0 = (w, \mathcal{D}_f 0) = \int_\Omega Q\nabla\mathcal{D}_w 0 \cdot \nabla\mathcal{D}_f 0 \\ \int_\Omega Q\nabla\mathcal{N}_f 0 \cdot \nabla\mathcal{D}_w 0 = (f, \mathcal{D}_w 0) = \int_\Omega Q\nabla\mathcal{D}_f 0 \cdot \nabla\mathcal{D}_w 0 \end{aligned} \tag{2.5}$$

and the self-adjoint property of $\mathbf{T}$ now follows directly from (2.3)-(2.5).

(ii) Denoting by $\xi := f - w$, we get from (1.10) that

$$(\mathbf{L}(f) - \mathbf{L}(w), f - w) = (\mathbf{T}(f) - \mathbf{T}(w), f - w) = (\mathbf{T}(\xi), \xi) = \int_\Omega Q\nabla\mathcal{N}_\xi 0 \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0). \tag{2.6}$$

We further have from (1.10)–(1.11) that

$$\int_\Omega Q\nabla\mathcal{D}_\xi 0 \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) = 0. \tag{2.7}$$

Combining (2.6) with (2.7), we arrive at

$$(\mathbf{L}(f) - \mathbf{L}(w), f - w) = \int_\Omega Q\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) \cdot \nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) \ge 0,$$

which finishes the proof. □

We now briefly discuss the character of the null-space $\ker(\mathbf{T})$ of the operator $\mathbf{T}$ occurring in the above lemma. For simplicity we assume that the known matrix $Q = I_d$, the $d \times d$-unit matrix. Then the counterexample of [3] shows that $\ker(\mathbf{T}) \neq \{0\}$. Furthermore, one can easily see that for any $\Psi \in C_c^2(\Omega)$, the space of all functions having second-order derivatives with compact support in Ω, the negative Laplace $-\Delta\Psi$ of Ψ satisfies $-\Delta\Psi \in \ker(\mathbf{T})$.

Finding an element $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$, i.e. a solution to the identification problem $(\mathcal{IP})$, is equivalent to solving an operator equation (see for notations and details Lemma 2.2 above)

$$F(f, j, g) = 0, \quad \text{where} \quad F(f, j, g) := \mathbf{L}(f), \quad \text{and} \quad j := j^\dagger,\ g := g^\dagger. \tag{2.8}$$

Such an implicit inverse problem model was generally introduced in the monograph [7, Section 1.2]. Due to Lemma 2.2, the forward operator F in the operator equation (2.8) is affine linear with respect to f and of implicit nonlinear structure with respect to the data. We can rewrite it as $\mathbf{T}(f) + B(j^\dagger, g^\dagger) = 0$, where for fixed $(j^\dagger, g^\dagger)$ the magnitude $B(j^\dagger, g^\dagger)$ is a constant function in $L^2(\Omega)$ and $\mathbf{T}$ is a linear, bounded and self-adjoint operator mapping in $L^2(\Omega)$ such that $\mathbf{L}$ is a continuous, affine linear and monotone operator. The non-standard character comes from the fact that data j and g in the operator equation do not occur separated in a right-hand side, but implicitly and nonlinearly embedded in the forward operator itself. Nevertheless the monotonicity of $\mathbf{L}$ allows us to apply Lavrentiev regularization for stabilizing the inverse problem for given noisy data (j_δ, g_δ) by using the solution of the singular perturbed version

$$F(f, j_\delta, g_\delta) + \rho\,(f - f^*) = \mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta + \rho\,(f - f^*) = 0 \tag{2.9}$$

of the original operator equation as regularized solution. One simply sees that the uniquely determined Lavrentiev-regularized solution satisfying (2.9) and the Tikhonov-regularized solution $f_{\rho,\delta}$ satisfying (1.15) coincide. Since the null-space $\ker(\mathbf{T})$ of the operator $\mathbf{T}$ is non-trivial, the operator equation (2.8) with affine linear forward operator is locally ill-posed everywhere, see [13] and in particular Definition 1.1 ibidem. Taking into account the results from [36] Lavrentiev-regularized solutions cannot converge to the f^*-minimum norm solution with a convergence rate better than $\|f_{\rho,\delta} - f^*\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ as $\delta \to 0$. As Corollary 5.5 will show, we achieve this optimal rate with our method.

Lemma 2.3. *(i) If the sequence $(f_n) \subset L^2(\Omega)$ converges weakly in $L^2(\Omega)$ to an element f, then the sequence $(\mathcal{N}_{f_n} j_\delta, \mathcal{D}_{f_n} g_\delta)$ converges weakly to $(\mathcal{N}_f j_\delta, \mathcal{D}_f g_\delta)$ in $H^1(\Omega) \times H^1(\Omega)$.*

(ii) The function $\mathcal{J}_\delta$ defined by (1.14) is convex and weakly sequentially lower semi-continuous.

Proof. (i) Since the sequence (f_n) converges weakly in $L^2(\Omega)$, so is bounded in the $L^2(\Omega)$-norm. Thus, it follows from the estimate (2.1) that the sequence $(\mathcal{N}_{f_n} j_\delta) \subset H^1_\diamond(\Omega)$ is bounded in the $H^1(\Omega)$-norm. Then a subsequence which is not relabelled and an element $\Phi_\mathcal{N} \in H^1_\diamond(\Omega)$ exist such that $(\mathcal{N}_{f_n} j_\delta)$ converges weakly to $\Phi_\mathcal{N}$ in $H^1(\Omega)$. For all $\varphi \in H^1_\diamond(\Omega)$ and $n \in N$ we have that $\int_\Omega Q\nabla\mathcal{N}_{f_n} j_\delta \cdot \nabla\varphi = \langle j_\delta, \gamma\varphi\rangle + (f_n, \varphi)$. Sending n to ∞, we obtain $\int_\Omega Q\nabla\Phi_\mathcal{N} \cdot \nabla\varphi = \langle j_\delta, \gamma\varphi\rangle + (f, \varphi)$, which means that $\Phi_\mathcal{N} = \mathcal{N}_f j_\delta$.

Next, we have for all $n \in N$ that $\mathcal{D}_{f_n} g_\delta = D_n + \gamma^{-1} g_\delta$, where $D_n \in H^1_0(\Omega)$ is the unique solution to the following variational equation

$$\int_\Omega Q\nabla D_n \cdot \nabla\psi = (f_n, \psi) - \int_\Omega Q\nabla\gamma^{-1} g_\delta \cdot \nabla\psi \tag{2.10}$$

for all $\psi \in H^1_0(\Omega)$. Since the sequence $(\mathcal{D}_{f_n} g_\delta)$ is bounded in the $H^1(\Omega)$-norm, so is the sequence (D_n). A subsequence not relabelled and an element $D \in H^1_0(\Omega)$ exist such that (D_n) converges weakly to D in $H^1(\Omega)$. Then, sending n to ∞ in (2.10), we obtain the identity $\int_\Omega Q\nabla\left(D + \gamma^{-1} g_\delta\right) \cdot \nabla\psi = (f, \psi)$ for all $\psi \in H^1_0(\Omega)$. Therefore, $\mathcal{D}_f g_\delta = D + \gamma^{-1} g_\delta$ and the sequence $(\mathcal{D}_{f_n} g_\delta)$ converges weakly to $\mathcal{D}_f g_\delta$ in $H^1(\Omega)$.

(ii) Due to the inequality

$$\frac{C_\Omega \underline{q}}{1 + C_\Omega}\|\varphi\|^2_{H^1(\Omega)} \le \int_\Omega Q\nabla\varphi \cdot \nabla\varphi \le d\|Q\|_{L^\infty(\Omega)^{d\times d}}\|\varphi\|^2_{H^1(\Omega)}, \quad \forall\varphi \in H^1_\diamond(\Omega), \tag{2.11}$$

the expression

$$[u, v] := \int_\Omega Q\nabla u \cdot \nabla v \tag{2.12}$$

generates a scalar inner product on the space $H^1_\diamond(\Omega)$ which is equivalent to the usual one. This results $(\mathcal{N}_{f_n} j_\delta, \mathcal{D}_{f_n} g_\delta) \rightharpoonup (\mathcal{N}_f j_\delta, \mathcal{D}_f g_\delta)$ weakly in $H^1_\diamond(\Omega) \times H^1_\diamond(\Omega)$ with respect to the scalar product (2.12). Finally, any norm is weakly lower semi-continuous, we then obtain

$$\mathcal{J}_\delta(f) = [\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta, \mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta] \le \liminf_{n\to\infty} [\mathcal{N}_{f_n} j_\delta - \mathcal{D}_{f_n} g_\delta, \mathcal{N}_{f_n} j_\delta - \mathcal{D}_{f_n} g_\delta] = \liminf_{n\to\infty} \mathcal{J}_\delta(f_n),$$

which follows that $\mathcal{J}_\delta$ is weakly sequentially lower semi-continuous. Finally, we show the convexity of $\mathcal{J}_\delta$. In deed, due to Lemma 2.1, we from (1.12) have for all $\xi \in L^2(\Omega)$ that

$$\begin{aligned}\mathcal{J}'_\delta(f)\xi &= 2\int_\Omega Q\nabla(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta)\cdot\nabla\left(\mathcal{N}'_f j_\delta(\xi) - \mathcal{D}'_f g_\delta(\xi)\right) = 2\int_\Omega Q\nabla(\mathcal{N}_f j_\delta - \mathcal{D}_f g_\delta)\cdot\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0)\\ &= 2\int_\Omega Q\nabla(\mathcal{N}_f 0 - \mathcal{D}_f 0)\cdot\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) + 2\int_\Omega Q\nabla(\mathcal{N}_0 j_\delta - \mathcal{D}_0 g_\delta)\cdot\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0),\end{aligned}$$

and so

$$\mathcal{J}''_\delta(f)(\xi,\xi) = 2\int_\Omega Q\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0)\cdot\nabla(\mathcal{N}_\xi 0 - \mathcal{D}_\xi 0) \ge 0,$$

which shows the function $\mathcal{J}_\delta$ in fact is convex. The proof is completed. □

Proposition 2.4. *The minimization problem $(\mathcal{P}_{\rho,\delta})$ attains a unique solution $f_{\rho,\delta}$, which as regularized solution represents an approximation of the f^*-minimum-norm solution $f^\dagger$ to the identification problem $(\mathcal{IP})$.*

Proof. The proof of existence of solutions is based on Lemma 2.3 in combining with arguments of [39, Proposition 4.1], therefore omitted here. Furthermore, since the cost function of $(\mathcal{P}_{\rho,\delta})$ is strictly convex, the minimizer is unique. □

3 Finite element discretization

Let $\left(\mathcal{T}^h\right)_{0<h<1}$ be a family of regular and quasi-uniform triangulations of the domain $\overline{\Omega}$ with the mesh size h. For the definition of the discretization space of the state functions let us denote

$$\mathcal{V}^h_1 := \left\{v^h \in C(\overline{\Omega}) \mid {v^h}_{|T} \in \mathcal{P}_1(T),\ \forall T \in \mathcal{T}^h\right\}$$

and

$$\mathcal{V}^h_{1,\diamond} := \mathcal{V}^h_1 \cap H^1_\diamond(\Omega) \text{ and } \mathcal{V}^h_{1,0} := \mathcal{V}^h_1 \cap H^1_0(\Omega) \subset \mathcal{V}^h_{1,\diamond},$$

where $\mathcal{P}_1$ consists all polynomial functions of degree less than or equal to 1.

