Heat flow regularity, Bismut–Elworthy–Li's derivative formula, and pathwise couplings on Riemannian manifolds with Kato bounded Ricci curvature

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Abstract We prove that if the Ricci curvature Ric of a geodesically complete Riemannian manifold M, endowed with the Riemannian distance ρ and the Riemannian measure $\mathfrak{m}$, is bounded from below by a continuous function $k\colon \mathrm{M} \to \mathbb{R}$ whose negative part k^- satisfies, for every $t > 0$, the exponential integrability condition

$$\sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X_r^x)/2\,\mathrm{d}r}\,\mathbb{1}_{\{t<\zeta^x\}}\Big] < \infty,$$

then the lifetime ζ^x of Brownian motion X^x on M starting in any $x \in \mathrm{M}$ is a.s. infinite. This assumption on k holds if k^- is in the Kato class of M. We also derive a Bismut–Elworthy–Li derivative formula for $\nabla \mathsf{P}_t f$ for every $f \in L^\infty(\mathrm{M})$ and $t > 0$ along the heat flow $(\mathsf{P}_t)_{t\geq 0}$ with generator $-\Delta/2$, yielding its L^∞-Lip-regularization as a corollary.

Moreover, without any assumption on k except continuity, we prove the equivalence of lower boundedness of Ric by k to the existence, given any $x, y \in \mathrm{M}$, of a Markovian coupling (X^x, X^y) of Brownian motions on M starting in (x, y) such that a.s.,

$$\rho\big(X_t^x, X_t^y\big) \leq \mathrm{e}^{-\int_s^t \underline{k}(X_r^x, X_r^y)/2\,\mathrm{d}r}\,\rho\big(X_s^x, X_s^y\big)$$

holds for every $s, t \geq 0$ with $s \leq t$, involving the "average" $\underline{k}(u, v) := \inf_\gamma \int_0^1 k(\gamma_r)\,\mathrm{d}r$ of k along geodesics from u to v.

Our results generalize to weighted Riemannian manifolds, where the Ricci curvature is replaced by the corresponding Bakry–Émery Ricci tensor.

Contents

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1 Main results

Let (M,g) be a smooth, geodesically complete, noncompact, connected Riemannian manifold without boundary. The metric $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$ induces the Riemannian distance ρ and the Riemannian measure $\mathfrak{m}$. W.r.t. ρ, we write $B_r(x)$ for the open ball of radius $r > 0$ around $x \in \mathrm{M}$, $\mathrm{Lip}(\mathrm{M})$ for the space of real-valued Lipschitz functions on M, and $\mathrm{Lip}(f)$ for the Lipschitz constant of any $f \in \mathrm{Lip}(\mathrm{M})$. Without further notice, all functions and sections of bundles are assumed to be real-valued, and all appearing Lebesgue and Sobolev spaces are understood w.r.t. $\mathfrak{m}$. With the usual abuse of notation, the fiberwise norm both on $T\mathrm{M}$ and $T^*\mathrm{M}$ is $|\cdot| := \langle\cdot,\cdot\rangle^{1/2}$. Let ∇ be the Levi-Civita connection on M and Ric be the induced Ricci curvature. We recall that by geodesic completeness, the Laplace–Beltrami operator Δ is an essentially self-adjoint operator in $L^2(\mathrm{M})$ when defined initially on smooth compactly supported functions, and it admits a unique – non-relabeled – self-adjoint extension. Let $(\mathsf{P}_t)_{t\geq 0}$ be the heat flow in $L^2(\mathrm{M})$ with generator $-\Delta/2$, i.e. $\mathsf{P}_t := \mathrm{e}^{t\Delta/2}$ via spectral calculus. For every $x \in \mathrm{M}$, let $X^x\colon [0,\zeta^x)\times\Omega \to \mathrm{M}$ be a corresponding adapted Brownian motion starting at $x \in \mathrm{M}$ with lifetime ζ^x, defined on a filtered probability space $(\Omega,\mathscr{F}_*,\mathbb{P})$, see [Elw82, Hsu02a, IW81, Wan14] for particular constructions of X^x.

Throughout, we fix a continuous function $k\colon \mathrm{M}\to\mathbb{R}$. We write "$\mathrm{Ric} \geq k$ on M" if

$$\mathrm{Ric}(x)(\xi,\xi) \geq k(x)\,|\xi|^2 \quad \text{for every } x \in \mathrm{M},\ \xi \in T_x\mathrm{M}.$$

The goal of this paper is to study the previous condition, where the negative part k^- of k, where $k^-(x) := -\min\{k(x),0\}$, obeys the integrability

$$\sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X_r^x)/2\,\mathrm{d}r}\,\mathbb{1}_{\{t<\zeta^x\}}\Big] < \infty \quad \text{for every } t > 0. \tag{1.1}$$

Our main results come in two groups. First, we study analytic and probabilistic *consequences* of the assumption $\mathrm{Ric} \geq k$ on M if k satisfies (1.1), as described in Section 1.1 and stated in Theorem 1.1 and Theorem 1.5. Along with this, we treat an explicit class of k for which (1.1) holds, the so-called *Kato decomposable* ones, and highlight a general condition for k to obey the latter property, Theorem 1.3. Second, we give equivalent *characterizations* of the condition $\mathrm{Ric} \geq k$ on M, which are summarized in Section 1.2, see Theorem 1.6 therein, and mostly do not even require (1.1).

Besides [ER+20, GvR20], our article is among the first to systematically study analytic and probabilistic consequences of variable lower Ricci bounds – and equivalent characterizations of these – which are not uniformly bounded from below and do not underlie geometric growth conditions. We also stress our novel general recipe from Theorem 1.5 to determine whether a given variable Ricci curvature lower bound is Kato decomposable, while the – albeit more general – condition (1.1) is in general hard to verify directly. Lastly, our equivalence result improves upon previously known ones especially because it involves a pathwise coupling estimate which has just recently been introduced in a slightly different framework [BHS19].

1.1 Consequences of variable lower Ricci bounds

To formulate our first result, given an initial point $x \in \mathrm{M}$, let $/\!\!/^x \colon [0, \zeta^x) \times \Omega \to (X^x)^*(T\mathrm{M})$, i.e. $/\!\!/^x_t \colon T_x\mathrm{M} \to T_{X^x_t}\mathrm{M}$ for all $t \in [0, \zeta^x)$, denote the stochastic parallel transport w.r.t. ∇ along the sample paths of X^x, let the process $Q^x \colon [0, \zeta^x) \times \Omega \to \mathrm{End}(T_x\mathrm{M})$ be defined as the unique solution to the pathwise ordinary differential equation

$$\mathrm{d}Q^x_s = -\frac{1}{2}\, Q^x_s (/\!\!/^x_s)^{-1} \mathrm{Ric}(X^x_s) /\!\!/^x_s \,\mathrm{d}s, \quad Q^x_0 = \mathrm{Id}_{T_x\mathrm{M}}, \tag{1.2}$$

and let $W^x \colon [0, \zeta^x) \times \Omega \to T_x\mathrm{M}$ denote the anti-development of X^x, a canonically given Euclidean Brownian motion on $T_x\mathrm{M}$. See [Elw82, Hsu02a, IW81, Wan14] for the details.

Theorem 1.1. *Let $k \colon \mathrm{M} \to \mathbb{R}$ be a continuous function satisfying* (1.1) *and assume that* $\mathrm{Ric} \geq k$ *on* M*. Then*

(i) M *is* stochastically complete*, i.e.*

$$\mathbb{P}\big[\zeta^x = \infty\big] = 1 \quad \textit{for every } x \in \mathrm{M},$$

(ii) *for every $f \in L^\infty(\mathrm{M})$ and every $t > 0$, we have* Bismut–Elworthy–Li's derivative formula

$$\big\langle \nabla \mathsf{P}_t f(x), \xi \big\rangle = \frac{1}{t}\, \mathbb{E}\Big[f(X^x_t) \int_0^t \langle Q^x_s \xi, \mathrm{d}W^x_s \rangle \Big] \quad \textit{for every } x \in \mathrm{M},\ \xi \in T_x\mathrm{M},$$

where the stochastic integral inside the expectation is understood in Itô's sense, and

(iii) *for every $t > 0$, one has the* L^∞-Lip-regularization property $\mathsf{P}_t \colon L^\infty(\mathrm{M}) \to \mathrm{Lip}(\mathrm{M})$ *with*

$$\mathrm{Lip}(\mathsf{P}_t f) \leq \sqrt{8}\, t^{-1/2} \sup_{x \in \mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X^x_r)/2\,\mathrm{d}r} \Big]\, \|f\|_{L^\infty} \quad \textit{for every } f \in L^\infty(\mathrm{M}).$$

Before further commenting on Theorem 1.1 and its proof, in order to make more refined statements, we introduce the following definition.

