# The best constant in Sobolev's inequality

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# Sobolev's inequality

Aim : compute the best constant $A > 0$ in Sobolev's inequality

$$\boxed{\|\varphi\|_{L^{2^*}(\mathbb{R}^d)} \leq A \|\nabla\varphi\|_{L^2(\mathbb{R}^d)}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d),}$$

where $d \geq 3$ and $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$.

- ineq. [Sobolev, 38], simplified by [Gagliardo, 58] and [Nirenberg, 59]
- Best constant : [Bliss, 30] for radial functions, [Rodemich, 66], [Aubin, 76] and [Talenti, 76] for the g'al case, by symetrization

$$\boxed{A^2 = \frac{4}{d(d-2)} |\mathbb{S}^d|^{-\frac{2}{d}}}$$

# Plan of the talk

1. Sobolev's inequality
2. The Caffarelli-Kohn-Nirenberg inequality
3. Bakry-Emery's $\Gamma$-calculus, curvature-dimension condition
4. Gradient flows in the Euclidean space $\mathbb{R}^m$
5. Wasserstein space, Otto's calculus
6. Gradient flows in the Wasserstein space: two Rényi entropies and a fast diffusion equation

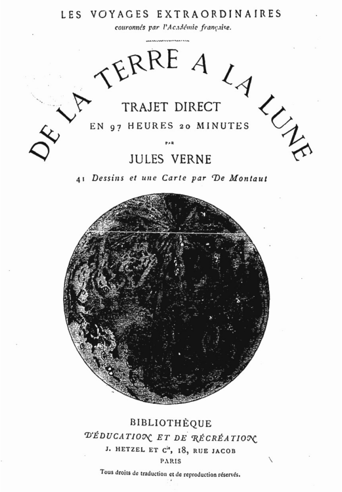
1. Sobolev's inequality

# Enters the sphere

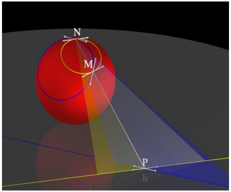
Recall that

$$A^2 = \frac{4}{d(d-2)} |\mathbb{S}^d|^{-\frac{2}{d}}$$

Why does the area of the sphere $|\mathbb{S}^d|$ enter in the optimal constant ?



# Stereographic projection




Source : Wikipedia.

The stereographic projection $\pi$ is a **conformal map**. In addition,

$$\boxed{g^{ij} = \frac{(1+|x|^2)^2}{4}\delta^{ij}}$$

# Using the conformal invariance of the inequality

Sobolev's inequality reads

$$\boxed{\left(\int_{\mathbb{S}^d} |v|^{2^*} d\mu\right)^{2/2^*} \leq \frac{4}{d(d-2)} \int_{\mathbb{S}^d} |\nabla_{\mathbb{S}^d} v|^2 d\mu + \int_{\mathbb{S}^d} |v|^2 d\mu.}$$

where $d\mu = \frac{1}{|\mathbb{S}^d|} d\mathrm{Vol}_g$ is the normalized volume on $\mathbb{S}^d$.

Since $2^* - 2 = \frac{2d}{d-2} - 2 = \frac{4}{d-2}$, we can rephrase as follows :

$$\boxed{\frac{||v||^2_{L^{2^*}(\mathbb{S}^d)} - ||v||^2_{L^2(\mathbb{S}^d)}}{2^* - 2} \leq \frac{1}{d} ||\nabla v||^2_{L^2(\mathbb{S}^d)}.}$$

# From Sobolev to Beckner inequalities

Sobolev's inequality can be seen as a limiting case of a family of interpolation inequalities [Bidaut-Véron, Véron, 91], [Beckner, 93], [Demange, 04] : given $q \in [1, 2^*]$, $q \neq 2$,

$$\boxed{\frac{1}{q-2}\left(\|v\|^2_{L^q(\mathbb{S}^d)} - \|v\|^2_{L^2(\mathbb{S}^d)}\right) \le \frac{1}{d}\|\nabla v\|^2_{L^2(\mathbb{S}^d)}.}$$