Proposition 3.1. *(i) Let f be in $L^2(\Omega)$ and j be in $H^{-1/2}(\partial\Omega)$. Then the variational equation*

$$\int_\Omega Q\nabla u^h \cdot \nabla\varphi^h = \left(f, \varphi^h\right) + \left\langle j, \gamma\varphi^h\right\rangle \text{ for all } \varphi^h \in \mathcal{V}^h_{1,\diamond} \tag{3.1}$$

admits a unique solution $u^h \in \mathcal{V}^h_{1,\diamond}$. Furthermore, the prior estimate

$$\left\|u^h\right\|_{H^1(\Omega)} \le C_\mathcal{N}\left(\|f\|_{L^2(\Omega)} + \|j\|_{H^{-1/2}(\partial\Omega)}\right) \tag{3.2}$$

is satisfied. The map $\mathcal{N}^h : L^2(\Omega) \to \mathcal{V}^h_{1,\diamond}$ from each $f \in L^2(\Omega)$ to the unique solution $u^h =: \mathcal{N}^h_f j$ of (3.1) is then called the discrete Neumann operator.

(ii) Let f be in $L^2(\Omega)$ and g be in $H^{1/2}_\diamond(\partial\Omega)$. The equation

$$\int_\Omega Q\nabla v^h \cdot \nabla\psi^h = \left(f, \psi^h\right) \text{ for all } \psi^h \in \mathcal{V}^h_{1,0} \tag{3.3}$$

has a unique solution $v^h \in \mathcal{V}^h_{1,\diamond}$ with $\gamma v^h = g$. Furthermore, the inequality

$$\left\|v^h\right\|_{H^1(\Omega)} \leq C_{\mathcal{D}} \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right) \tag{3.4}$$

is satisfied. The map $\mathcal{D}^h : L^2(\Omega) \to \mathcal{V}^h_{1,\diamond}$ from each $f \in L^2(\Omega)$ to the unique solution $v^h =: \mathcal{D}^h_f g$ of (3.3) is called the discrete Dirichlet operator.

Similar to Lemma 2.1, one sees that the discrete operators $\mathcal{N}^h$, $\mathcal{D}^h$ are Fréchet differentiable on $L^2(\Omega)$. For each $f \in L^2(\Omega)$ the Fréchet derivatives $\mathcal{N}^{h\prime}(f)\xi =: \mathcal{N}^{h\prime}_f j(\xi) \in \mathcal{V}^h_{1,\diamond}$ and $\mathcal{D}^{h\prime}(f)\xi =: \mathcal{D}^{h\prime}_f g(\xi) \in \mathcal{V}^h_{1,0}$ in the direction $\xi \in L^2(\Omega)$ satisfy the equations

$$\int_\Omega Q\nabla \mathcal{N}^{h\prime}_f j(\xi) \cdot \nabla \varphi^h = \left(\xi, \varphi^h\right) \tag{3.5}$$

and

$$\int_\Omega Q\nabla \mathcal{D}^{h\prime}_f g(\xi) \cdot \nabla \psi^h = \left(\xi, \psi^h\right) \tag{3.6}$$

for all $\varphi^h \in \mathcal{V}^h_{1,\diamond}$ and $\psi^h \in \mathcal{V}^h_{1,0}$.

We now can introduce the strictly convex, discrete cost function

$$\Upsilon^h_{\rho,\delta}(f) := \mathcal{J}^h_\delta(f) + \rho\|f - f^*\|^2_{L^2(\Omega)} \text{ with } \mathcal{J}^h_\delta(f) := \int_\Omega Q\nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right).$$

Theorem 3.2. *The problem*

$$\min_{f \in L^2(\Omega)} \Upsilon^h_{\rho,\delta}(f) \tag{$\mathcal{P}^h_{\rho,\delta}$}$$

attains a unique minimizer f which satisfies the equation

$$f - f^* = -\frac{1}{\rho}\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right). \tag{3.7}$$

Remark 3.3. Since $\mathcal{N}^h_f j_\delta$ and $\mathcal{D}^h_f g_\delta$ are both in $\mathcal{V}^1_h$, so is f, provided that $f^* \in \mathcal{V}^1_h$. Thus, taking this into account, a discretization of the set $L^2(\Omega)$ can be avoided.

Proof of Theorem 3.2. The existence and uniqueness of a minimizer to the problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ are exactly obtained as in the continuous case, therefore omitted here. It remains to show (3.7).

Let $f \in L^2(\Omega)$ be the minimizer to $\left(\mathcal{P}^h_{\rho,\delta}\right)$. The first-order optimality condition yields that

$$\Upsilon^h_{\rho,\delta}{}'(f)\xi = \mathcal{J}^{h\prime}_\delta(f)\xi + 2\rho(\xi, f - f^*) = 0 \tag{3.8}$$

for all $\xi \in L^2(\Omega)$, where

$$\begin{aligned}\mathcal{J}^{h\prime}_\delta(f)\xi &= 2\int_\Omega Q\nabla\left(\mathcal{N}^{h\prime}_f j_\delta(\xi) - \mathcal{D}^{h\prime}_f g_\delta(\xi)\right) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)\\ &= 2\int_\Omega Q\nabla\mathcal{N}^{h\prime}_f j_\delta(\xi) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) + 2\int_\Omega Q\nabla\mathcal{D}^h_f g_\delta \cdot \nabla\mathcal{D}^{h\prime}_f g_\delta(\xi) - 2\int_\Omega Q\nabla\mathcal{N}^h_f j_\delta \cdot \nabla\mathcal{D}^{h\prime}_f g_\delta(\xi).\end{aligned}$$

By (3.5), it follows that

$$\int_\Omega Q\nabla\mathcal{N}^{h\prime}_f j_\delta(\xi) \cdot \nabla\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) = \left(\xi, \mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right)$$

while (3.3) and (3.1) yield

$$\int_\Omega Q\nabla\mathcal{D}^h_f g_\delta \cdot \nabla\mathcal{D}^{h\prime}_f g_\delta(\xi) = \left(f, \mathcal{D}^{h\prime}_f g_\delta(\xi)\right)$$

and

$$\int_\Omega Q\nabla\mathcal{N}_f^h j_\delta\cdot\nabla\mathcal{D}_f^{h'} g_\delta(\xi) = \Big(f,\mathcal{D}_f^{h'} g_\delta(\xi)\Big) + \Big\langle j_\delta,\gamma\mathcal{D}_f^{h'} g_\delta(\xi)\Big\rangle = \Big(f,\mathcal{D}_f^{h'} g_\delta(\xi)\Big).$$

We thus infer that

$$\mathcal{J}_\delta^{h'}(f)(\xi) = 2\left(\xi,\mathcal{N}_f^h j_\delta - \mathcal{D}_f^h g_\delta\right) \tag{3.9}$$

and so obtain the equation

$$\left(\xi,\frac{1}{\rho}\left(\mathcal{N}_f^h j_\delta - \mathcal{D}_f^h g_\delta\right) + f - f^*\right) = 0 \tag{3.10}$$

for all $\xi\in L^2(\Omega)$, which finishes the proof. □

4 Convergence

From now on C is a generic positive constant which is independent of the mesh size h of $\mathcal{T}^h$, the noise level δ and the regularization parameter ρ. Before presenting the convergence of finite element approximations we here state some auxiliary results.

Lemma 4.1. *An interpolation operator $\Pi_\diamond^h : L^1(\Omega)\to\mathcal{V}_{1,\diamond}^h$ exists such that*

$$\Pi_\diamond^h\varphi^h = \varphi^h \textit{ for all } \varphi^h\in\mathcal{V}_{1,\diamond}^h \textit{ and } \Pi_\diamond^h\big(H_0^1(\Omega)\big)\subset\mathcal{V}_{1,0}^h\subset\mathcal{V}_{1,\diamond}^h.$$

Furthermore, it satisfies the properties

$$\lim_{h\to0}\left\|\vartheta-\Pi_\diamond^h\vartheta\right\|_{H^1(\Omega)} = 0 \quad \textit{for all } \vartheta\in H_\diamond^1(\Omega) \tag{4.1}$$

and

$$\left\|\vartheta-\Pi_\diamond^h\vartheta\right\|_{H^1(\Omega)} \le Ch\|\vartheta\|_{H^2(\Omega)} \textit{ for all } \vartheta\in H_\diamond^1(\Omega)\cap H^2(\Omega). \tag{4.2}$$

Proof. Let $\Pi^h : L^1(\Omega)\to\mathcal{V}_1^h$ be the Clement's mollification interpolation operator, see [18] and some generalizations [11, 12, 40]. We then define the operator

$$\Pi_\diamond^h\vartheta := \Pi^h\vartheta - \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma\Pi^h\vartheta\in\mathcal{V}_{1,\diamond}^h, \quad \forall\vartheta\in L^1(\Omega)$$

which has the properties (4.1) and (4.2). The proof is completed. □

On the basis of (4.1) and (4.2) we introduce for each $\Phi\in H_\diamond^1(\Omega)$

$$\varrho_\Phi^h := \left\|\Phi-\Pi_\diamond^h\Phi\right\|_{H^1(\Omega)}. \tag{4.3}$$

We note that $\lim_{h\to0}\varrho_\Phi^h = 0$ and

$$0\le\varrho_\Phi^h\le Ch \tag{4.4}$$

in case $\Phi\in H^2(\Omega)$. Furthermore, let $(f,j,g)\in L^2(\Omega)\times H^{-1/2}(\partial\Omega)\times H_\diamond^{1/2}(\partial\Omega)$ be fixed, we denote by

$$\alpha_{f,j}^h = \left\|\mathcal{N}_f^h j - \mathcal{N}_f j\right\|_{H^1(\Omega)} \text{ and } \beta_{f,g}^h = \left\|\mathcal{D}_f^h g - \mathcal{D}_f g\right\|_{H^1(\Omega)}. \tag{4.5}$$

Then

$$\lim_{h\to0}\alpha_{f,j}^h = \lim_{h\to0}\beta_{f,g}^h = 0. \tag{4.6}$$

In particular, if $\mathcal{N}_f j\in H^2(\Omega)$ and $\mathcal{D}_f g\in H^2(\Omega)$, the error estimates

$$\alpha_{f,j}^h\le Ch \text{ and } \beta_{f,g}^h\le Ch \tag{4.7}$$

are satisfied (cf. [14, 17]).