Definition 1.2. (i) *The* Kato class $\mathcal{K}(\mathrm{M})$ *of* M *is the linear space of all Borel functions* $v \colon \mathrm{M} \to \mathbb{R}$ *such that*

$$\lim_{t \downarrow 0} \sup_{x \in \mathrm{M}} \int_0^t \mathbb{E}\big[\big|v(X^x_r)\big|\big]\,\mathrm{d}r = 0.$$

(ii) *A Borel function* $v\colon \mathrm{M} \to \mathbb{R}$ *is called* Kato decomposable *if it belongs to* $L^1_{\mathrm{loc}}(\mathrm{M})$ *and* v^- *belongs to* $\mathcal{K}(\mathrm{M})$.

Kato (decomposable) functions have been studied in great detail in the literature, see [AS82, Gün17a, MO20, RS20, SV96, Stu94] and the references therein. In connection with lower Ricci bounds, they have been introduced in [GP16] in the context of BV functions and turn out to arise very naturally in different frameworks, see e.g. [GvR20] (which treats some probabilistic and geometric aspects of molecular Schrödinger operators). For the convenience of the reader, we have collected some important properties of Kato decomposable functions in Appendix A. In particular, note that in view of

$$\mathbb{E}\big[|v(X^x_r)|\big]\,\mathrm{d}r \leq \|v\|_{L^\infty} \quad \text{for every } x \in \mathrm{M},\ r > 0,$$

it follows that $L^\infty(\mathrm{M}) \subset \mathcal{K}(\mathrm{M})$. More generally, paired with an explicit Example A.7, we provide the following criterion in Section A.3, for which we denote by $\Xi\colon \mathrm{M} \to \mathbb{R}$ the function $\Xi(x) := \mathrm{vol}[B_1(x)]^{-1}$, and by $L^p_\Xi(\mathrm{M})$ the p-th order Lebesgue space w.r.t. $\Xi\,\mathfrak{m}$.

Theorem 1.3. *Assume that* $\dim(\mathrm{M}) \geq 2$*, that* M *is quasi-isometric to a complete Riemannian manifold whose Ricci curvature is bounded from below by a constant, and that* $k^- \in L^p_\Xi(\mathrm{M}) + L^\infty(\mathrm{M})$ *for some* $p \in (\dim(\mathrm{M})/2, \infty)$*. Then* k *is Kato decomposable.*

One key feature for us about functions $v \in \mathcal{K}(\mathrm{M})$ is that they always satisfy

$$\sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t v(X^x_r)/2\,\mathrm{d}r}\,\mathbb{1}_{\{t<\zeta^x\}}\Big] < \infty \quad \text{locally uniformly in } t \in [0,\infty).$$

This is known as *Khasminskii's lemma*, see Lemma A.4. In particular, since $\mathcal{K}(\mathrm{M})$ is a linear space, we have the following link of Kato decomposability to (1.1).

Lemma 1.4. *Assume that* k *is a Kato decomposable function. Then for every* $p \in [1,\infty)$*, the exponential integrability* (1.1) *holds with* k *replaced by* pk.

This is ultimately the key behind the following result which states that in this case, Bismut–Elworthy–Li's derivative formula holds on an L^p-scale.

Theorem 1.5. *Assume* $k\colon \mathrm{M} \to \mathbb{R}$ *is a continuous Kato decomposable function satisfying* $\mathrm{Ric} \geq k$ *on* M*. Then* (1.1) *is satisfied, and moreover, Bismut–Elworthy–Li's derivative formula from Theorem 1.1 holds for every* $p \in (1,\infty]$ *and every* $f \in L^p(\mathrm{M})$.

The proof of (i) in Theorem 1.1 can be found in Section 3.1, while (ii) and (iii) as well as Theorem 1.5 are studied in Section 3.2.

Let us collect some bibliographical comments on Theorem 1.1 and Theorem 1.5.

In the framework of uniform bounds from below on the Ricci curvature, (i) in Theorem 1.1 is due to [Yau78]. On weighted Riemannian manifolds – on which the Ricci tensor is always replaced by the corresponding Bakry–Émery Ricci tensor, see Section 1.3 – the non-explosion for the induced diffusion processes under uniform lower Ricci bounds has been obtained by [Bak86]. In connection with (1.1), also for weighted Riemannian manifolds, the latter result has been extended by [Li94] using an approach via stochastic and Hessian flows. Once we have established all necessary intermediate results, our proof then closely follows the lines in [Bak86] (which is also worked towards in [Li94]). For different, more geometric non-explosion criteria in terms of distance functions, see [Wan14] and the references therein. A nonsmooth result similar to (i) – however assuming a Kato- or rather a Dynkin-type [SV96, Stu94] lower bound instead of only (1.1) – has recently been treated in [ER$^+$20].

Formula (ii) in Theorem 1.1 has first appeared in [Bis84] in the compact case. In the noncompact case, this result, as well as Theorem 1.5, have been proven in [EL94] under more general assumptions than (1.1) using the slightly different technique of stochastic derivative flows. We also refer to [DT01] for similar treatises for heat semigroups over vector bundles, and also [Hsu02a, Wan14] for similar results under more geometric conditions on the lower bound of Ric. Remarkably, localized versions of the Bismut–Elworthy–Li derivative formula hold without any assumptions on the geometry of the manifold, see e.g. [Tha97, TW98, TW11].

The L^∞-Lip-regularization (iii) from Theorem 1.1 is a corollary of (ii), thus indicating the importance of the latter in studying further regularity properties of $(\mathsf{P}_t)_{t\geq 0}$. In fact, (iii) is already known even without the assumption (1.1) on k [TW98, Wan14], with slightly different and more rough estimates on $\mathrm{Lip}(\mathsf{P}_t f)$ involving locally uniform lower bounds on Ric. (The proof uses the above mentioned local derivative formula.) Outside the smooth scope, a similar property is known on $\mathrm{RCD}(K,\infty)$ spaces [AGS15]. This setting allows for more flexibility in the variety of spaces (metric measure spaces), but is still restricted to uniform lower Ricci bounds, formulated in a synthetic sense [LV09, Stu06].

1.2 Characterizations of variable lower Ricci bounds

We now come to our second main result, i.e. several equivalent characterizations of lower Ricci bounds, which we shortly introduce.

The closest characterization of $\mathrm{Ric} \geq k$ on M is the L^1-*Bochner inequality* which, for $f \in C^\infty(\mathrm{M})$, is related to the Ricci curvature of M by the well-known Bochner formula

$$\Delta \frac{|\nabla f|^2}{2} = \langle \nabla \Delta f, \nabla f \rangle + |\operatorname{Hess} f|^2 + \mathrm{Ric}(\nabla f, \nabla f). \tag{1.3}$$

We also derive a one-to-one connection between lower Ricci bounds by k and the existence of certain couplings of Brownian motions on M. Here, if M is stochastically complete, then given $x, y \in \mathrm{M}$, by a *coupling of Brownian motions starting in* (x, y), we understand an $\mathrm{M}\times\mathrm{M}$-valued stochastic process $(X^x, X^y)\colon [0,\infty)\times\Omega \to \mathrm{M}\times\mathrm{M}$ on some filtered probability space $(\Omega, \mathscr{F}_*, \mathbb{P})$ such that X^x and X^y are Brownian motions on M starting in x and y, respectively. To formulate an appropriate pathwise coupling estimate, we denote by $\mathrm{Geo}(\mathrm{M})$ the set of minimizing geodesics $\gamma\colon [0,1] \to \mathrm{M}$, and define the lower semicontinuous function $\underline{k}\colon \mathrm{M}\times\mathrm{M} \to \mathbb{R}$ by

$$\underline{k}(u,v) := \inf\Big\{ \int_0^1 k(\gamma_r)\,\mathrm{d}r : \gamma \in \mathrm{Geo}(\mathrm{M}),\ \gamma_0 = u,\ \gamma_1 = v \Big\}. \tag{1.4}$$

The key feature about $\underline{k}$ is that it provides a way to avoid mentioning cut-loci.