- When $q = 1$, this is Poincaré's inequality, achieved by $v(\omega) = \omega_1$
- When $q \neq 1$, in the limit $v = 1 + \epsilon w$, $\epsilon \to 0$, the inequality linearizes to Poincaré's inequality and so the constant $1/d$ is again optimal
- The limit $q \to 2$ yields the following log-Sobolev inequality

$$\int_{\mathbb{S}^d} v^2 \ln\left(\frac{v^2}{\|v\|^2_{L^2(\mathbb{S}^d)}}\right) d\mu \le \frac{1}{d}\|\nabla v\|^2_{L^2(\mathbb{S}^d)}.$$

# From the sphere to its curvature

## Theorem (Ilias, 83)

*Let $(M,g)$ be a smooth connected compact Riemannian manifold of dimension $d$, $d \geq 3$. Assume that its Ricci curvature is bounded below by $\rho > 0$. Then, for every $v \in C^\infty(M)$,*

$$\frac{||v||^2_{L^{2^*}(M)} - ||v||^2_{L^2(M)}}{2^* - 2} \leq \frac{1}{d}\frac{d-1}{\rho}||\nabla v||^2_{L^2(M)},$$

*where we normalized the volume of $M$.*

In other words, $d$ and $\rho$ are the only two parameters needed to obtain an explicit constant in Sobolev's inequality.

Could this be more general ?

2. The CKN inequality and the CKN spaces

# Smooth metric measure spaces

## Definition

A smooth metric measure space $(M, g, \mu)$ is a (compact) smooth Riemannian manifold $(M, g)$ together with a weighted measure

$$d\mu = e^{-W} dVol_g$$

We would like to define a (meaningful) notion of dimension $n \in \overline{\mathbb{R}}$ and curvature bound $\rho \in \overline{\mathbb{R}}$ on $(M, g, \mu)$. First observe

## Proposition

*Let $L = \Delta - \nabla W \cdot \nabla$. Then,*

$$\int_M u(-Lv) d\mu = \int_M \nabla u \cdot \nabla v \, d\mu$$

Indeed,

$$\int_M u(-Lv)d\mu = \int_M u(-\Delta v + \nabla W \cdot \nabla v)e^{-W} dVol_g = \int_M u(-\nabla \cdot (e^{-W} \nabla v))dVol_g = \int_M \nabla u \cdot \nabla v \, d\mu$$

# Example : the Caffarelli-Kohn-Nirenberg inequality

For any $v \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\boxed{\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}}\,dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}}\,dx,}$$

where $d \ge 3$, $a \le b \le a+1$, $a < a_c = \frac{d-2}{2}$ and

$$p = \frac{2d}{d-2+2(b-a)} \le 2^* = \frac{2d}{d-2}.$$

Define $n \ge d$ by

$$\boxed{p = \frac{2n}{n-2}}$$

- if $a = b = 0$, this is Sobolev's inequality
- if $a = 0$ and $b = 1$, then $p = 2$ and we find Hardy's inequality
- the value of $p$ and the restriction $a \le b \le a+1$ are necessary (just use scaling)
- the restriction $a < a_c = \frac{d-2}{2}$ is also needed for the integrals to be finite

# The CKN Euclidean space

Consider the measure $\boxed{d\hat{\mu} = |x|^{-bp}dx} =: e^{-W}dx$ so that

$$\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}}dx = \int_{\mathbb{R}^d} |v|^p d\hat{\mu}$$

Conformally deform $\mathbb{R}^d$ by setting for some $\alpha$

$$\boxed{\hat{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)}\delta^{ij}.}$$

Then, $|\hat{\nabla} v|^2_{\hat{\mathfrak{g}}} = |x|^{2(1-\alpha)}|\nabla v|^2$, $dVol_{\hat{\mathfrak{g}}} = |x|^{d(\alpha-1)}dx$ and if $2(1-\alpha) - bp = -2a$,

$$\int_{\mathbb{R}^d} |\hat{\nabla} v|^2_{\hat{\mathfrak{g}}} d\hat{\mu} = \int_{\mathbb{R}^d} |x|^{2(1-\alpha)-bp}|\nabla v|^2 dx = \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}}dx$$