Lemma 4.2. *Let (f_1, j_1, g_1) and (f_2, j_2, g_2) be arbitrary in $L^2(\Omega) \times H^{-1/2}(\partial\Omega) \times H_\diamond^{1/2}(\partial\Omega)$. Then the estimates*

$$\left\|\mathcal{N}_{f_1}^h j_1 - \mathcal{N}_{f_2}^h j_2\right\|_{H^1(\Omega)} \le C_{\mathcal{N}} \left(\|f_1 - f_2\|_{L^2(\Omega)} + \|j_1 - j_2\|_{H^{-1/2}(\partial\Omega)} \right) \tag{4.8}$$

and

$$\left\|\mathcal{D}_{f_1}^h g_1 - \mathcal{D}_{f_2}^h g_2\right\|_{H^1(\Omega)} \le C_{\mathcal{D}} \left(\|f_1 - f_2\|_{L^2(\Omega)} + \|g_1 - g_2\|_{H^{1/2}(\partial\Omega)} \right) \tag{4.9}$$

hold for all $h > 0$.

Proof. According to the definition of the discrete Neumann operator, we have for all $\varphi^h \in \mathcal{V}_{1,\diamond}^h$ that

$$\int_\Omega Q\nabla\mathcal{N}_{f_i}^h j_i \cdot \nabla\varphi^h = \left\langle j_i, \gamma\varphi^h \right\rangle + \left(f_i, \varphi^h\right) \text{ with } i = 1, 2.$$

Thus, $\Phi_{\mathcal{N}}^h := \mathcal{N}_{f_1}^h j_1 - \mathcal{N}_{f_2}^h j_2$ is the unique solution to the variational problem

$$\int_\Omega Q\nabla\Phi_{\mathcal{N}}^h \cdot \nabla\varphi^h = \left\langle j_1 - j_2, \gamma\varphi^h \right\rangle + \left(f_1 - f_2, \varphi^h\right)$$

for all $\varphi^h \in \mathcal{V}_{1,\diamond}^h$ and so that (4.8) is satisfied. Likewise, we also obtain (4.9). The proof is completed. □

Lemma 4.3. *Let $\left(\mathcal{T}^{h_n}\right)$ be a sequence of triangulations with $\lim_{n\to\infty} h_n = 0$. Assume that $(j_{\delta_n}, g_{\delta_n})$ is a sequence in $H^{-1/2}(\partial\Omega) \times H_\diamond^{1/2}(\partial\Omega)$ convergent to (j_δ, g_δ) in the $H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$-norm and (f_n) is a sequence in $L^2(\Omega)$ weakly convergent in $L^2(\Omega)$ to f, then there holds the inequality*

$$\liminf_{n\to\infty} \mathcal{J}_{\delta_n}^{h_n}(f_n) \ge \mathcal{J}_\delta(f). \tag{4.10}$$

Proof. In view of (3.2), the sequence $(\mathcal{N}_{f_n}^{h_n} j_{\delta_n})_n \subset H_\diamond^1(\Omega)$ is bounded in the $H^1(\Omega)$-norm, a subsequence not relabelled and an element $\Phi_{\mathcal{N}} \in H_\diamond^1(\Omega)$ exist such that $(\mathcal{N}_{f_n}^{h_n} j_{\delta_n})$ converges weakly in $H^1(\Omega)$ to $\Phi_{\mathcal{N}}$. Similar to Proof of Lemma 2.3, in order to establish (4.10), it is sufficient to show that $\Phi_{\mathcal{N}} = \mathcal{N}_f j_\delta$. Using the operator $\Pi_\diamond^{h_n}$ in Lemma 4.1, for all $\varphi \in H_\diamond^1(\Omega)$ we have that

$$\langle j_{\delta_n}, \gamma\varphi\rangle + (f_n, \varphi) = \left\langle j_{\delta_n}, \gamma\Pi_\diamond^{h_n}\varphi\right\rangle + \left(f_n, \Pi_\diamond^{h_n}\varphi\right) + \left\langle j_{\delta_n}, \gamma\left(\varphi - \Pi_\diamond^{h_n}\varphi\right)\right\rangle + \left(f_n, \varphi - \Pi_\diamond^{h_n}\varphi\right),$$

where $\Pi_\diamond^{h_n}\varphi \in \mathcal{V}_{1,\diamond}^{h_n}$. We note that

$$\begin{aligned}
\left|\left\langle j_{\delta_n}, \gamma\left(\varphi - \Pi_\diamond^{h_n}\varphi\right)\right\rangle\right| &\le \|j_{\delta_n}\|_{H^{-1/2}(\partial\Omega)} \left\|\gamma\left(\varphi - \Pi_\diamond^{h_n}\varphi\right)\right\|_{H^{1/2}(\partial\Omega)} \\
&\le \|j_{\delta_n}\|_{H^{-1/2}(\partial\Omega)} \|\gamma\|_{\mathcal{L}\left(H^1(\Omega), H^{1/2}(\partial\Omega)\right)} \left\|\varphi - \Pi_\diamond^{h_n}\varphi\right\|_{H^1(\Omega)} \\
&\le C\left\|\varphi - \Pi_\diamond^{h_n}\varphi\right\|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Likewise, $\lim_{n\to\infty} \left|\left(f_n, \varphi - \Pi_\diamond^{h_n}\varphi\right)\right| = 0$. We thus get for all $\varphi \in H_\diamond^1(\Omega)$ that

$$\begin{aligned}
&\langle j_\delta, \gamma\varphi\rangle + (f, \varphi) = \lim_{n\to\infty} \left(\langle j_{\delta_n}, \gamma\varphi\rangle + (f_n, \varphi)\right) = \lim_{n\to\infty} \left(\left\langle j_{\delta_n}, \gamma\Pi_\diamond^{h_n}\varphi\right\rangle + \left(f_n, \Pi_\diamond^{h_n}\varphi\right)\right) \\
&= \lim_{n\to\infty} \int_\Omega Q\nabla\mathcal{N}_{f_n}^{h_n} j_{\delta_n} \cdot \nabla\Pi_\diamond^{h_n}\varphi = \lim_{n\to\infty} \int_\Omega Q\nabla\mathcal{N}_{f_n}^{h_n} j_{\delta_n} \cdot \nabla\varphi + \lim_{n\to\infty} \int_\Omega Q\nabla\mathcal{N}_{f_n}^{h_n} j_{\delta_n} \cdot \nabla\left(\Pi_\diamond^{h_n}\varphi - \varphi\right) \\
&= \lim_{n\to\infty} \int_\Omega Q\nabla\mathcal{N}_{f_n}^{h_n} j_{\delta_n} \cdot \nabla\varphi = \int_\Omega Q\nabla\Phi_{\mathcal{N}} \cdot \nabla\varphi
\end{aligned}$$

and so that $\Phi_{\mathcal{N}} = \mathcal{N}_f j_\delta$. The proof is completed. □

Lemma 4.4. *The problem*

$$\min_{f\in\mathcal{I}(j^\dagger, g^\dagger)} \|f - f^*\|_{L^2(\Omega)}^2 \tag{$\mathcal{IP}-MN$}$$

attains a unique solution, which is called the f^*-minimum-norm solution of the identification problem.

Proof. Since $\left(j^\dagger, g^\dagger\right)$ is the exact data, the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is nonempty. Furthermore, it is a closed subset of $L^2(\Omega)$. Indeed, let $(f_n) \subset \mathcal{I}\left(j^\dagger, g^\dagger\right)$ be a sequence convergent in the $L^2(\Omega)$-norm to f. In view of Lemma 2.3 the sequence $\left(\mathcal{N}_{f_n} j^\dagger, \mathcal{D}_{f_n} g^\dagger\right)$ then converges weakly in $H^1(\Omega)\times H^1(\Omega)$ to $\left(\mathcal{N}_f j^\dagger, \mathcal{D}_f g^\dagger\right)$. Since $\mathcal{N}_{f_n} j^\dagger = \mathcal{D}_{f_n} g^\dagger$ for all $n \in N$, it follows that $\mathcal{N}_f j^\dagger = \mathcal{D}_f g^\dagger$ and so that $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$.

We will see that the set $\mathcal{I}\left(j^\dagger, g^\dagger\right)$ is in fact convex. Let $f_1, f_2 \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ and $c_1, c_2 \in [0, 1]$ with $c_1 + c_2 = 1$. We show that $f := c_1 f_1 + c_2 f_2 \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. Indeed, since $f_1, f_2 \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$, there are $u_1, u_2 \in H^1_\diamond(\Omega)$ with $\gamma u_1 = \gamma u_2 = g^\dagger$ such that

$$\int_\Omega Q\nabla u_1 \cdot \nabla\varphi = \left\langle j^\dagger, \gamma\varphi\right\rangle + (f_1, \varphi) \text{ and } \int_\Omega Q\nabla u_2 \cdot \nabla\varphi = \left\langle j^\dagger, \gamma\varphi\right\rangle + (f_2, \varphi)$$

for all $\varphi \in H^1_\diamond(\Omega)$. Setting $\Phi := c_1 u_1 + c_2 u_2 \in H^1_\diamond(\Omega)$, we then have that

$$\gamma\Phi = c_1\gamma u_1 + c_2\gamma u_2 = c_1 g^\dagger + c_2 g^\dagger = (c_1 + c_2) g^\dagger = g^\dagger.$$

Furthermore, we get for all $\varphi \in H^1_\diamond(\Omega)$ that

$$\int_\Omega Q\nabla\Phi \cdot \nabla\varphi = \left\langle j^\dagger, \gamma\varphi\right\rangle + (f, \varphi)$$

which infers $\Phi = \mathcal{N}_f j^\dagger$ with $\gamma\Phi = g^\dagger$, thus $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. Consequently, the problem $(\mathcal{IP})$ has a unique minimizer, which finishes the proof. □

We now show the convergence of finite element approximations to the identification problem.