Theorem 1.6. *Let* $k\colon \mathrm{M} \to \mathbb{R}$ *be a continuous function satisfying* (1.1)*. Then the following conditions are equivalent:*

(i) *we have* $\mathrm{Ric} \geq k$ *on* M,

(ii) *the* L^1-Bochner inequality w.r.t. k *is satisfied, i.e. for every* $f \in C_c^\infty(\mathrm{M})$,

$$\Delta|\nabla f| - |\nabla f|^{-1}\,\langle \nabla\Delta f, \nabla f\rangle \geq k\,|\nabla f| \quad \textit{on } \{|\nabla f| \neq 0\}, \tag{1.5}$$

(iii) *we have the* pathwise coupling property w.r.t. k*, i.e.* M *is stochastically complete and for every* $x, y \in \mathrm{M}$*, there exists a Markovian coupling* (X^x, X^y) *of Brownian motions on* M *starting in* (x, y) *such that a.s., we have*

$$\rho(X_t^x, X_t^y) \leq \mathrm{e}^{-\int_s^t \underline{k}(X_r^x, X_r^y)/2\,\mathrm{d}r}\,\rho(X_s^x, X_s^y) \quad \textit{for every } s, t \in [0, \infty) \textit{ with } s \leq t.$$

Here and in the sequel, the Markov property for every respective process under consideration is understood w.r.t. its canonically given filtration. The statement of Theorem 1.6 is still true if one does not require the Markov property of (X^x, X^y).

Remark 1.7. Thanks to the local, respectively pathwise, nature of the statements, the implications "(iii) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (i)" are even true without (1.1). Moreover, under the *a priori* assumption of stochastic completeness, "(i) $\Longrightarrow$ (iii)" is satisfied without (1.1). ■

We prove "(ii) $\Longrightarrow$ (i)" in Section 4.1, "(i) $\Longrightarrow$ (iii)" in Section 4.2 and "(iii) $\Longrightarrow$ (ii)" in Section 4.3. For Kato decomposable functions k, another equivalent characterization of $\mathrm{Ric} \geq k$ on M in terms of the L^1*-gradient estimate* is discussed in Section A.1.

Again, some bibliographical comments are in order.

In the abstract framework of [ER+20], the equivalence "(i) $\Longleftrightarrow$ (ii)" – with (ii) in a weak formulation – together with their equivalence to (a nonsmooth version of) the L^1-gradient estimate from Theorem A.1 has been shown independently.

The pathwise estimate appearing in (iii), as well as the equivalence of (iii) to lower Ricci bounds, extends similar results from [BHS19] (the Markov property of the constructed coupling has not been established therein), where analogous equivalences have been established in the synthetic framework of $\mathrm{CD}(k, \infty)$ spaces with lower semicontinuous, lower bounded variable Ricci bounds $k\colon \mathrm{M} \to \mathbb{R}$ (see also [Stu15]). Even for the smooth case, the stated pathwise inequality involving the function $\underline{k}$ has been firstly introduced in [BHS19]. (Although it is quite straightforward to detect the place where $\underline{k}$ enters from the construction of the coupling, see Section 4.2, the function $\underline{k}$ was seemingly never mentioned explicitly in the literature before [BHS19].) In the Riemannian case, Theorem 1.6 establishes a similar result in full generality without any lower boundedness assumption on k. We point out that, in contrast to [BHS19], the coupling technique on manifolds does not require any notion of "Wasserstein contractivity" for the dual heat flow to $(\mathsf{P}_t)_{t\geq 0}$ on the space of Borel probability measures on M. It is rather provided in a direct way by the method of *coupling by parallel displacement* [Cra91, Ken86], see [ACT11] for a treatise in the case of constant k. Let us also point out [Vey11], which provides a coupling (X^x, X^y) such that for every $t > 0$, even

$$\rho(X_t^x, X_t^y) = \mathrm{e}^{-\int_0^t \kappa(X_r^x, X_r^y)/2\,\mathrm{d}r}\,\rho(x, y)$$

holds on the event that (X_r^x, X_r^y) does not belong to the cut-locus of M for all $r \in [0, t]$. The real-valued function κ, the so-called "coarse curvature" of M, is defined outside the diagonal of $\mathrm{M} \times \mathrm{M}$ and is slightly larger than $\underline{k}$.

1.3 Extensions to possible other settings

Recent results [GvR20] suggest a detailed study of weighted Riemannian manifolds having Kato-type lower bounds on their Bakry–Émery Ricci tensor. In this context, we note that Theorem 1.1, Theorem 1.5 and Theorem 1.6 remain valid if for some $\Phi \in C^2(\mathrm{M})$, we replace

- $\mathfrak{m}$ by the weighted measure $\mathrm{e}^{-2\Phi}\,\mathfrak{m}$,
- Δ by the drift Laplacian $\Delta - 2\,\langle\nabla\Phi, \nabla\cdot\rangle$,
- Ric by the Bakry–Émery Ricci tensor $\mathrm{Ric} + 2\,\mathrm{Hess}\,\Phi$,
- $(\mathsf{P}_t)_{t\geq 0}$ by the semigroup generated by $-\Delta/2 + \langle\nabla\Phi, \nabla\cdot\rangle$, noting that the latter is again essentially self-adjoint [BMS02], and
- X^x by the diffusion generated by the operator $-\Delta/2 + \langle\nabla\Phi, \nabla\cdot\rangle$, see e.g. [Wan14, Chapter 3] for the particular form of the corresponding stochastic differential equation and the construction of its solution.

Other appropriate changes compared to the non-weighted setting, if needed, will always be indicated in the sequel.

It would also be interesting to study Theorem 1.1, Theorem 1.5 and Theorem 1.6 in the context of lower bounds on the Bakry–Émery Ricci curvature $\mathrm{Ric}_Z := \mathrm{Ric} + 2\,\nabla Z$ which is associated to a C^1-vector field Z on M not necessarily of gradient-type. See [Wan14] and the references therein for a summary of similar statements under different, more geometric conditions. Given appropriate interpretations of the involved analytic objects, see [Wan14] for details, some of the results immediately carry over with trivial modifications (for instance, the chain "(iii) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (i)" in Theorem 1.6). On the other hand, many of our arguments, e.g. Theorem 2.1 and thus (i) in Theorem 1.1, are implicitly based on self-adjointness of the semigroup $(\mathsf{P}_t)_{t\geq 0}$ and the heat flow on 1-forms. The latter properties lack in this generality, which is why we restricted ourselves to gradient vector fields.

Finally, a further possible (but highly nontrivial) direction of investigation is the case of manifolds with boundary, taking the heat flow with Neumann boundary conditions. See [CF12, Hsu02b, Wan14] and the references therein for an account on diffusion processes on these. The key difficulty in this context will be to take into account the local time of the boundary appropriately, a highly subtle business.

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2 Preliminaries

For more details on the heat flows on functions and on 1-forms collected in this chapter, we refer the reader to [Gri09, Gün17a, Hsu02a, Ros97] and the references therein. For details on their connection with the underlying stochastic processes, see [IW81, Mal97, Wan14].

All objects and results presented here have counterparts in the weighted case outlined in Section 1.3: the heat flow on functions [Gri09], Brownian motion (or rather the corresponding Ornstein–Uhlenbeck process) [IW81, Wan14], and the heat flow on 1-forms [Gün17a].