And so, CKN's inequality is exactly Sobolev's inequality on

$$\boxed{\text{the CKN Euclidean space} \quad (\mathbb{R}^d, \hat{\mathfrak{g}}, d\hat{\mu})}$$

# The CKN spherical and hyperbolic spaces

Recall that the classical Sobolev inequality was usefully rewritten on the sphere. For CKN, we conformally deform $\mathbb{R}^d$ by setting

$$\boxed{\bar{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)} \frac{(1+|x|^{2\alpha})^2}{4} \delta^{ij}}$$

and choosing the reference measure

$$\boxed{d\bar{\mu} = \frac{|x|^{-bp}}{(1+|x|^{2\alpha})^n} dx}$$

$$\boxed{\text{the CKN spherical space is} \quad (\mathbb{R}^d, \bar{\mathfrak{g}}, d\bar{\mu})}$$

Note that the CKN sphere is the round sphere in the case $\alpha = 1$. Similarly, the CKN hyperbolic space is defined by $(B_1, \tilde{\mathfrak{g}}, d\tilde{\mu})$ where

$$\tilde{\mathfrak{g}}^{ij} = |x|^{2(1-\alpha)}(1-|x|^{2\alpha})^2 \delta^{ij}, \qquad d\tilde{\mu} = \frac{|x|^{-bp}}{(1-|x|^{2\alpha})^n} dx$$

## Theorem (D-Gentil-Zugmeyer, 21)

*Sobolev's inequality holds on the three CKN spaces in the form*

$$\left(\int |v|^p d\mu\right)^{2/p} \leq C\left[\int S v^2 d\mu + \int |\nabla v|^2_{\mathfrak{g}} d\mu\right],$$

*where $C$ is the optimal constant, $S = 0$ for the Euclidean CKN space, $S = \frac{n(n-2)}{4}\alpha^2$ for the CKN spherical space and $S = -\frac{n(n-2)}{4}\alpha^2$ for the CKN hyperbolic space. The test function $v$ is supported in $\mathbb{R}^d \setminus \{0\}$ in the Euclidean and spherical cases and in $B_1$ in the hyperbolic case.*

### Theorem (Bakry-Gentil-Ledoux, 13)

*Let $(M, \mathfrak{g}, d\mu)$ be a smooth metric measure space, $n > 2$ and $\gamma \in \mathbb{R}$. Then, there exists $\beta_n(\gamma), \theta_n(\gamma)$ such that*

$$S_\gamma(M, \mathfrak{g}, d\mu) = \theta_n(\gamma)[sc_\mathfrak{g} - \gamma\Delta_\mathfrak{g} W + \beta_n(\gamma)|\nabla_\mathfrak{g} W|^2_\mathfrak{g}]$$

*is an n-conformal invariant i.e. if $p = 2n/(n-2)$, the inequality*

$$\left(\int |v|^p d\mu\right)^{2/p} \leq C\left(\int S_\gamma(M, \mathfrak{g}, d\mu)v^2 d\mu + \int |\nabla_\mathfrak{g} u|^2_\mathfrak{g} d\mu\right),$$

*is invariant under the transformation $\mathfrak{g} \to c^2\mathfrak{g}$, $\mu \to c^{-n}\mu$, $c \in C^\infty(M, \mathbb{R}^*_+)$.*

### Theorem (D-Gentil-Zugmeyer, 21)

*Let $(M, \hat{\mathfrak{g}}, d\hat{\mu})$ be the CKN Euclidean space. There exists $\gamma \in \mathbb{R}$ s.t. $\boxed{S_\gamma(M, \hat{\mathfrak{g}}, d\hat{\mu}) = 0}$. And so, $S_\gamma(M, \bar{\mathfrak{g}}, d\bar{\mu}) = \frac{n(n-2)}{4}\alpha^2$ and $S_\gamma(M, \tilde{\mathfrak{g}}, d\tilde{\mu}) = -\frac{n(n-2)}{4}\alpha^2$.*

All the inequalities are equivalent but we have proved none of them yet !