Theorem 4.5. *Let $f^\dagger$ be the unique f^*-minimum-norm solution to the identification problem $(\mathcal{IP})$, which solves the minimization problem $(\mathcal{IP} - MN)$. Assume that $\lim_{n\to\infty} h_n = 0$ and (δ_n) and (ρ_n) any positive sequences such that*

$$\rho_n \to 0,\ \frac{\delta_n}{\sqrt{\rho_n}} \to 0,\ \frac{\alpha^{h_n}_{f^\dagger, j^\dagger}}{\sqrt{\rho_n}} \to 0 \text{ and } \frac{\beta^{h_n}_{f^\dagger, g^\dagger}}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty, \tag{4.11}$$

where $\alpha^{h_n}_{f^\dagger, j^\dagger}$ and $\beta^{h_n}_{f^\dagger, g^\dagger}$ are defined by (4.5)–(4.6). Furthermore, assume that $(j_{\delta_n}, g_{\delta_n})$ is a sequence in $H^{-1/2}(\partial\Omega) \times H^{1/2}_\diamond(\partial\Omega)$ satisfying

$$\left\|j_{\delta_n} - j^\dagger\right\|_{H^{-1/2}(\partial\Omega)} + \left\|g_{\delta_n} - g^\dagger\right\|_{H^{1/2}(\partial\Omega)} \le \delta_n$$

and $f_n := f^{h_n}_{\rho_n, \delta_n}$ is the unique minimizer of $\left(\mathcal{P}^{h_n}_{\rho_n, \delta_n}\right)$ for each $n \in N$. Then:

(i) The sequence (f_n) converges in the $L^2(\Omega)$-norm to $f^\dagger$.

(ii) The corresponding state sequences $\left(\mathcal{N}^{h_n}_{f_n} j_{\delta_n}\right)$ and $\left(\mathcal{D}^{h_n}_{f_n} g_{\delta_n}\right)$ converge in the $H^1(\Omega)$-norm to the unique weak solution $\Phi^\dagger = \Phi^\dagger\left(f^\dagger, j^\dagger, g^\dagger\right)$ of the boundary value problem (1.1)–(1.3).

Before going to prove the theorem, we make the following short remark.

Remark 4.6. In case the weak solution $\Phi^\dagger = \Phi^\dagger\left(f^\dagger, j^\dagger, g^\dagger\right)$ of (1.1)–(1.3) belonging to $H^2(\Omega)$, the estimate (4.7) shows that $0 \le \alpha^{h_n}_{f^\dagger, j^\dagger},\ \beta^{h_n}_{f^\dagger, g^\dagger} \le Ch_n$. Therefore, in view of (4.11), the above convergences (i) and (ii) are obtained if the sequence (ρ_n) is chosen such that

$$\rho_n \to 0,\ \frac{\delta_n}{\sqrt{\rho_n}} \to 0 \text{ and } \frac{h_n}{\sqrt{\rho_n}} \to 0 \text{ as } n \to \infty.$$

By regularity theory for elliptic boundary value problems, the regularity assumption $\Phi^\dagger \in H^2(\Omega)$ is satisfied if the diffusion matrix $Q \in C^{0,1}(\Omega)^{d\times d}$, $j^\dagger \in H^{1/2}(\partial\Omega)$, $g^\dagger \in H^{3/2}(\partial\Omega)$ and either $\partial\Omega$ is smooth of the class $C^{0,1}$ or the domain Ω is convex (see, for example, [21, 43]).

Proof of Theorem 4.5. We have from the optimality of f_n that

$$\mathcal{J}^{h_n}_{\delta_n}(f_n) + \rho_n \|f_n - f^*\|^2_{L^2(\Omega)} \le \mathcal{J}^{h_n}_{\delta_n}(f^\dagger) + \rho_n \|f^\dagger - f^*\|^2_{L^2(\Omega)}. \tag{4.12}$$

Since at $f^\dagger$ there holds the equation $\mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$, we infer from Lemma 4.2 that

$$\begin{aligned}
&\mathcal{J}^{h_n}_{\delta_n}(f^\dagger) \le C \left\| \mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n} \right\|^2_{H^1(\Omega)} \\
&= C \left\| \mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger + \mathcal{D}^{h_n}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n} + \mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger + \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger \right\|^2_{H^1(\Omega)} \\
&\le C \Bigg(\left\| \mathcal{N}^{h_n}_{f^\dagger} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger \right\|_{H^1(\Omega)} + \left\| \mathcal{D}^{h_n}_{f^\dagger} g_{\delta_n} - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger \right\|_{H^1(\Omega)} \\
&\quad + \left\| \mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger \right\|_{H^1(\Omega)} + \left\| \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}^{h_n}_{f^\dagger} g^\dagger \right\|_{H^1(\Omega)} \Bigg)^2 \\
&\le C \left(\|j_{\delta_n} - j^\dagger\|_{H^{-1/2}(\partial\Omega)} + \|g_{\delta_n} - g^\dagger\|_{H^{1/2}(\partial\Omega)} + \alpha^{h_n}_{f^\dagger, j^\dagger} + \beta^{h_n}_{f^\dagger, g^\dagger} \right)^2 \le C \left(\delta_n^2 + \left(\alpha^{h_n}_{f^\dagger, j^\dagger}\right)^2 + \left(\beta^{h_n}_{f^\dagger, g^\dagger}\right)^2 \right)
\end{aligned} \tag{4.13}$$

which implies from (4.12) that

$$\lim_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(f_n) = 0 \tag{4.14}$$

and, by the assumption (4.11),

$$\limsup_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)}. \tag{4.15}$$

So that the sequence (f_n) is bounded in the $L^2(\Omega)$-norm. A subsequence not relabelled and an element $\widehat{f} \in L^2(\Omega)$ exist such that (f_n) converges weakly in $L^2(\Omega)$ to $\widehat{f}$ and

$$\left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)}. \tag{4.16}$$

For any $f \in L^2(\Omega)$ we denote by $\mathcal{J}_0(f) := \int_\Omega Q\nabla\left(\mathcal{N}_f j^\dagger - \mathcal{D}_f g^\dagger\right) \cdot \nabla\left(\mathcal{N}_f j^\dagger - \mathcal{D}_f g^\dagger\right)$. By (1.7), we have

$$\left\| \mathcal{N}_{\widehat{f}} j^\dagger - \mathcal{D}_{\widehat{f}} g^\dagger \right\|^2_{H^1(\Omega)} \le \frac{1 + C_\Omega}{C_\Omega \underline{q}} \mathcal{J}_0\big(\widehat{f}\big). \tag{4.17}$$

Furthermore, applying Lemma 4.3, we have that $\mathcal{J}_0\big(\widehat{f}\big) \le \liminf_{n\to\infty} \mathcal{J}^{h_n}_{\delta_n}(f_n) = 0$, here we used (4.14). Combining this with (4.17), we get $\mathcal{N}_{\widehat{f}} j^\dagger = \mathcal{D}_{\widehat{f}} g^\dagger$ which infers $\widehat{f} \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$. Now we show $\widehat{f} = f^\dagger$ and the sequence (f_n) converges to $\widehat{f}$ in the $L^2(\Omega)$-norm. By the definition of the f^*-minimum-norm solution and (4.15)–(4.16), we get that

$$\left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)} \le \left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} \le \liminf_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \limsup_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)} \le \left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)}$$

and so that $\left\|f^\dagger - f^*\right\|^2_{L^2(\Omega)} = \left\|\widehat{f} - f^*\right\|^2_{L^2(\Omega)} = \lim_{n\to\infty} \|f_n - f^*\|^2_{L^2(\Omega)}$. By the uniqueness of the minimum-norm solution and the sequence (f_n) weakly converging in $L^2(\Omega)$ to $\widehat{f}$, we conclude that $\widehat{f} = f^\dagger$ and the sequence (f_n) in fact converges in the $L^2(\Omega)$-norm to $\widehat{f}$.

Finally, we show the sequences $(\mathcal{N}^{h_n}_{f_n} j_{\delta_n})$ and $(\mathcal{D}^{h_n}_{f_n} g_{\delta_n})$ converge to $\Phi^\dagger = \mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$ in the $H^1(\Omega)$-norm. Indeed, by Lemma 4.2, we obtain that

$$\begin{aligned}
\left\| \mathcal{N}^{h_n}_{f_n} j_{\delta_n} - \mathcal{N}_{f^\dagger} j^\dagger \right\|_{H^1(\Omega)} &\le \left\| \mathcal{N}^{h_n}_{f_n} j_{\delta_n} - \mathcal{N}^{h_n}_{f^\dagger} j^\dagger \right\|_{H^1(\Omega)} + \left\| \mathcal{N}^{h_n}_{f^\dagger} j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger \right\|_{H^1(\Omega)} \\
&\le C \left(\left\|j_{\delta_n} - j^\dagger\right\|_{H^{-1/2}(\partial\Omega)} + \left\|f_n - f^\dagger\right\|_{L^2(\Omega)} + \alpha^{h_n}_{f^\dagger, j^\dagger} \right) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Similarly, we also get $\left\| \mathcal{D}^{h_n}_{f_n} g_{\delta_n} - \mathcal{D}_{f^\dagger} g^\dagger \right\|_{H^1(\Omega)} \le C \left(\left\|g_{\delta_n} - g^\dagger\right\|_{H^{1/2}(\partial\Omega)} + \left\|f_n - f^\dagger\right\|_{L^2(\Omega)} + \beta^{h_n}_{f^\dagger, g^\dagger} \right) \to 0$ as n tends to ∞, which finishes the proof. □

5 Convergence rates

In this section we investigate convergence rates for Tikhonov regularization of our identification problem. Let us start with the following remark concerning the unique f^*-minimum-norm solution of the identification problem.

At this point it should be noted that due to $\mathbf{L}(w) = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger$ for $j = j^\dagger$ and $g = g^\dagger$ the condition (1.16) is a *range-type source condition* in the sense that $f - f^*$ belongs to the range of the affine linear forward operator $\mathbf{L}$. In the standard model of Lavrentiev regularization with linear forward operator this the typical source condition for obtaining the optimal convergence rate $\|f_{\rho,\delta} - f\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta})$ (cf. [42] and [25]) and that such a source condition can only hold if f is an f^*-minimum norm solution. The latter assertion will be confirmed for our modified model by the subsequent remark.