Heat flow on functions The operator $-\Delta/2$ is the generator of the strongly local, regular Dirichlet form $\mathscr{E}\colon L^2(\mathrm{M}) \to [0,\infty]$ given by

$$\mathscr{E}(f) := \frac{1}{2}\int_{\mathrm{M}} |\nabla f|^2 \,\mathrm{d}\mathfrak{m} \quad \text{if } f \in W^{1,2}(\mathrm{M}), \quad \mathscr{E}(f) := \infty \quad \text{otherwise.}$$

The *heat semigroup* or *heat flow* $(\mathsf{P}_t)_{t\geq 0}$ introduced in the beginning of this article is directly linked to $\mathscr{E}$ by spectral calculus and is a strongly continuous, positivity preserving contraction semigroup of linear, self-adjoint operators in $L^2(M)$. Powerful L^2-L^∞-regularization properties of the heat flow on relatively compact subsets of M, an exhaustion procedure and bootstrapping of regularity imply the existence of the so-called *minimal heat kernel* $\mathsf{p} \in C^\infty((0,\infty)\times \mathrm{M}\times\mathrm{M};(0,\infty))$ on M, the smallest positive fundamental solution to the heat operator $\partial/\partial t - \Delta/2$. It has the property that for every $f \in L^2(\mathrm{M})$ and $t > 0$, (a version of) $\mathsf{P}_t f$ can be represented by

$$\mathsf{P}_t f(x) := \int_{\mathrm{M}} \mathsf{p}_t(x,y)\, f(y)\,\mathrm{d}\mathfrak{m}(y) \quad \text{for every } x \in \mathrm{M}.$$

Actually, $(\mathsf{P}_t)_{t\geq 0}$ extends to a contraction semigroup of linear operators from $L^p(\mathrm{M})$ into $L^p(\mathrm{M})$ for every $p \in [1,\infty]$ which is strongly continuous if $p < \infty$. Moreover, the previous representation formula is still valid for every $p \in [1,\infty]$ and $f \in L^p(\mathrm{M})$. For such f, the above properties of the heat kernel show that $\mathsf{P}_\cdot f \in C^\infty((0,\infty)\times\mathrm{M})$ solves the heat equation

$$\frac{\partial}{\partial t}\mathsf{P}_t f = \frac{1}{2}\Delta \mathsf{P}_t f \quad \text{in } (0,\infty)\times \mathrm{M} \tag{2.1}$$

with inital condition f in the classical sense. In addition, we have $\mathsf{P}_\cdot f \in C^\infty([0,\infty)\times\mathrm{M})$ if f is also smooth.

Brownian motion Given a locally compact Polish space Y we denote by $C([0,\infty);Y)$ the space of continuous maps $\gamma\colon [0,\infty) \to Y$, equipped with the topology of locally uniform convergence and the induced Borel σ-algebra. Let $Y_\partial := Y \cup \{\partial\}$ denote the one-point compactification of Y.

Given a point $x \in \mathrm{M}$, any stochastic process X with sample paths in $C([0,\infty);\mathrm{M}_\partial)$ which is defined on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ (i.e. the map $t \mapsto X_t(\omega)$ belongs to $C([0,\infty);\mathrm{M}_\partial)$ for all $\omega \in \Omega$) is termed *Brownian motion on* M *starting in* x if its law equals the Wiener measure $\mathbb{P}_x$ on $C([0,\infty);\mathrm{M}_\partial)$ concentrated at paths starting in x. (Usually we want to underline the dependency of X from its starting point x, whence we shall often write X^x.) Recall that $\mathbb{P}_x$ is the uniquely determined probability measure on $C([0,\infty);\mathrm{M}_\partial)$ with $(\mathsf{ev}_0)_\sharp\mathbb{P}_x = \delta_x$ (where $\mathsf{ev}_0(\gamma) := \gamma_0$ is the evaluation map at 0) and whose transition density is given by the function $\mathsf{q}\colon (0,\infty)\times\mathrm{M}_\partial\times\mathrm{M}_\partial \to [0,\infty)$ defined by setting, for every $y, y' \in \mathrm{M}$,

$$\mathsf{q}_t(y,y') = \mathsf{p}_t(y,y'), \quad \mathsf{q}_t(\partial,y') := 0, \quad \mathsf{q}_t(\partial,\partial) := 1, \quad \mathsf{q}_t(y,\partial) := 1 - \int_{\mathrm{M}} \mathsf{p}_t(y,z)\,\mathrm{d}\mathfrak{m}(z).$$

Now let $\zeta^x := \inf\{t \geq 0 : X_t^x = \partial\}$ denote the *explosion time* of X^x, with the usual convention that $\inf\emptyset := \infty$. Since the Wiener measure is concentrated on paths having ∂ as a trap, for every $p \in [1,\infty]$ and $f \in L^p(\mathrm{M})$ one has

$$\mathsf{P}_t f(x) = \mathbb{E}\big[f(X_t^x)\,\mathbb{1}_{\{t<\zeta^x\}}\big] \quad \text{for every } x \in \mathrm{M},\ t \geq 0. \tag{2.2}$$

Therefore, M is stochastically complete if and only if

$$\mathbb{P}\big[t < \zeta^x\big] = \mathsf{P}_t \mathbb{1}_{\mathrm{M}}(x) = \int_{\mathrm{M}} \mathsf{p}_t(x,y)\,\mathrm{d}\mathfrak{m}(y) = 1 \quad \text{for every } x \in \mathrm{M},\ t > 0. \tag{2.3}$$

If $(\Omega, \mathscr{F}_*, \mathbb{P})$ is filtered and X^x adapted to the given filtration, then X^x is called an *adapted Brownian motion*. In this case, X^x is a semimartingale on M in the sense that for every $f \in C^\infty(\mathrm{M})$, the real-valued process $f \circ X^x$ is a semimartingale up to the explosion time ζ^x. The *stochastic parallel transport* along X^x w.r.t. ∇ started in $x \in \mathrm{M}$ constitutes a process $/\!\!/^x \colon [0, \zeta^x) \times \Omega \to (X^x)^*(T\mathrm{M})$, the latter being the pullback bundle of $T\mathrm{M}$ along the paths of X^x, such that $/\!\!/^x_t \colon T_x\mathrm{M} \to T_{X^x_t}\mathrm{M}$ is a.s. orthogonal for every $t \in [0, \zeta^x)$.

Heat flow on 1-forms In the sequel, Borel equivalence classes of 1-forms on M having a certain regularity $\mathscr{R}$ are denoted by $\Gamma_{\mathscr{R}}(T^*\mathrm{M})$, and similarly $\Gamma_{\mathscr{R}}(T\mathrm{M})$ for Borel equivalence classes of vector fields with regularity $\mathscr{R}$. For instance, given $p \in [1, \infty]$, we get the Banach space $\Gamma_{L^p}(T^*\mathrm{M})$ given by all Borel equivalence classes ω of sections in $T^*\mathrm{M}$ such that $|\omega| \in L^p(\mathrm{M})$. Let $\vec{\Delta} := \mathrm{d}^*\mathrm{d} + \mathrm{d}\,\mathrm{d}^*$ denote the Hodge–de Rham Laplacian. When defined initially on $\Gamma_{C_c^\infty}(T^*\mathrm{M})$, by geodesic completeness this operator has a unique self-adjoint extension in the Hilbert space $\Gamma_{L^2}(T^*\mathrm{M})$, which will be denoted with the same symbol again. Note our sign convention: $\vec{\Delta}$ is nonnegative, while Δ is nonpositive. The heat semigroup $(\vec{\mathsf{P}}_t)_{t\geq 0}$ on 1-forms given by $\vec{\mathsf{P}}_t := \mathrm{e}^{-t\vec{\Delta}/2}$ in $\Gamma_{L^2}(T^*\mathrm{M})$ is smooth, in the sense for every $\omega \in \Gamma_{L^2}(T^*\mathrm{M})$ one has a jointly smooth representative $\vec{\mathsf{P}}_\cdot\omega$ which solves the heat equation

$$\frac{\partial}{\partial t}\vec{\mathsf{P}}_t\omega = -\frac{1}{2}\vec{\Delta}\vec{\mathsf{P}}_t\omega \quad \text{in } (0,\infty) \times \mathrm{M}$$

on 1-forms with initial condition ω (and in $[0,\infty) \times \mathrm{M}$ if ω is also smooth).