3. Bakry-Emery's $\Gamma$-calculus, curvature-dimension condition

# Γ-calculus and the Bochner formula

In $\mathbb{R}^d$, the Laplacian of the product of two functions is given by

$$\Delta v^2 = 2v\Delta v + 2|\nabla v|^2$$

In other words, the Laplacian fails to satisfy the chain rule and the defect is measured by the **carré du champ**

$$\Gamma(v) := \frac{1}{2}\Delta(v^2) - v\Delta v = |\nabla v|^2$$

Repeat this once more and measure the defect in the chain rule applied to the quadratic form Γ. We get the **iterated carré du champ**

$$\begin{aligned}\Gamma_2(v) &:= \frac{1}{2}\Delta(\Gamma(v)) - \Gamma(v, \Delta v)\\ &= \frac{1}{2}\Delta(|\nabla v|^2) - \nabla v \cdot \nabla \Delta v\\ &= \nabla \cdot (\nabla^2 v \nabla v) - \nabla v \cdot \nabla \Delta v = \|\nabla^2 v\|^2_{H.S.}\end{aligned}$$

On a Riemannian manifold, the Bochner-Lichnerowitz-Weizenboch formula states that

$$\boxed{\Gamma_2(v) = \mathrm{Ric}_g(\nabla v, \nabla v) + \|\nabla^2 v\|^2_{H.S.}}$$

# Γ-calculus and the Bochner formula

Recall that

$$\Gamma_2(v) = \mathrm{Ric}_g(\nabla v, \nabla v) + \|\nabla^2 v\|^2_{H.S.}$$

If $\mathrm{Ric}_g \geq \rho g$, Cauchy's inequality implies that

$$\boxed{\Gamma_2(v) \geq \rho|\nabla v|^2_g + \frac{1}{d}(\Delta_g v)^2}$$

More generally, consider a smooth metric measure space $(M, g, \mu)$ and the associated elliptic operator $L = \Delta - \nabla W \cdot \nabla$. Define its carré du champ and iterated carré du champ by

$$\Gamma(v) = \frac{1}{2}L(v^2) - vLv, \qquad \Gamma_2(v) = \frac{1}{2}L(\Gamma v) - \Gamma(v, Lv)$$

$(M, g, \mu)$ satisfies the $\boxed{CD(\rho, n) \text{ curvature-dimension condition}}$ if

$$\boxed{\Gamma_2(v) \geq \rho\Gamma(v) + \frac{1}{n}(Lv)^2}$$

# A basic example : the Gaussian space

**Example 1** : consider $(\mathbb{R}^d, \delta_{ij}, e^{-W}dx)$ with $W(x) = |x|^2/2$. Then, $L = \Delta - x \cdot \nabla$ on $M = \mathbb{R}^d$, $\Gamma(v) = |\nabla v|^2$ and

$$\begin{aligned}\Gamma_2(v) &:= \frac{1}{2}L(\Gamma(v)) - \Gamma(v, Lv)\\ &= \|\nabla^2 v\|^2_{H.S.} - \frac{1}{2}x \cdot \nabla|\nabla v|^2 + \nabla v \cdot \nabla(x \cdot \nabla v)\\ &= \|\nabla^2 v\|^2_{H.S.} + |\nabla v|^2 \geq 1 \cdot |\nabla v|^2 + \frac{1}{\infty}(Lv)^2\end{aligned}$$

In this example, whereas $(\mathbb{R}^d, \delta_{ij}, dx)$ is flat, the potential $W(x) = |x|^2/2$ is confining, yielding the curvature bound $\rho = 1$.

Note : regarding dimension, the Gaussian space can be seen as the limit of a $(d+m)$-dimensional sphere (of normalized radius), as $m \to +\infty$.

# The Sobolev inequality on smooth metric measure spaces

## Theorem (Bakry-Ledoux, 96)

*Assume that* $(M, \mathfrak{g}, d\mu)$ *verifies the curvature-dimension condition* $CD(\rho, n)$ *for some* $\rho > 0$ *and* $n \in [d, +\infty)$, $n > 2$. *Then,*

$$||v||^2_{L^{\frac{2n}{n-2}}(M)} \leq \frac{4}{n(n-2)} \frac{n-1}{\rho} ||\nabla v||^2_{L^2(M)} + ||v||^2_{L^2(M)},$$

*under the normalization* $\mu(M) = 1$.