Remark 5.1. Assume that $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ and a function $w \in L^2(\Omega)$ exists such that $f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger$. Then f is the *unique* f^*-minimum-norm solution of the problem $(\mathcal{IP} - MN)$.

Indeed, we have with $\xi \in \left\{\xi \in L^2(\Omega) \mid \mathcal{N}_\xi j^\dagger = \mathcal{D}_\xi g^\dagger\right\}$ that

$$\begin{aligned}
&\frac{1}{2}\|\xi - f^*\|^2_{L^2(\Omega)} - \frac{1}{2}\|f - f^*\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\|\xi - f\|^2_{L^2(\Omega)} + (f - f^*, \xi - f) \geq (f - f^*, \xi - f) = \left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger, \xi - f\right) \\
&= \left(\xi, \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) - \left(f, \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) \\
&= \int_\Omega Q\nabla\mathcal{N}_\xi j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) - \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right)\right\rangle \\
&\quad - \int_\Omega Q\nabla\mathcal{N}_f j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) + \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right)\right\rangle \\
&= \int_\Omega Q\nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right).
\end{aligned}$$

Since $\gamma\mathcal{N}_\xi j^\dagger = \gamma\mathcal{N}_f j^\dagger = g^\dagger$, it follows that $\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger \in H^1_0(\Omega)$. We thus obtain from the last inequality

$$\begin{aligned}
\frac{1}{2}\|\xi - f^*\|^2_{L^2(\Omega)} - \frac{1}{2}\|f - f^*\|^2_{L^2(\Omega)} &\geq \int_\Omega Q\nabla\mathcal{N}_w j^\dagger \cdot \nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) - \int_\Omega Q\nabla\mathcal{D}_w g^\dagger \cdot \nabla\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) \\
&= \left(w, \mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) + \left\langle j^\dagger, \gamma\left(\mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right)\right\rangle - \left(w, \mathcal{N}_\xi j^\dagger - \mathcal{N}_f j^\dagger\right) = 0,
\end{aligned}$$

which finishes the proof.

We are now in a position to state the main result of this section.

Theorem 5.2. *Assume that for $f \in \mathcal{I}\left(j^\dagger, g^\dagger\right)$ there exists a function $w \in L^2(\Omega)$ such that the source condition*

$$f - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger \tag{5.1}$$

holds true. Then, according to Remark 5.1, f is the uniquely determined f^*-minimum-norm solution $f^\dagger$ and minimizer of the problem $(\mathcal{IP} - MN)$. Moreover, we have the error estimate and convergence rate

$$\begin{aligned}
&\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{D}^h_{f^h} g_\delta\right\|^2_{H^1(\Omega)} + \rho\left\|f^h - f^\dagger\right\|^2_{L^2(\Omega)} \\
&\quad = \mathcal{O}\left(\delta^2 + \left(\alpha^h_{f^\dagger, j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger, g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \delta\rho + \rho^2\right),
\end{aligned} \tag{5.2}$$

where $f^h := f^h_{\rho,\delta}$ is the unique minimizer of $\left(\mathcal{P}^h_{\rho,\delta}\right)$ and $\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger$ is the unique weak solution to the Dirichlet problem

$$-\nabla \cdot (Q\nabla v) = 0 \text{ in } \Omega \text{ and } v = \gamma\mathcal{N}_w j^\dagger - g^\dagger \text{ on } \partial\Omega$$

and $\alpha^h_{f^\dagger, j^\dagger}$, $\beta^h_{f^\dagger, g^\dagger}$, $\varrho^h_{\mathcal{N}_w j^\dagger}$, $\varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}$ and $\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}$ come from (4.3) and (4.5).

Remark 5.3. In case (cf. Remark 4.6) $\mathcal{N}_{f^\dagger} j^\dagger, \mathcal{N}_w j^\dagger, \mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger \in H^2(\Omega)$, by (4.4) and (4.7), we have

$$0 \le \alpha^h_{f^\dagger,j^\dagger}, \beta^h_{f^\dagger,g^\dagger}, \varrho^h_{\mathcal{N}_w j^\dagger}, \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}, \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} \le Ch$$

and so that the convergence rate

$$\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{D}^h_{f^h} g_\delta\right\|^2_{H^1(\Omega)} + \rho\|f^h - f^\dagger\|^2_{L^2(\Omega)} = \mathcal{O}\left(\delta^2 + h^2 + h\rho + \delta\rho + \rho^2\right)$$

is obtained.

Remark 5.4. Let $\Phi^\dagger = \Phi^\dagger(f^\dagger, j^\dagger, g^\dagger)$ be the weak solution of (1.1)–(1.3). Then the convergence rate

$$\begin{aligned}&\left\|\mathcal{N}^h_{f^h} j_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} + \left\|\mathcal{D}^h_{f^h} g_\delta - \Phi^\dagger\right\|^2_{H^1(\Omega)} \\ &= \mathcal{O}\left(\delta^2\rho^{-1} + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2\rho^{-1} + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\rho^{-1} + \varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \delta + \rho + \alpha^h_{f^\dagger,j^\dagger} + \beta^h_{f^\dagger,g^\dagger}\right)\end{aligned}$$

is also established. Indeed, the desired equation directly follows from (5.2) and the following inequalities

$$\begin{aligned}\left\|\mathcal{N}^h_{f^h} j_\delta - \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} &\le C\left(\|j_\delta - j^\dagger\|_{H^{-1/2}(\partial\Omega)} + \|f^h - f^\dagger\|_{L^2(\Omega)} + \alpha^h_{f^\dagger,j^\dagger}\right) \\ &\le C\left(\delta + \|f^h - f^\dagger\|_{L^2(\Omega)} + \alpha^h_{f^\dagger,j^\dagger}\right)\end{aligned}$$

and $\left\|\mathcal{D}^h_{f^h} g_\delta - \mathcal{D}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)} \le C\left(\delta + \|f^h - f^\dagger\|_{L^2(\Omega)} + \beta^h_{f^\dagger,g^\dagger}\right)$, here we used Lemma 4.2.

Proof of Theorem 5.2. In view of (4.13) we first have that

$$\mathcal{J}^h_\delta(f^\dagger) \le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right).$$

We have from the optimality of f^h that

$$\mathcal{J}^h_\delta(f^h) + \rho\|f^h - f^*\|^2_{L^2(\Omega)} \le \mathcal{J}^h_\delta(f^\dagger) + \rho\|f^\dagger - f^*\|^2_{L^2(\Omega)}$$

which yields

$$\begin{aligned}\rho\|f^h - f^*\|_{L^2(\Omega)} &\le C\rho^{1/2}\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right)^{1/2} + \rho\|f^\dagger - f^*\|_{L^2(\Omega)} \\ &\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\right). \qquad (5.3)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}^h_\delta(f^h) + \rho\|f^h - f^\dagger\|^2_{L^2(\Omega)} &\le \mathcal{J}^h_\delta(f^\dagger) + \rho\left(\|f^\dagger - f^*\|^2_{L^2(\Omega)} - \|f^h - f^*\|^2_{L^2(\Omega)} + \|f^h - f^\dagger\|^2_{L^2(\Omega)}\right) \\ &\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2\right) + 2\rho\left(f^\dagger - f^*, f^\dagger - f^h\right). \qquad (5.4)\end{aligned}$$

Since $\mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$, it follows from (5.1) that

$$\begin{aligned}\left(f^\dagger - f^*, f^\dagger - f^h\right) &= \left(f^\dagger - f^h, \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) \\ &= \left(f^\dagger - f^h, \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger + \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) \\ &= \left(f^\dagger - f^h, \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) + \left(f^\dagger - f^h, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right). \qquad (5.5)\end{aligned}$$

By (1.10), we infer

$$\left(f^\dagger, \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) = \int_\Omega Q\nabla\mathcal{N}_{f^\dagger} j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) - \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right)\right\rangle$$

and

$$\left(f^h, \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) = \int_\Omega Q\nabla \mathcal{N}_{f^h} j^\dagger \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) - \left\langle j^\dagger, \gamma\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right)\right\rangle$$

which follow that

$$\begin{aligned}
&\left(f^\dagger - f^h, \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) \\
&= \int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{N}_{f^h} j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) \\
&= \int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) + \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{N}_{f^h} j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right). \qquad (5.6)
\end{aligned}$$

Since $\gamma\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) = 0$, it follows from (1.11) that

$$\begin{aligned}
\left(f^\dagger - f^h, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) &= \left(f^\dagger, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) - \left(f^h, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) \\
&= \int_\Omega Q\nabla \mathcal{D}_{f^\dagger} g^\dagger \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) - \int_\Omega Q\nabla \mathcal{D}_{f^h} g^\dagger \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right). \qquad (5.7)
\end{aligned}$$

We thus infer from (5.5)–(5.7) that

$$\begin{aligned}
&\left(f^\dagger - f^*, f^\dagger - f^h\right) \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{N}_{f^h} j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) + \int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right). \qquad (5.8)
\end{aligned}$$

We note again that $\mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger$ and $\gamma\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) = 0$. Then, together with (1.10) and (1.11), the last two term in the right hand side of (5.8) is written as

$$\begin{aligned}
&\int_\Omega Q\nabla\left(\mathcal{N}_{f^\dagger} j^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right) + \int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_w g^\dagger\right) \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger\right) \\
&= \int_\Omega Q\nabla \mathcal{N}_w j^\dagger \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) - \int_\Omega Q\nabla \mathcal{D}_w g^\dagger \cdot \nabla\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) \\
&= \left(w, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) + \left\langle j^\dagger, \gamma\left(\mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right)\right\rangle - \left(w, \mathcal{D}_{f^\dagger} g^\dagger - \mathcal{D}_{f^h} g^\dagger\right) = 0.
\end{aligned}$$

Thus, we obtain from (5.8) that

$$\left(f^\dagger - f^*, f^\dagger - f^h\right) = \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{N}_{f^h} j^\dagger\right) \cdot \nabla\left(\mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger\right).$$