On exact forms, $\vec{\mathsf{P}}_t$ can be represented by the heat operator P_t; more precisely, for every $f \in C_c^\infty(\mathrm{M})$, it has been discussed in detail in [Gün17a] that

$$\vec{\mathsf{P}}_t\mathrm{d}f = \mathrm{d}\mathsf{P}_t f \quad \text{for every } t \geq 0. \tag{2.4}$$

This relation may fail on noncomplete Riemannian manifolds [Tha98].

Lastly, a key result is Feynman–Kac's formula, for which we recall the process Q^x from (1.2). Compare with Section A.1. Note that the asserted estimate in the theorem follows from Gronwall's inequality.

Theorem 2.1 [DT01, Theorem B.4]**.** *Let $t > 0$ and suppose that* Ric $\geq k$ *on* M *for some continuous $k \colon \mathrm{M} \to \mathbb{R}$. Assume that for every compact $K \subset \mathrm{M}$, we have*

$$\mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X^x_r)/2\,\mathrm{d}r}\, \mathbb{1}_{\{X^x_t \in K\}}\, \mathbb{1}_{\{t<\zeta^x\}}\Big] < \infty \quad \textit{for every } x \in \mathrm{M}.$$

Then for every $\omega \in \Gamma_{C_c^\infty}(T^*\mathrm{M})$, we have the Feynman–Kac formula

$$\vec{\mathsf{P}}_t\omega(x) = \mathbb{E}\big[Q^x_t (/\!\!/^x_t)^{-1}\omega^\sharp(X^x_t)\,\mathbb{1}_{\{t<\zeta^x\}}\big]^\flat \quad \textit{for every } x \in \mathrm{M}$$

and, in particular,

$$\big|\vec{\mathsf{P}}_t\omega(x)\big| \leq \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X^x_r)/2\,\mathrm{d}r}\,\big|\omega(X^x_t)\big|\,\mathbb{1}_{\{t<\zeta^x\}}\Big] \quad \textit{for every } x \in \mathrm{M}.$$

Note that on weighted Riemannian manifolds, one has to replace $(\vec{\mathsf{P}}_t)_{t\geq 0}$ by the semigroup on $\Gamma_{L^2}(T^*\mathrm{M})$ which is generated by the Schrödinger-type operator $-\nabla^*\nabla/2+\mathrm{Ric}/2+\mathrm{Hess}\,\Phi$, where $\nabla^*\nabla$ is the Bochner Laplacian on M, and Ric and $\mathrm{Hess}\,\Phi$ are identified with their induced endomorphisms $T^*\mathrm{M}\to T^*\mathrm{M}$ [Gün17a].

3 Proof of Theorem 1.1 and Theorem 1.5

This chapter treats the stochastic completeness of M, Bismut–Elworthy–Li's derivative formula, and the L^∞-Lip-regularization of the heat semigroup $(\mathsf{P}_t)_{t\geq 0}$ if we have $\mathrm{Ric}\geq k$ on M for some continuous function $k\colon \mathrm{M}\to\mathbb{R}$ satisfying (1.1).

3.1 Stochastic completeness

A key tool for proving stochastic completeness under geodesic completeness, already used in [Bak86], are sequences of first-order cutoff-functions [Str83, Chapter 2]. Their existence is equivalent to the geodesic completeness of M [Gün16, Theorem 2.2].

Lemma 3.1. *There exists a sequence* $(\psi_n)_{n\in\mathbb{N}}$ *in* $C_c^\infty(\mathrm{M})$ *satisfying*

(i) $\psi_n(\mathrm{M})\subset[0,1]$ *for every* $n\in\mathbb{N}$,

(ii) *for all compact* $K\subset\mathrm{M}$, *there exists* $N\in\mathbb{N}$ *such that* $\psi_n|_K=\mathbb{1}_K$ *for every* $n\geq N$, *and*

(iii) $\big\| |\mathrm{d}\psi_n| \big\|_{L^\infty}\to 0$ *as* $n\to\infty$.

Proof of (i) *in Theorem 1.1.* We are going to show the statement (2.3), i.e. that $\mathsf{P}_t\mathbb{1}_\mathrm{M}=\mathbb{1}_\mathrm{M}$ for every $t>0$. Let $\phi\in C_c^\infty(\mathrm{M})$, and let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of first-order cutoff functions provided by Lemma 3.1. Then Theorem 2.1 applied to the 1-form $\omega:=\mathrm{d}\psi_n$ for every $n\in\mathbb{N}$ as well as (2.4) give

$$\begin{aligned}\big\| |\vec{\mathsf{P}}_s\mathrm{d}\psi_n| \big\|_{L^\infty} &\leq \sup_{x\in\mathrm{M}}\mathbb{E}\Big[\mathrm{e}^{-\int_0^s k(X_r^x)/2\,\mathrm{d}r}\,|\mathrm{d}\psi_n(X_s^x)|\,\mathbb{1}_{\{s<\zeta^x\}}\Big]\\ &\leq \sup_{x\in\mathrm{M}}\mathbb{E}\Big[\mathrm{e}^{\int_0^s k^-(X_r^x)/2\,\mathrm{d}r}\,\mathbb{1}_{\{s<\zeta^x\}}\Big]\,\big\||\mathrm{d}\psi_n|\big\|_{L^\infty}\leq C\,\big\||\mathrm{d}\psi_n|\big\|_{L^\infty},\end{aligned}$$

uniformly in $s\in[0,t]$, where $C>0$ is the quantity from (1.1). Since $\mathsf{P}_\cdot\psi_n$ solves the heat equation (2.1), also using Fubini's theorem, integration by parts as well as the commutation rule (2.4) we arrive at

$$\begin{aligned}\int_\mathrm{M}(\mathsf{P}_t\psi_n-\psi_n)\,\phi\,\mathrm{d}\mathfrak{m} &= \frac{1}{2}\int_\mathrm{M}\int_0^t \phi\,\Delta\mathsf{P}_s\psi_n\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}\\ &= -\frac{1}{2}\int_0^t\int_\mathrm{M}\langle\mathrm{d}\phi,\mathrm{d}\mathsf{P}_s\psi_n\rangle\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s = -\frac{1}{2}\int_0^t\int_\mathrm{M}\langle\mathrm{d}\phi,\vec{\mathsf{P}}_s\mathrm{d}\psi_n\rangle\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\Big|\int_\mathrm{M}(\mathsf{P}_t\mathbb{1}_\mathrm{M}-\mathbb{1}_\mathrm{M})\,\phi\,\mathrm{d}\mathfrak{m}\Big| &= \lim_{n\to\infty}\Big|\int_\mathrm{M}(\mathsf{P}_t\psi_n-\psi_n)\,\phi\,\mathrm{d}\mathfrak{m}\Big|\\ &\leq \limsup_{n\to\infty}\frac{1}{2}\int_0^t\int_\mathrm{M}|\mathrm{d}\phi|\,|\vec{\mathsf{P}}_s\mathrm{d}\psi_n|\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\\ &\leq C\,\|\mathrm{d}\phi\|_{L^1}\,\limsup_{n\to\infty}\big\||\mathrm{d}\psi_n|\big\|_{L^\infty}=0.\end{aligned}$$

Since ϕ was arbitrary, this proves the claim. □

3.2 Bismut–Elworthy–Li's derivative formula and the Lipschitz smoothing property

In view of proving Bismut–Elworthy–Li's derivative formula and the L^∞-Lip-regularization property of $(\mathsf{P}_t)_{t\geq 0}$, for convenience we cite the following version of the Burkholder–Davis–Gundy inequality for $q \in [1,\infty)$ (although we only need the upper bounds, respectively), which improves the classically known constants to better ones.