## Remark

*The result extends to the nonsmooth setting : [Profeta, 15] for* $RCD^*(K, N)$*-spaces, [Cavaletti-Mondino, 17] for* $CD^*(K, N)$*-spaces.*

# Liouville-type theorem

## Theorem (D-Gentil-Zugmeyer, 20)

*Assume $(M, \mathfrak{g}, d\mu)$ verifies the curvature-dimension condition $CD(\rho, n)$ for some $\rho > 0$ and $n \in [d, +\infty)$, $n > 2$. Set $q = \frac{2n}{n-2}$. Let $A > 0$ and $f \in C^1(\mathbb{R}_+^*; \mathbb{R}_+^*)$ nonincreasing.*
*Assume that $v \in C^2(M)$, $v > 0$ is a nonconstant solution to*

$$-A\, Lv + v = v^{q-1} f(v) \quad \text{in } M,$$

*Then, $A \leq A^* := \frac{4(n-1)}{n(n-2)\rho}$.*
*If $A = A^*$, then $f = cste$ and if $W = 0$, $\Phi = v^{-\frac{q-2}{2}}$ solves $\nabla^2 \Phi = \frac{\Delta \Phi}{d} g$.*

## Theorem (Tashiro 65, Obata 71, Nobili-Violo 21)

*Assume $A = A^*$, $W = 0$. Then, $(M, g)$ is isometric to the round sphere and for some $\lambda > 1$ and $x_0 \in M$,*

$$v(x) = (\lambda - \cos r)^{-\frac{d-2}{2}}, \qquad \text{where } r = d(x, x_0).$$

# Curvature-dimension condition for the CKN sphere

## Theorem (D-Gentil-Zugmeyer, 21)

*When*

$$\alpha^2 \leq \frac{d-2}{n-2}$$

*is nonnegative, the CKN sphere satisfies the $CD(\rho, n)$ condition with*

$$\rho = \alpha^2(n-1).$$

## Remark

*The above parameter zone does not coincide with the known region for symmetry for extremals of the inequality. In fact, as we shall see, only an integrated version of $CD(\rho, n)$ is needed to prove sharp inequalities, which holds true if and only if*

$$\alpha^2 \leq \frac{d-1}{n-1}$$

*Surprise, this is equivalent to $0 \leq \alpha \leq 1$ !*

# 3. Gradient flows in Euclidean space

# Gradient flows in Euclidean space

Let $F : \mathbb{R}^m \mapsto \mathbb{R}$ of class $C^2$, strictly convex and coercive i.e. $\lim_{|x|\to+\infty} F(x) = +\infty$. Then, $F$ has a unique critical point $x^*$. in addition,

$$F(x^*) = \inf_{x\in\mathbb{R}^m} F(x).$$

In order to locate the point of minimum $x^*$, start from an arbitrary point $x \in \mathbb{R}^m$ and follow the gradient flow of $F$ :

$$\begin{cases} \frac{d}{dt} S_t(x) = -\nabla F(S_t(x)) \\ S_0(x) = x. \end{cases}$$

Since

$$\frac{d}{dt} F(S_t(x)) = \nabla F(S_t(x)) \cdot \frac{d}{dt} S_t(x) = -|\nabla F(S_t(x))|^2 \leq 0$$

one proves easily that

$$\lim_{t\to\infty} S_t(x) = x^*.$$

# Entropy-entropy production inequality

If in addition $F$ is strongly convex, i.e. $\nabla^2 F \geq \rho\,\mathrm{Id}$ where $\rho > 0$, then we obtain the optimal speed of convergence to equilibrium of the **entropy $F$ along its gradient flow** : compute $\frac{d^2}{dt^2} F(S_t(x))$ and easily get

$$F(S_t(x)) - F(x^*) \leq e^{-2\rho t}\,(F(x) - F(x^*))$$

Since equality holds when $t = 0$, we can differentiate the inequality at $t = 0$ and deduce that

$$F(x) - F(x^*) \leq \frac{1}{2\rho} |\nabla F(x)|^2$$

This **entropy-entropy production** inequality is optimal, in the sense that $F(x) = \rho |x|^2/2$ saturates it. More generally,

$$\boxed{G(x^*) \leq \frac{1}{2\rho} |\nabla F(x)|^2 + G(x),}$$

under the assumption

$$\boxed{\nabla F \cdot \nabla^2 F \, \nabla F \geq -\rho \nabla F \cdot \nabla G}$$