Next, for simplicity we denote by $W = \mathcal{N}_w j^\dagger - \mathcal{N}_{f^\dagger} j^\dagger$ and note

$$\gamma W = \gamma \mathcal{N}_w j^\dagger - g^\dagger. \qquad (5.9)$$

Then we have that

$$\begin{aligned}
\left(f^\dagger - f^*, f^\dagger - f^h\right) &= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{N}_{f^h} j^\dagger\right) \cdot \nabla W \\
&= \int_\Omega Q\nabla\left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right) \cdot \nabla W - \int_\Omega Q\nabla\left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right) \cdot \nabla W \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g_\delta\right) \cdot \nabla W - \int_\Omega Q\nabla\left(\mathcal{N}^h_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j_\delta\right) \cdot \nabla W \\
&\quad + \int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h} g_\delta - \mathcal{N}^h_{f^h} j_\delta\right) \cdot \nabla W := I_1 + I_2 + I_3. \qquad (5.10)
\end{aligned}$$

We can write

$$\begin{aligned}
&\int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla W\\
&= \int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \mathcal{D}_0\gamma W + \int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right)\\
&\quad + \int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \left(W - \mathcal{D}_0\gamma W - \Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right)\right).
\end{aligned}$$

Since $\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger \in H^1_0(\Omega)$, we then get that

$$\int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \mathcal{D}_0\gamma W = \int_\Omega Q\nabla \mathcal{D}_0\gamma W\cdot \nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right) = 0.$$

Since

$$\gamma\left(W - \mathcal{D}_0\gamma W\right) = \gamma W - \gamma \mathcal{D}_0\gamma W = \gamma W - \gamma W = 0,$$

we infer $\Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right) \in \mathcal{V}^h_{1,0} = \mathcal{V}^h_1 \cap H^1_0(\Omega)$ and then obtain from (1.11) and (3.3) that

$$\int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right) = 0.$$

Hence we have

$$\begin{aligned}
&\left|\int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla W\right|\\
&= \left|\int_\Omega Q\nabla \left(\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right)\cdot \nabla \left(W - \mathcal{D}_0\gamma W - \Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right)\right)\right|\\
&\le C\left\|\mathcal{D}_{f^h} g^\dagger - \mathcal{D}^h_{f^h} g^\dagger\right\|_{H^1(\Omega)} \left\|W - \mathcal{D}_0\gamma W - \Pi^h_\diamond \left(W - \mathcal{D}_0\gamma W\right)\right\|_{H^1(\Omega)}\\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right)\left(\left\|W - \Pi^h_\diamond W\right\|_{H^1(\Omega)} + \left\|\mathcal{D}_0\gamma W - \Pi^h_\diamond \mathcal{D}_0\gamma W\right\|_{H^1(\Omega)}\right)\\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right)\\
&\quad \cdot\left(\left\|\mathcal{N}_w j^\dagger - \Pi^h_\diamond \mathcal{N}_w j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{N}_{f^\dagger} j^\dagger - \Pi^h_\diamond \mathcal{N}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} + \left\|\mathcal{D}_0\gamma W - \Pi^h_\diamond \mathcal{D}_0\gamma W\right\|_{H^1(\Omega)}\right)\\
&= C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|g^\dagger\right\|_{H^{1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right),
\end{aligned} \tag{5.11}$$

here we used (5.9). Similarly, since $\Pi^h_\diamond W \in \mathcal{V}^h_\diamond$ and by (1.10) and (3.1), we get that

$$\begin{aligned}
&\left|\int_\Omega Q\nabla \left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot \nabla W\right|\\
&= \left|\int_\Omega Q\nabla \left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot \nabla \Pi^h_\diamond W + \int_\Omega Q\nabla \left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot \nabla \left(W - \Pi^h_\diamond W\right)\right|\\
&= \left|\int_\Omega Q\nabla \left(\mathcal{N}_{f^h} j^\dagger - \mathcal{N}^h_{f^h} j^\dagger\right)\cdot \nabla \left(W - \Pi^h_\diamond W\right)\right|\\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|j^\dagger\right\|_{H^{-1/2}(\partial\Omega)}\right)\left\|W - \Pi^h_\diamond W\right\|_{H^1(\Omega)}\\
&\le C\left(\left\|f^h\right\|_{L^2(\Omega)} + \left\|j^\dagger\right\|_{H^{-1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger} j^\dagger}\right).
\end{aligned} \tag{5.12}$$

Combining (5.11) with (5.12), we obtain, by (5.3), that

$$
\begin{aligned}
\rho|I_1| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g^\dagger\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j^\dagger\right)\cdot\nabla W\right|\\
&\le C\rho\left(\|f^h\|_{L^2(\Omega)} + \|g^\dagger\|_{H^{1/2}(\partial\Omega)} + \|j^\dagger\|_{H^{-1/2}(\partial\Omega)}\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right)\\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\right)\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right)\\
&\quad + C\rho\left(\varrho^h_{\mathcal{N}_w j^\dagger} + \varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right)\\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger}\right). \qquad (5.13)
\end{aligned}
$$

Now, using Lemma 4.2, we arrive at

$$
\begin{aligned}
\rho|I_2| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g_\delta\right)\cdot\nabla W - \int_\Omega Q\nabla\left(\mathcal{N}^h_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j_\delta\right)\cdot\nabla W\right|\\
&\le C\rho\left(\left\|\mathcal{D}^h_{f^h}g^\dagger - \mathcal{D}^h_{f^h}g_\delta\right\|_{H^1(\Omega)} + \left\|\mathcal{N}^h_{f^h}j^\dagger - \mathcal{N}^h_{f^h}j_\delta\right\|_{H^1(\Omega)}\right)\\
&\le C\rho\left(\|g_\delta - g^\dagger\|_{H^{1/2}(\partial\Omega)} + \|j_\delta - j^\dagger\|_{H^{-1/2}(\partial\Omega)}\right) \le C\delta\rho. \qquad (5.14)
\end{aligned}
$$

Since for a.e. in Ω the matrix $Q(x)$ is positive definite, the root $Q(x)^{1/2}$ is then well defined. Thus, using the Cauchy-Schwarz inequality and Young's inequality, we have that

$$
\begin{aligned}
\rho|I_3| &= \rho\left|\int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot\nabla W\right|\\
&= \rho\left|\int_\Omega Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot Q^{1/2}\nabla W\right|\\
&\le \rho\left(\int_\Omega Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot Q^{1/2}\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\right)^{1/2}\left(\int_\Omega Q^{1/2}\nabla W\cdot Q^{1/2}\nabla W\right)^{1/2}\\
&= \rho\left(\int_\Omega Q\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\cdot\nabla\left(\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right)\right)^{1/2}\left(\int_\Omega Q\nabla W\cdot\nabla W\right)^{1/2}\\
&\le C\rho\left(\mathcal{J}^h_\delta(f^h)\right)^{1/2} \le C^2\rho^2 + \frac{1}{4}\mathcal{J}^h_\delta(f^h) \le C\rho^2 + \frac{1}{4}\mathcal{J}^h_\delta(f^h). \qquad (5.15)
\end{aligned}
$$

It follows from (5.10) and (5.13)–(5.15) that

$$
\begin{aligned}
&2\rho\left(f^\dagger - f^*, f^\dagger - f^h\right)\\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \rho\delta + \rho^2\right) + \frac{1}{2}\mathcal{J}^h_\delta(f^h)
\end{aligned}
$$

which together with (5.4) follows that

$$
\begin{aligned}
&\frac{1}{2}\mathcal{J}^h_\delta(f^h) + \rho\|f^h - f^\dagger\|^2_{L^2(\Omega)}\\
&\le C\left(\delta^2 + \left(\alpha^h_{f^\dagger,j^\dagger}\right)^2 + \left(\beta^h_{f^\dagger,g^\dagger}\right)^2 + \rho\varrho^h_{\mathcal{N}_w j^\dagger} + \rho\varrho^h_{\mathcal{N}_{f^\dagger}j^\dagger} + \rho\varrho^h_{\mathcal{D}_0\gamma\mathcal{N}_w j^\dagger - g^\dagger} + \rho\delta + \rho^2\right). \qquad (5.16)
\end{aligned}
$$

Since $\left\|\mathcal{D}^h_{f^h}g_\delta - \mathcal{N}^h_{f^h}j_\delta\right\|^2_{H^1(\Omega)} \le C\mathcal{J}^h_\delta(f^h)$, (5.2) now directly follows from (5.16), which finishes the proof. □

For showing the optimality character of the rate result in Theorem 5.2 it is of interest to work out the rate situation in the continuous (non-discretized, $h = 0$) but noisy ($\delta > 0$) case. We will do this in the following corollary. The result is an immediate consequence of formula (5.2) if we omit the misfit term on the left-hand side, divide both sides by ρ and take the square root.

Corollary 5.5. *Assume that, for the f^*-minimum-norm solution $f^\dagger$, there exists a function $w \in L^2(\Omega)$ such that the source condition $f^\dagger - f^* = \mathcal{N}_w j^\dagger - \mathcal{D}_w g^\dagger$ holds true. Then we have the convergence rate*

$$\|f_{\rho,\delta} - f^\dagger\|_{L^2(\Omega)} = \mathcal{O}(\sqrt{\delta}) \qquad as \qquad \delta \to 0$$

whenever the regularization parameter ρ is chosen a priori as

$$\underline{c}\,\delta \le \rho(\delta) \le \overline{c}\,\delta$$

with some constants $0 < \underline{c} \le \overline{c} < \infty$.

6 Conjugate gradient method

In this section we will utilize the conjugate gradient (CG) method (see, for example, [22, 27]) to find the minimizes of the strictly convex, discrete regularized problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$.