Lemma 3.2 [Ren08, Theorem 2]**.** *Let $(M_r)_{r\geq 0}$ be a real-valued continuous local martingale with $M_0 = 0$, and let $q \in [1,\infty)$. Then*

$$(8q)^{-q/2}\,\mathbb{E}\big[[M]_\tau^{q/2}\big] \leq \mathbb{E}\Big[\sup_{r\in[0,\tau]} |M_r|^q\Big] \leq (8q)^{q/2}\,\mathbb{E}\big[[M]_\tau^{q/2}\big]$$

for every stopping time τ, where $([M]_r)_{r\geq 0}$ denotes the quadratic variation process of $(M_r)_{r\geq 0}$.

Again recall the process Q^x defined by (1.2) and taking values in $T_x\mathrm{M}$.

Proof of (ii) *in Theorem 1.1.* Fix $x \in \mathrm{M}$, $t > 0$ and $\xi \in T_x\mathrm{M}$. It suffices to assume $|\xi| \leq 1$. We first assume that $f \in C_\mathrm{c}^\infty(\mathrm{M})$. By [DT01, Proposition 3.2], the process N^x given by

$$N_r^x := \Big\langle Q_r^x (/\!\!/_r^x)^{-1} \nabla \mathsf{P}_{t-r} f(X_r^x), \frac{t-r}{t}\,\xi\Big\rangle + \frac{1}{t}\,\mathsf{P}_{t-r}f(X_r^x)\int_0^r \langle Q_s^x \xi, \mathrm{d}W_s^x\rangle,$$

$r \in [0,t]$, is a local martingale. We show that under the given assumption on k, this process is even a martingale. Indeed, estimating $|Q_r^x|$ by (3.2) below and using the commutation rule (2.4), Theorem 2.1 as well as Lemma 1.4, for all $r \in [0,t]$ one a.s. has

$$\begin{aligned}
|N_r^x| &\leq \mathrm{e}^{\int_0^t k^-(X_s^x)/2\,\mathrm{d}s}\,|\vec{\mathsf{P}}_{t-r}\,\mathrm{d}f(X_r^x)| + \frac{\|f\|_{L^\infty}}{t}\,\Big|\int_0^r \langle Q_s^x\xi,\,\mathrm{d}W_s^x\rangle\Big|\\
&\leq \mathrm{e}^{\int_0^t k^-(X_s^x)/2\,\mathrm{d}s}\,\mathbb{E}_{X_r^x}\Big[\mathrm{e}^{-\int_0^{t-r} k(Y_s)/2\,\mathrm{d}s}\,|\mathrm{d}f(Y_{t-r})|\Big] + \frac{\|f\|_{L^\infty}}{t}\,\Big|\int_0^r \langle Q_s^x\xi, \mathrm{d}W_s^x\rangle\Big|\\
&\leq \mathrm{e}^{\int_0^t k^-(X_s^x)/2\,\mathrm{d}s}\,\sup_{y\in\mathrm{M}}\mathbb{E}\Big[\mathrm{e}^{-\int_0^{t-r} k(X_s^y)/2\,\mathrm{d}s}\Big]\,\big\||\mathrm{d}f|\big\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{t}\,\Big|\int_0^r \langle Q_s^x\xi, \mathrm{d}W_s^x\rangle\Big|\\
&\leq \mathrm{e}^{\int_0^t k^-(X_s)/2\,\mathrm{d}s}\,C_1\,\mathrm{e}^{C_2 t}\,\big\||\mathrm{d}f|\big\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{t}\,\Big|\int_0^r \langle Q_s^x\xi, \mathrm{d}W_s^x\rangle\Big|.
\end{aligned}$$

Here, we denote by Y a generic path in $\mathrm{C}([0,\infty);\mathrm{M})$. It follows that

$$\mathbb{E}\Big[\sup_{r\in[0,t]}|N_r^x|\Big] \leq \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X_s^x)/2\,\mathrm{d}s}\Big]\,C_1\,\mathrm{e}^{C_2 t}\,\big\||\mathrm{d}f|\big\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{t}\,\mathbb{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r \langle Q_s^x\xi, \mathrm{d}W_s^x\rangle\Big|\Big].$$

Estimating the second summand by (3.3) below for $q = 1$, the previous right-hand side is finite by Lemma 1.4. It follows that $(N_r^x)_{r\geq 0}$ is a true martingale, and thus

$$\langle \nabla \mathsf{P}_t f(x), \xi\rangle = \mathbb{E}[N_0^x] = \mathbb{E}[N_t^x] = \frac{1}{t}\,\mathbb{E}\Big[f(X_t^x)\int_0^t \langle Q_s^x\xi, \mathrm{d}W_s^x\rangle\Big]. \tag{3.1}$$

The claimed equality for bounded $f \in C^\infty(\mathrm{M})$ follows by replacing f by $\psi_n f$ in (3.1) for every $n \in \mathbb{N}$, where $(\psi_n)_{n\in\mathbb{N}}$ is as in Lemma 3.1, and letting $n \to \infty$ (together with the dominated convergence theorem on the right-hand side). In turn, if only $f \in L^\infty(\mathrm{M})$, a similar procedure works by replacing f by $\mathsf{P}_\varepsilon f$ in (3.1), where $\varepsilon > 0$, and letting $\varepsilon \to 0$. □

Proof of (iii) *in Theorem 1.1.* Using Bismut–Elworthy–Li's formula proved above and (3.3) below for $q = 1$, for every $x \in M$, $t > 0$ and $\xi \in T_x M$ with $|\xi| \leq 1$, we get

$$|\langle \nabla \mathsf{P}_t f(x), \xi\rangle| \leq \frac{1}{t}\,\mathbb{E}\Big[\Big|\int_0^t \langle Q_s^x \xi,\, \mathrm{d}W_s^x\rangle\Big|\Big]\,\|f\|_{L^\infty} \leq \sqrt{8}\,t^{-1/2} \sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X_r^x)/2\,\mathrm{d}r}\Big]\,\|f\|_{L^\infty},$$

and duality gives

$$\mathrm{Lip}(\mathsf{P}_t f) \leq \sqrt{8}\,t^{-1/2} \sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t k^-(X_r^x)/2\,\mathrm{d}r}\Big]\,\|f\|_{L^\infty}.$$ □

Now we assume Kato decomposability of k in the rest of this chapter, devoting ourselves to the proof of Theorem 1.5. In this situation, one has to guarantee that the right-hand side of Bismut–Elworthy–Li's formula is well-defined for $f \in L^p(\mathrm{M})$, where $p \in (1, \infty)$, which is essentially the content of the following lemma.

Lemma 3.3. *Let $t \geq 0$ and $V \in \Gamma_{L^\infty}(T\mathrm{M})$. Then for every $f \in L^\infty(\mathrm{M})$ and $x \in \mathrm{M}$, the random variable $f(X_t^x)\int_0^t \langle Q_s^x V(x),\, \mathrm{d}W_s^x\rangle$ is integrable. Moreover, for every $p \in (1,\infty]$, the operator E_t^V given on functions $f \in L^\infty(\mathrm{M}) \cap L^p(\mathrm{M})$ in terms of*

$$\mathsf{E}_t^V f(x) := \mathbb{E}\Big[f(X_t^x)\int_0^t \langle Q_s^x V(x), \mathrm{d}W_s^x\rangle\Big] \quad \textit{for every } x \in \mathrm{M}$$

extends to a bounded linear operator from $L^p(\mathrm{M})$ into $L^p(\mathrm{M})$, and the previous representation is valid and well-defined for every $f \in L^p(\mathrm{M})$.