4. Wasserstein space, Otto's calculus

# Wasserstein space

Equip the set of prob. measures $\mathcal{P}_2(M)$ with the Wasserstein distance

$$W_2(\mu,\nu) = \inf \sqrt{\iint \mathbf{d}(x,y)^2 d\pi(x,y)},$$

where the inf. is taken on all transport plans $\pi \in \mathcal{P}(M \times M)$ with marginals $\mu$ and $\nu$ and where $\mathbf{d}$ is the Riemannian distance on $M$.
A path $[0,1] \ni t \mapsto \nu_t \in \mathcal{P}_2(M)$ is a.c. w.r.t $W_2$ if

$$|\dot{\nu}_t| := \limsup_{s \to t} \frac{W_2(\nu_t,\nu_s)}{|t-s|} \in L^1([0,1]).$$

## Theorem (Ambrosio-Gigli-Savaré, 08)

*Given any a.c. path, there exists a unique vector field $(t,x) \mapsto V_t(x)$ s.t. $\int |V_t|^2 d\nu_t < \infty$ and $|\dot{\nu}_t|^2 = \int |V_t|^2 d\nu_t$ a.e. in $[0,1]$. In addition, $V_t$ is the limit of a sequence $(\nabla \varphi_n)$ in $L^2(\nu_t)$ and*

$$\boxed{\partial_t \nu_t + \nabla \cdot (\nu_t V_t) = 0} \quad \text{in } \mathcal{D}'(M \times (0,1)).$$

# Tangent space, Otto's metric

In other words, for a.e. $t \in [0,1]$, $V_t$ identifies with a tangent vector to the path $(\nu_t)_{t\in[0,1]}$. We write

$$\dot{\nu}_t := V_t$$

and we call $\dot{\nu}_t$ the velocity of the path $(\nu_t)_{t\in[0,1]}$ at time $t$. The tangent space at the point $\mu \in \mathcal{P}_2(M)$ is thus defined by

$$\boxed{T_\mu \mathcal{P}_2(M) = \overline{\{\nabla\varphi,\ \varphi : M \mapsto \mathbb{R}, \varphi \in C^\infty(M)\}}^{L^2(\mu)}}$$

and a natural Riemannian metric is the following

$$\boxed{\langle \nabla\varphi, \nabla\psi \rangle_\mu = \int \nabla\varphi \cdot \nabla\psi \, d\mu = \int \Gamma(\varphi,\psi) d\mu}, \qquad \text{for } \nabla\varphi, \nabla\psi \in T_\mu \mathcal{P}_2(M).$$

5. Gradient flows in Wasserstein space : Rényi entropies and fast diffusion eq.

# Fast diffusion is...

Let $(\mu_t)_{t\geq 0}$ s.t. $\mu_0 = \mu$ eand

$$\partial_t \mu_t = \frac{1}{\alpha}\Delta\mu_t^\alpha = \nabla\cdot\left(\mu_t \frac{1}{\alpha-1}\nabla\mu_t^{\alpha-1}\right),$$

with $\alpha > 0$, $\alpha \neq 1$. By the continuity eq., the velocity of the flow is

$$\dot{\mu}_t = -\frac{1}{\alpha-1}\nabla\mu_t^{\alpha-1}$$

Consider Rényi's entropy

$$\mathcal{R}_\alpha(\mu) = \frac{1}{\alpha(\alpha-1)}\int \mu^\alpha$$

and differentiate $\mathcal{R}_\alpha$ along the flow.

$$\frac{d}{dt}\mathcal{R}_\alpha(\mu_t) = \frac{1}{(\alpha-1)}\int \mu_t^{\alpha-1}\partial_t\mu_t = \frac{1}{\alpha-1}\int \mu_t^{\alpha-1}\nabla\cdot\left(\mu_t\frac{1}{\alpha-1}\nabla\mu_t^{\alpha-1}\right)$$

$$= \int\left(\nabla\frac{\mu_t^{\alpha-1}}{\alpha-1}\right)\cdot\dot{\mu}_t\,d\mu_t = \left\langle\frac{1}{\alpha-1}\nabla\mu_t^{\alpha-1},\dot{\mu}_t\right\rangle_{\mu_t}$$