Let

$$\nabla \Upsilon^h_{\rho,\delta}(f) = 2\left(\mathcal{N}^h_f j_\delta - \mathcal{D}^h_f g_\delta\right) + 2\rho(f - f^*)$$

be the L^2-gradient of the cost function $\Upsilon^h_{\rho,\delta}$ at f (see Proof of Theorem 3.2), where $f^* \in \mathcal{V}^h_1$. Then the sequence of iterates via this algorithm is generated by $f^0 \in L^2(\Omega) \cap \mathcal{V}^h_1$ and, for $k \ge 0$

$$f^{k+1} := f^k + t^k d^k,$$

where

$$d^k := \begin{cases} -\nabla \Upsilon^h_{\rho,\delta}(f^k) & \text{if } k = 0, \\ -\nabla \Upsilon^h_{\rho,\delta}(f^k) + \beta^k d^{k-1} & \text{if } k > 0 \end{cases} \text{ with } \beta^k := \frac{\|\nabla \Upsilon^h_{\rho,\delta}(f^k)\|^2}{\|\nabla \Upsilon^h_{\rho,\delta}(f^{k-1})\|^2} \text{ and } t^k := \arg\min_{t \ge 0} \Upsilon^h_{\rho,\delta}(f^k + t d^k).$$

A short computation shows that

$$\begin{aligned} t^k &= -\frac{\int_\Omega Q\nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) \cdot \nabla\left(\mathcal{N}^h_{f^k} j_\delta - \mathcal{D}^h_{f^k} g_\delta\right) + \rho\left(d^k, f^k - f^*\right)}{\int_\Omega Q\nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) \cdot \nabla\left(\mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\,\|d^k\|^2_{L^2(\Omega)}} \\ &= -\frac{1}{2}\frac{\left(d^k, \nabla \Upsilon^h_{\rho,\delta}(f^k)\right)}{\left(d^k, \mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\,\|d^k\|^2_{L^2(\Omega)}}. \end{aligned}$$

Consequently, the CG method then reads as follows: giving an initial approximation $f^0 \in \mathcal{V}^h_1$, number of iterations N and a positive constants τ_1, τ_2. Computing

$$\nabla \Upsilon^h_{\rho,\delta}(f^0) = 2\left(\mathcal{N}^h_{f^0} j_\delta - \mathcal{D}^h_{f^0} g_\delta\right) + 2\rho(f^0 - f^*),\ d^0 = -\nabla \Upsilon^h_{\rho,\delta}(f^0),\ t^0 = \frac{1}{2}\frac{\|d^0\|^2_{L^2(\Omega)}}{\left(d^0, \mathcal{N}^h_{d^0}0 - \mathcal{D}^h_{d^0}0\right) + \rho\,\|d^0\|^2_{L^2(\Omega)}}$$

and setting

$$f^1 = f^0 + t^0 d^0 \text{ and } k = 1, \quad \text{Tolerance} := \left\|\nabla \Upsilon^h_{\rho,\delta}\left(f^k\right)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2\left\|\nabla \Upsilon^h_{\rho,\delta}\left(f^0\right)\right\|_{L^2(\Omega)}.$$

while $(Tolerance > 0)$ & $(k \le N)$ **do**

$$\overline{r} = \left\|\nabla \Upsilon^h_{\rho,\delta}(f^{k-1})\right\|^2_{L^2(\Omega)}, \quad r = \left\|\nabla \Upsilon^h_{\rho,\delta}(f^{k})\right\|^2_{L^2(\Omega)}, \quad \beta^k = \frac{r}{\overline{r}}$$

$$d^k = -\nabla \Upsilon^h_{\rho,\delta}(f^k) + \beta^k d^{k-1}, \quad t^k = -\frac{1}{2}\frac{\left(d^k, \nabla \Upsilon^h_{\rho,\delta}(f^k)\right)}{\left(d^k, \mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\left\|d^k\right\|^2_{L^2(\Omega)}}$$

if $t^k \ge 0$ **then**

$$f^{k+1} = f^k + t^k d^k$$

else

$$d^k = -\nabla \Upsilon^h_{\rho,\delta}(f^k), \quad t^k = \frac{1}{2}\frac{\left\|d^k\right\|^2_{L^2(\Omega)}}{\left(d^k, \mathcal{N}^h_{d^k}0 - \mathcal{D}^h_{d^k}0\right) + \rho\left\|d^k\right\|^2_{L^2(\Omega)}}$$

$$f^{k+1} = f^k + t^k d^k$$

end

$$k := k+1, \quad \text{Tolerance} := \left\|\nabla \Upsilon^h_{\rho,\delta}\big(f^k\big)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2\left\|\nabla \Upsilon^h_{\rho,\delta}\big(f^0\big)\right\|_{L^2(\Omega)}$$

end

Printing $(Tolerance, k, f^k)$

Algorithm 1: CG iteration

7 Numerical test

In this section we illustrate the theoretical result with a numerical example. For this purpose we consider the the boundary value problem

$$-\nabla \cdot \big(Q\nabla\Phi\big) = f^\dagger \text{ in } \Omega, \tag{7.1}$$

$$Q\nabla\Phi \cdot \vec{n} = j^\dagger \text{ on } \partial\Omega \text{ and} \tag{7.2}$$

$$\Phi = g^\dagger \text{ on } \partial\Omega \tag{7.3}$$

with $\Omega = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1, x_2 < 1\}$. We assume that entries of the known symmetric diffusion matrix Q are discontinuous which are defined as

$$q_{11} = 3\chi_{\Omega_{11}} + \chi_{\Omega\setminus\Omega_{11}}, \; q_{12} = \chi_{\Omega_{12}}, \; q_{22} = 4\chi_{\Omega_{22}} + 2\chi_{\Omega\setminus\Omega_{22}},$$

where χ_D is the characteristic function of the Lebesgue measurable set D and

$\Omega_{11} := \big\{(x_1, x_2) \in \Omega \;\big|\; |x_1| \le 1/2 \text{ and } |x_2| \le 1/2\big\}$, $\Omega_{12} := \big\{(x_1, x_2) \in \Omega \;\big|\; |x_1| + |x_2| \le 1/2\big\}$ and
$\Omega_{22} := \big\{(x_1, x_2) \in \Omega \;\big|\; x_1^2 + x_2^2 \le 1/4\big\}$.

The Neumann boundary condition $j^\dagger \in H^{-1/2}(\partial\Omega)$ in (7.2) is chosen with

$$j^\dagger(x) = \begin{cases} 5\sin \pi x_1 & \text{if } -1 \le x_1 \le 1 \text{ and } x_2 = \pm 1, \\ 3\cos \pi x_2 & \text{if } x_1 = \pm 1 \text{ and } -1 < x_2 < 1. \end{cases}$$

The identified source function $f^\dagger \in L^2(\Omega)$ in (7.1) is assumed to be discontinuous and defined as

$$f^\dagger = 2\chi_{\Omega_1} - \chi_{\Omega_2} + \frac{5\pi}{7\pi - 192}\chi_{\Omega\setminus(\Omega_1\cup\Omega_2)},$$

where

$$\Omega_1 := \big\{(x_1, x_2) \in \Omega \;\big|\; 9(x_1 + 1/2)^2 + 16(x_2 - 1/2)^2 \le 1\big\} \text{ and}$$
$$\Omega_2 := \big\{(x_1, x_2) \in \Omega \;\big|\; (x_1 - 1/2)^2 + (x_2 + 1/2)^2 \le 1/16\big\}.$$

Then the Dirichlet boundary condition $g^\dagger$ in (7.3) is defined as $g^\dagger = \gamma \mathcal{N}_{f^\dagger} j^\dagger$ with $\mathcal{N}_{f^\dagger} j^\dagger$ being the unique weak solution to the Neumann problem (7.1)–(7.2).

For the discretization we divide the interval $(-1,1)$ into ℓ equal segments and so that the domain $\Omega = (-1,1)^2$ is divided into $2\ell^2$ triangles, where the diameter of each triangle is $h_\ell = \frac{\sqrt{8}}{\ell}$. In the minimization problem $\left(\mathcal{P}^h_{\rho,\delta}\right)$ we take $h = h_\ell$ and $\rho = \rho_\ell = 0.001 h_\ell$. For observations with noise we assume that

$$(j_{\delta_\ell}, g_{\delta_\ell}) = \left(j^\dagger + \overline{\delta_\ell} \cdot \overline{j}, g^\dagger + \overline{\delta_\ell} \cdot \overline{g}\right),$$

where $\overline{\delta_\ell} = h_\ell \sqrt{\rho_\ell}$ and

$$\overline{j}(x) = \begin{cases} \cos 2\pi x_1 & \text{if } -1 < x_1 < 1,\ x_2 = \pm 1, \\ \cos 5\pi x_2 & \text{if } x_1 = \pm 1,\ -1 \le x_2 \le 1 \end{cases} \text{ and } \overline{g}(x) = \begin{cases} \sin 3\pi x_1 & \text{if } -1 < x_1 < 1,\ x_2 = \pm 1, \\ \sin 4\pi x_2 & \text{if } x_1 = \pm 1,\ -1 \le x_2 \le 1. \end{cases}$$

The measurement error is then computed as $\delta_\ell = \left\|j_{\delta_\ell} - j^\dagger\right\|_{L^2(\partial\Omega)} + \left\|g_{\delta_\ell} - g^\dagger\right\|_{L^2(\partial\Omega)}$.

We use Algorithm 1 which is described in the last paragraph of Section 6 for computing the numerical solution of the problem $\left(\mathcal{P}^{h_\ell}_{\rho_\ell,\delta_\ell}\right)$. We start with the coarsest level $\ell = 6$. As an a-priori estimate and the initial approximation we choose $f^* := 0$ and

$$f^0_\ell(x) := \begin{cases} -1 & \text{if } (x_1,x_2) \in [-1,0] \times [-1,1], \\ 1 & \text{if } (x_1,x_2) \in (0,1] \times [-1,1]. \end{cases}$$

At each iteration k we compute

$$\text{Tolerance} := \left\|\nabla \Upsilon^{h_\ell}_{\rho_\ell,\delta_\ell}\left(f^k_\ell\right)\right\|_{L^2(\Omega)} - \tau_1 - \tau_2 \left\|\nabla \Upsilon^{h_\ell}_{\rho_\ell,\delta_\ell}\left(f^0_\ell\right)\right\|_{L^2(\Omega)},$$

where $\tau_1 := 10^{-6} h_\ell^{1/2}$ and $\tau_2 := 10^{-4} h_\ell^{1/2}$. Then the iteration was stopped if Tolerance ≤ 0 or the number of iterations reached the maximum iteration count of 500.