Proof. Let $V \in \Gamma_{L^\infty}(T\mathrm{M})$ and $f \in L^\infty(\mathrm{M})$, for which we assume without loss of generality that $\big\||V|\big\|_{L^\infty} \leq 1$ and $\|f\|_{L^\infty} \leq 1$. Fix t and x as above. Given any $s \geq 0$, it follows from Gronwall's inequality and $\mathrm{Ric} \geq k$ on M that a.s.,

$$|Q_s^x| \leq \mathrm{e}^{-\int_0^s k(X_r^x)/2\,\mathrm{d}r}, \tag{3.2}$$

so that for every $q \in [1, \infty)$, we obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r \langle Q_s^x V(x), \mathrm{d}W_s^x\rangle\Big|^q\Big] &\leq (8q)^{q/2}\,\mathbb{E}\Big[\Big(\int_0^t |Q_s^x|^2\,\mathrm{d}s\Big)^{q/2}\Big]\\ &\leq (8q)^{q/2}\,t^{1/2}\sup_{y\in\mathrm{M}}\mathbb{E}\Big[\mathrm{e}^{\int_0^t qk^-(X_r^y)/2\,\mathrm{d}r}\Big]\end{aligned} \tag{3.3}$$

by Lemma 3.2. This inequality for $q = 1$ and Lemma 1.4 directly show the claimed integrability of the random variable $f(X_t^x)\int_0^t \langle Q_s^x V(x),\, \mathrm{d}W_s^x\rangle$. These facts also prove that E_t^V is a bounded linear operator from $L^\infty(\mathrm{M})$ into $L^\infty(\mathrm{M})$.

If $p \in (1,\infty)$, then Hölder's inequality, (3.3) for $q = p/(p-1)$, contractivity of $(\mathsf{P}_t)_{t\geq 0}$ and Theorem A.5 show that there exist finite constants $C_1, C_2 \geq 0$ depending only on k^- and p such that for every $f \in L^p(\mathrm{M}) \cap L^\infty(\mathrm{M})$,

$$\big\|\mathsf{E}_t^V f\big\|_{L^p} \leq C_1\,t^{(p-1)/2p}\,\mathrm{e}^{C_2 t}\,\|f\|_{L^p},$$

and we conclude by an approximation argument as in the proof of Theorem A.5. □

Proof of Theorem 1.5. Trivially, $L^\infty(\mathrm{M}) \cap L^p(\mathrm{M})$ is dense in $L^p(\mathrm{M})$. Note that, given $p \in (1,\infty)$, and $f \in L^p(\mathrm{M})$, it follows from the divergence theorem as well as Lemma 3.3 – replacing ξ by a smooth and bounded vector field $V \in \Gamma(T\mathrm{M})$ such that $V(x) = \xi$ – that both sides of (3.1) are continuous in f w.r.t. convergence in $L^p(\mathrm{M})$. In particular, the desired pointwise identity follows. □

4 Proof of Theorem 1.6

We turn to characterizations of continuous lower Ricci curvature bounds in terms of functional inequalities and existence couplings. Throughout this chapter, we assume that $k\colon \mathrm{M} \to \mathbb{R}$ is continuous, and only state explicitly if we need (1.1).

4.1 From the L^1-Bochner inequality to lower Ricci bounds

As already hinted, the key point in showing the implication "(ii) $\Longrightarrow$ (i)" in Theorem 1.6 is the well-known Bochner formula (1.3), subject to a clever choice of f as granted by the subsequent lemma, together with the chain rule to deduce $\mathrm{Ric} \geq k$ on M.

It is well-known in Riemannian geometry that, given any $x \in \mathrm{M}$, there exists an open subset $O_x \subset T_x\mathrm{M}$ such that the restriction of the exponential map to O_x provides a diffeomorphism $\exp_x\colon O_x \to \exp_x(O_x)$. We denote its inverse by $\exp_x^{-1}$.

Lemma 4.1 [vRS05, Lemma 3.2]**.** *Let $x \in \mathrm{M}$ and $\xi \in T_x\mathrm{M}$ with unit norm. Let $\mathscr{H} := \{\exp_x \eta : \eta \in O_x,\ \langle \eta, \xi\rangle = 0\}$ be the $(\dim(\mathrm{M}) - 1)$-dimensional hypersurface in M orthogonal to ξ at x. Then there exists an open neighborhood $U \subset \exp_x(O_x)$ of x such that the* signed distance function $\rho^\pm_{\mathscr{H}}\colon U \to \mathbb{R}$ *given by*

$$\rho^\pm_{\mathscr{H}}(y) := \rho(y,\mathscr{H})\,\mathrm{sgn}\langle \xi, \exp_x^{-1} y\rangle, \quad \textit{where} \quad \rho(y,\mathscr{H}) := \inf_{z\in\mathscr{H}} \rho(y,z),$$

obeys

$$\rho^\pm_{\mathscr{H}} \in C^\infty(U), \quad \nabla\rho^\pm_{\mathscr{H}}(x) = \xi, \quad |\nabla\rho^\pm_{\mathscr{H}}(U)| = \{1\}, \quad \mathrm{Hess}\,\rho^\pm_{\mathscr{H}}(x) = 0.$$

Proof of "(ii) $\Longrightarrow$ (i)*" in Theorem 1.6.* Let $x \in \mathrm{M}$, and let $\xi \in T_x\mathrm{M}$ obey $|\xi(x)| = 1$. Retaining the notation from Lemma 4.1, consider the function $\rho^\pm_{\mathscr{H}}$ provided therein. By Lemma 4.1, Bochner's formula (1.3) and the chain rule for Δ, we have

$$\begin{aligned}\mathrm{Ric}(x)(\xi,\xi) &= \Delta\frac{|\nabla\rho^\pm_{\mathscr{H}}|^2(x)}{2} - \langle \nabla\Delta\rho^\pm_{\mathscr{H}}(x), \nabla\rho^\pm_{\mathscr{H}}(x)\rangle \\ &= |\nabla\rho^\pm_{\mathscr{H}}(x)|\,\Delta|\nabla\rho^\pm_{\mathscr{H}}(x)| + \big|\nabla|\nabla\rho^\pm_{\mathscr{H}}(x)|\big|^2 - \langle \nabla\Delta\rho^\pm_{\mathscr{H}}(x), \nabla\rho^\pm_{\mathscr{H}}(x)\rangle \\ &\geq k(x)\,|\nabla\rho^\pm_{\mathscr{H}}(x)|^2 = k(x).\end{aligned}$$

The arbitrariness of ξ concludes the proof. □

4.2 From lower Ricci bounds to pathwise couplings

We start with the existence of a suitable coupling of Brownian motions under the inequality $\mathrm{Ric} \geq k$ on M, also assuming (1.1) in this section. (Note that the stochastic completeness of M is already known by Theorem 1.1.) The coupling technique is well-known and called *coupling by parallel displacement*, see [Cra91, Ken86, Wan14] and the references therein. See also [Wan94] for a "local" treatise on regular subdomains.

We first collect some notation. Denote by Cut_y the cut-locus of $y \in \mathrm{M}$, and by R the Riemannian curvature tensor of M. Put $d := \dim(\mathrm{M})$ and $\mathrm{Cut} := \{(x,y) \in \mathrm{M}\times\mathrm{M} : x \in \mathrm{Cut}_y\}$. Given any $(x,y) \in (\mathrm{M}\times\mathrm{M}) \setminus \mathrm{Cut}$, let $J_1, \dots, J_{d-1}$ be Jacobi fields along the unique minimal geodesic $\gamma\colon [0,\rho(x,y)] \to \mathrm{M}$ from x to y such that $\{J_1(r), \dots, J_{d-1}(r), \dot\gamma(z)\}$ is an orthonormal basis of $T_z\mathrm{M}$ both for $r=0$ and $z=x$ as well as $r=\rho(x,y)$ and $z=y$. Define

$$I(x,y) := \sum_{i=1}^{d-1} \int_0^{\rho(x,y)} \Big(\big|\nabla_{\dot\gamma_s} J_i(s)\big|^2 - \big\langle \mathrm{R}(\dot\gamma_s, J_i(s))\dot\gamma_s, J_i(s)\big\rangle\Big)\,\mathrm{d}s.$$

(In the weighted case $\Phi \neq 0$, the previous quantity has to be replaced by its weighted counterpart $I_\Phi(x,y) := I(x,y) - (\nabla\Phi)\rho(\cdot,y)(x) - (\nabla\Phi)\rho(\cdot,x)(y)$.)