# ...a gradient flow in Wasserstein space

The gradient of $\mathcal{R}_\alpha$ is thus given by

$$\boxed{\mathrm{grad}_\mu \mathcal{R}_\alpha := \frac{1}{\alpha - 1} \nabla \mu^{\alpha - 1}}$$

and the fast diffusion eq. is the gradient flow of the Rényi entropy in Wasserstein space :

$$\boxed{\dot{\mu}_t = -\mathrm{grad}_{\mu_t} \mathcal{R}_\alpha}$$

Differentiating twice, one finds that

$$\mathrm{Hess}_\mu \mathcal{R}_\alpha(\nabla\phi, \nabla\phi) = \frac{1}{\alpha} \int \left[(\alpha - 1)(\Delta\phi)^2 + \Gamma_2(\phi)\right] \mu^\alpha,$$

# Rényi entropy

Apply the computations in the finite dimensional setting with

$$F \equiv R_\alpha, \quad \alpha = 1 - \frac{1}{d} \quad \text{and} \quad G \equiv -R_\beta, \quad \beta = 1 - \frac{2}{d}.$$

Then, letting the pressure function be $\Phi = \frac{1}{\alpha-1}\mu^{\alpha-1}$, we find under $CD(\rho, d)$

$$\begin{aligned}
&\mathrm{Hess}_\mu \mathcal{R}_\alpha(\mathrm{grad}_\mu \mathcal{R}_\alpha, \mathrm{grad}_\mu \mathcal{R}_\alpha) \\
&\qquad = \frac{1}{\alpha}\int \left[(\alpha-1)(\Delta\Phi)^2 + \Gamma_2(\Phi)\right]\mu^\alpha \geq \frac{\rho}{\alpha}\int \Gamma(\Phi)\mu^\alpha
\end{aligned}$$

And since $\beta - 3 = 2\alpha - 4$,

$$\begin{aligned}
-\langle \mathrm{grad}_\mu \mathcal{R}_\alpha, \mathrm{grad}_\mu(-\mathcal{R}_\beta)\rangle_\mu &= \frac{1}{(\alpha-1)(\beta-1)}\int \nabla\mu^{\alpha-1}\nabla\mu^{\beta-1} d\mu \\
&= \int \mu^{\alpha+\beta-3}|\nabla\mu|^2 = \int \Gamma(\Phi)\mu^\alpha
\end{aligned}$$

and so

$$\boxed{\mathrm{Hess}_\mu \mathcal{R}_\alpha(\mathrm{grad}_\mu \mathcal{R}_\alpha, \mathrm{grad}_\mu \mathcal{R}_\alpha) \geq -\frac{\rho}{\alpha}\langle \mathrm{grad}_\mu \mathcal{R}_\alpha, \mathrm{grad}_\mu(-\mathcal{R}_\beta)\rangle_\mu}$$

# Sobolev's inequality, finally !

As in the finite dimensional case, the previous convexity inequality implies that

$$-\mathcal{R}_\beta(\mu^*) \leq \frac{\alpha}{2\rho}|\mathrm{grad}_\mu \mathcal{R}_\alpha|_\mu^2 - \mathcal{R}_\beta(\mu).$$

By definition of $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$, $\alpha$, $\beta$ and $\Phi$, we find

$$1 \leq \frac{4(d-1)}{\rho d(d-2)} \int \Gamma(\mu^{\frac{d-2}{2d}}) + \int \mu^{\frac{d-2}{d}},$$

Letting at last $|v| = \mu^{\frac{d-2}{2d}}$, it follows that

$$1 \leq \frac{4(d-1)}{\rho d(d-2)} \int \Gamma(v) + \int v^2,$$

under the normalization $\|v\|_{2^*} = 1$ (so that $\mu$ is a prob. measure). This is Sobolev's inequality.