After obtaining the numerical solution of the first iteration process with respect to the coarsest level $\ell = 6$, we use its interpolation on the next finer mesh $\ell = 12$ as an initial approximation f^0_ℓ for the algorithm on this finer mesh, and so on $\ell = 24, \ldots$

Let f_ℓ be the function which is obtained at *the final iterate* of Algorithm 1 corresponding to the refinement level ℓ. Furthermore, let $\mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$ and $\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell}$ denote *the computed numerical solution* to the Neumann problem

$$-\nabla \cdot (Q \nabla u) = f_\ell \text{ in } \Omega \text{ and } Q\nabla u \cdot \vec{n} = j_{\delta_\ell} \text{ on } \partial\Omega$$

and the Dirichlet problem

$$-\nabla \cdot (Q \nabla v) = f_\ell \text{ in } \Omega \text{ and } v = g_\ell \text{ on } \partial\Omega,$$

respectively. The notations $\mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger$ and $\mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger$ of *the exact numerical solutions* are to be understood similarly. We use the following abbreviations for the errors

$$L^2_f = \left\|f_\ell - f^\dagger\right\|_{L^2(\Omega)}, L^2_\mathcal{N} = \left\|\mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger\right\|_{L^2(\Omega)},\ H^1_\mathcal{N} = \left\|\mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger\right\|_{H^1(\Omega)} \text{ and}$$
$$L^2_\mathcal{D} = \left\|\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell} - \mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger\right\|_{L^2(\Omega)},\ H^1_\mathcal{D} = \left\|\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell} - \mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger\right\|_{H^1(\Omega)}.$$

The numerical results are summarized in Table 1 and Table 2, where we present the refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measured noise δ_ℓ, number of iterations, value of tolerances and errors L^2_f, $L^2_\mathcal{N}$, $L^2_\mathcal{D}$, $H^1_\mathcal{N}$ and $H^1_\mathcal{D}$. Their experimental order of convergence (EOC) is presented in Table 3, where

$$\text{EOC}_\Theta := \frac{\ln \Theta(h_1) - \ln \Theta(h_2)}{\ln h_1 - \ln h_2}$$

and $\Theta(h)$ is an error function with respect to the mesh size h.

All figures are here presented corresponding to $\ell = 96$. Figure 1 from left to right shows the graphs of the exact numerical states $\mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger$, $\mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger$ and the difference $\mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger - \mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger$. It is reminded in our setting that $\mathcal{N}_{f^\dagger} j^\dagger = \mathcal{D}_{f^\dagger} g^\dagger \in H^1_\diamond(\Omega)$. The computation shows

$$\left\|\mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger - \mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger\right\|_{L^\infty(\Omega)} = 9.0761\text{e-}15 \text{ while } \int_{\partial\Omega} \mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger = \int_{\partial\Omega} \mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger = -5.5164\text{e-}16.$$

Figure 2 from left to right we display the computed numerical states $\mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$, $\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell}$ of the algorithm at the final iteration and the difference $\mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell} - \mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$. The computation shows that

$$\left\|\mathcal{N}^{h_\ell}_{f_\ell} j_\ell - \mathcal{D}^{h_\ell}_{f_\ell} g_\ell\right\|_{L^\infty(\Omega)} = 1.2335\text{e-}7 \text{ while } \int_{\partial\Omega} \mathcal{N}^{h_\ell}_{f_\ell} j_\ell = -7.4073\text{e-}16, \int_{\partial\Omega} \mathcal{D}^{h_\ell}_{f_\ell} g_\ell = 1.2143\text{e-}17.$$

Figure 3 from left to right we perform the Lagrange interpolation $I^{h_\ell}_1 f^\dagger$ of the exact source, the computed numerical source f_ℓ of the algorithm at the final iteration, and the differences $\mathcal{N}^{h_\ell}_{f^\dagger} j^\dagger - \mathcal{N}^{h_\ell}_{f_\ell} j_{\delta_\ell}$ and $\mathcal{D}^{h_\ell}_{f^\dagger} g^\dagger - \mathcal{D}^{h_\ell}_{f_\ell} g_{\delta_\ell}$.

Convergence history					
ℓ	h_ℓ	ρ_ℓ	δ_ℓ	**Iterate**	**Tolerance**
6	0.4714	4.7140e-4	2.8949e-2	168	-3.6248e-5
12	0.2357	2.3570e-4	1.0235e-2	273	-9.6945e-7
24	0.1179	1.1785e-4	3.6186e-3	346	-2.6196e-7
48	0.0589	5.8926e-5	1.2794e-3	422	-9.6357e-8
96	0.0295	2.9463e-5	4.5233e-4	487	-1.0536e-9

Table 1: Refinement level ℓ, mesh size h_ℓ of the triangulation, regularization parameter ρ_ℓ, measured noise δ_ℓ, number of iterates and value of tolerances.

Convergence history					
ℓ	L^2_f	$L^2_\mathcal{N}$	$L^2_\mathcal{D}$	$H^1_\mathcal{N}$	$H^1_\mathcal{D}$
6	0.3854	5.0961e-3	5.0496e-3	1.7611e-2	1.7569e-2
12	0.1043	1.2819e-3	1.2783e-3	7.4580e-3	7.4509e-3
24	5.4161e-2	3.5054e-4	3.4928e-4	3.6375e-3	3.6246e-3
48	3.0747e-2	1.6778e-4	1.6766e-4	1.9741e-3	1.9720e-3
96	1.9757e-2	9.6405e-5	9.6378e-5	1.1375e-3	1.1367e-3

Table 2: Errors L^2_f, $L^2_\mathcal{N}$, $L^2_\mathcal{D}$, $H^1_\mathcal{N}$ and $H^1_\mathcal{D}$.

Experimental order of convergence					
ℓ	$\mathbf{EOC}_{L^2_f}$	$\mathbf{EOC}_{L^2_\mathcal{N}}$	$\mathbf{EOC}_{L^2_\mathcal{D}}$	$\mathbf{EOC}_{H^1_\mathcal{N}}$	$\mathbf{EOC}_{H^1_\mathcal{D}}$
6	–	–	–	–	–
12	1.8856	1.9911	1.9819	1.2396	1.2375
24	0.9454	1.8706	1.8718	1.0358	1.0396
48	0.8168	1.0630	1.0588	0.8818	0.8782
96	0.6381	0.7994	0.7988	0.7953	0.7948
Mean of EOC	1.0715	1.4310	1.4278	0.9881	0.9875

Table 3: Experimental order of convergence between finest and coarsest level for L^2_f, $L^2_\mathcal{N}$, $L^2_\mathcal{D}$, $H^1_\mathcal{N}$ and $H^1_\mathcal{D}$.

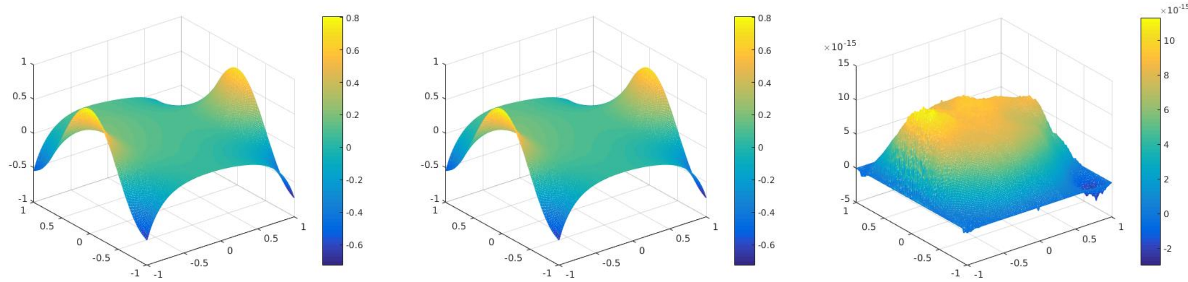

Figure 1: The exact numerical states $\mathcal{N}_{f^\dagger}^{h_\ell} j^\dagger$, $\mathcal{D}_{f^\dagger}^{h_\ell} g^\dagger$ and the difference $\mathcal{D}_{f^\dagger}^{h_\ell} g^\dagger - \mathcal{N}_{f^\dagger}^{h_\ell} j^\dagger$.

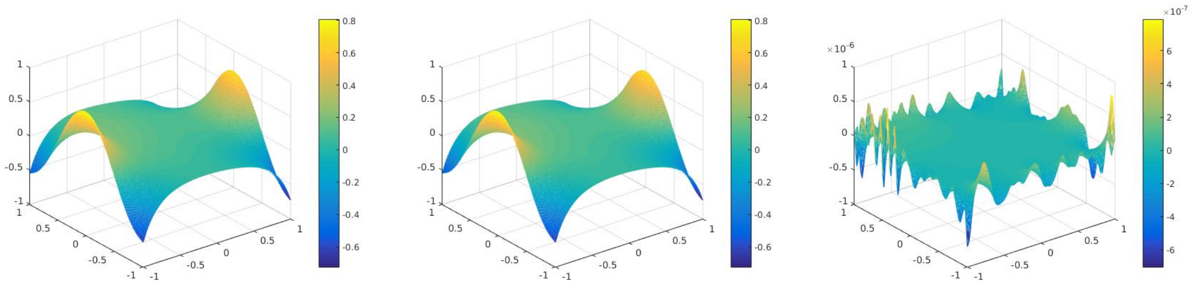

Figure 2: The computed numerical states $\mathcal{N}_{f_\ell}^{h_\ell} j_{\delta_\ell}$, $\mathcal{D}_{f_\ell}^{h_\ell} g_{\delta_\ell}$ of the algorithm at the final iteration and the difference $\mathcal{D}_{f_\ell}^{h_\ell} g_{\delta_\ell} - \mathcal{N}_{f_\ell}^{h_\ell} j_{\delta_\ell}$.

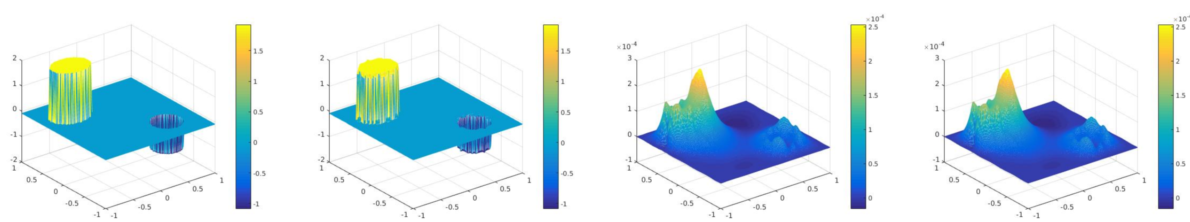

Figure 3: Interpolation $I_1^{h_\ell} f^\dagger$, computed numerical solution f_ℓ of the algorithm at the final iteration, and the differences $\mathcal{N}_{f^\dagger}^{h_\ell} j^\dagger - \mathcal{N}_{f_\ell}^{h_\ell} j_{\delta_\ell}$ and $\mathcal{D}_{f^\dagger}^{h_\ell} g^\dagger - \mathcal{D}_{f_\ell}^{h_\ell} g_{\delta_\ell}$.

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