Theorem 4.2 [Wan14, Theorem 2.3.2]**.** *For every $x,y \in \mathrm{M}$ with $x \neq y$, there exists a coupling (X^x, X^y) of Brownian motions on* M *starting in (x,y) which coincide past their coupling time $T(X^x,X^y) := \inf\{t \geq 0 : X_t^x = X_t^y\}$ and such that before $T(X^x,X^y)$, we have*

$$\mathrm{d}\rho\big(X_t^x, X_t^y\big) \leq \frac{1}{2}\, I\big(X_t^x, X_t^y\big)\,\mathrm{d}t.$$

The construction of this coupling is quite time- and space-demanding, whence we only sketch it; details are satisfactorily explained and motivated in [Cra91, Wan14]. Take a Brownian motion X^x on M starting in x. Using Brownian motion on the frame bundle and usual parallel transport, one can define an appropriate process $(X_t^y)_{t\in[0,\sigma(X^y))}$ until it hits the set $\mathrm{Cut}_{X_t^x}$ at time $\sigma(X^y) := \inf\{t > 0 : X_t^y \in \mathrm{Cut}_{X_t^x}\}$. It is then indeed a nontrivial task carried out in detail in [Cra91, Proposition 1] that X^y can be expanded past the critical time $\sigma(X^y)$ – so that in particular, $I\big(X_t^x, X_t^y\big)$ is well-defined for $t < T(X^x, X^y)$ – and the only additional effect on $\mathrm{d}\rho\big(X_t^x, X_t^y\big)$ is caused by a nonincreasing local time starting at 0. In view of bounding $\mathrm{d}\rho\big(X_t^x, X_t^y\big)$ from above, this contribution can thus be ignored. Moreover, (X^x, X^y) is a diffusion and therefore a Markov process [Hsu02a, Theorem 6.5.1].

A large part of the subsequent proof now follows [Wan14, Corollary 3.2.6].

Proof of "(i) $\Longrightarrow$ (iii)*" in Theorem 1.6.* By the index lemma, in the above notation, for all $x,y \in (\mathrm{M}\times\mathrm{M})\setminus\mathrm{Cut}$ we have

$$\begin{aligned} I(x,y) &\leq -\int_0^{\rho(x,y)} \Big[\sum_{i=1}^{d-1} \big\langle \mathrm{R}(\dot\gamma_s, J_i(s))\dot\gamma_s, J_i(s)\big\rangle\Big]\,\mathrm{d}s = -\int_0^{\rho(x,y)} \mathrm{Ric}(\gamma_s)(\dot\gamma_s,\dot\gamma_s)\,\mathrm{d}s \\ &\leq -\int_0^{\rho(x,y)} k(\gamma_s)\,\mathrm{d}s \leq -\rho(x,y)\,\underline{k}(x,y). \end{aligned}$$

Together with the integrated version of Theorem 4.2, this yields the claim. □

4.3 From pathwise couplings to the L^1-Bochner inequality

Even if k is smooth, the function $\underline{k}$ from (1.4) is in general only lower semicontinuous. A tool to bypass this lack of true continuity by approximation is the following fact, in which Lipschitz continuity on the product manifold $\mathrm{M}\times\mathrm{M}$ is understood w.r.t. the product metric ρ_2 given by $\rho_2^2((x,y),(x',y')) := \rho^2(x,x') + \rho^2(y,y')$.

Lemma 4.3. *Let $D \subset \mathrm{M}$ be a compact subset. Then, in $D \times D$, $\underline{k}$ is the pointwise limit of a pointwise increasing sequence of functions in* $\mathrm{Lip_b}(\mathrm{M}\times\mathrm{M})$ *which are everywhere not smaller than* $\inf \underline{k}(D\times D)$. *In particular, in D, k is the pointwise limit of a pointwise increasing sequence of functions in* $\mathrm{Lip_b}(\mathrm{M})$ *which are everywhere not smaller than* $\inf k(D)$.

Proof. Every lower semicontinuous, lower bounded function on $\mathrm{M}\times\mathrm{M}$ can be approximated pointwise on $\mathrm{M}\times\mathrm{M}$ by a pointwise increasing sequence of functions in $\mathrm{Lip_b}(\mathrm{M}\times\mathrm{M})$ which preserves uniform lower bounds, see [BHS19, Lemma 2.1] and the references therein. If $\underline{k}$ is not uniformly bounded from below, we apply the previous result to the function $\underline{\ell}\colon \mathrm{M}\times\mathrm{M}\to\mathbb{R}$ given by $\underline{\ell}(x,y) := \underline{k}(x,y)\,\mathbb{1}_{D\times D}(x,y) + \inf\underline{k}(D\times D)\,\mathbb{1}_{(D\times D)^{\mathrm{c}}}(x,y)$.

The second statement follows by noting that $k(x) = \underline{k}(x,x)$ for every $x\in\mathrm{M}$. □

The step from the pathwise coupling property w.r.t. k towards (1.5) requires a nontrivial extension of the arguments for [BHS19, Theorem 5.17] (which adapt the duality argument from [Kuw10] to the case of synthetic variable Ricci bounds and make crucial use of uniform lower boundedness of the Ricci curvature) for short times instead of fixed ones. This kind of localization argument was indeed used in [BHS19] in different variants at different instances. For this, the smoothness of Ric, allowing us to bound it locally uniformly from below apart from any information on the relation between Ric and k, plays a crucial role.

Given any $x\in\mathrm{M}$, let τ^x be the first exit time of Brownian motion starting in a fixed $x\in\mathrm{M}$ from $B_1(x)$. We will need the following exit time estimate for Brownian motion, which also holds for general gradient diffusions. It is a variant of [Wan14, Lemma 2.1.4], observing that the constant c_1 chosen in the proof therein can be chosen to be independent of x as long as x belongs to a compact subset of M, see [Hsu02a, Corollary 3.4.4, Corollary 3.4.5]. See also [Hsu02a, Theorem 3.6.1] and its proof.

Lemma 4.4. *For every compact $D\subset\mathrm{M}$, there exists a finite constant $c>0$ such that*

$$\mathbb{P}\big[\tau^x \le t\big] \le \mathrm{e}^{-c/t} \quad \textit{for every } x\in D,\ t\in(0,1].$$

Proof of "(iii) $\Longrightarrow$ (ii)*" in Theorem 1.6.* We start with some preparations. Let $f\in C_{\mathrm{c}}^\infty(\mathrm{M})$ and $x\in\mathrm{M}$ with $|\nabla f(x)|\neq 0$ be arbitrary, and let $\gamma\in\mathrm{Geo}(\mathrm{M})$ start in x with $\rho(x,\gamma_1)\le 1$. The continuity of k yields $k\ge K$ on $\overline{B}_6(x)$ for some $K\in\mathbb{R}$. Therefore, defining $D := \{z\in \mathrm{M} : z\in\overline{B}_1(\gamma_s) \text{ for some } s\in[0,1]\}$ we have

$$\underline{k}\ge K \quad \text{on } D\times D. \tag{4.1}$$

Finally, let $\underline{\ell}$ be any bounded Lipschitz function on $\mathrm{M}\times\mathrm{M}$ with $K\le\underline{\ell}\le\underline{k}$ on $D\times D$, see Lemma 4.3. Let us denote by (X^x, X^{γ_s}) a process starting in (x,γ_s) given by the pathwise coupling property w.r.t. k. This pair process still depends on s, but we suppress this dependency from the notation. Let τ^x and τ^{γ_s} denote the first exit times of the marginal Brownian motions X^x and X^{γ_s} from $B_1(x)$ and $B_1(\gamma_s)$, respectively. For every $s\in[0,1]$, a.s. we have

$$\begin{aligned}\rho\big(X_t^x, X_t^{\gamma_s}\big) \le \mathrm{e}^{-\int_0^t \underline{k}(X_r^x, X_r^{\gamma_s})/2\,\mathrm{d}r}\, s \le \mathrm{e}^{-\int_0^t \underline{\ell}(X_r^x, X_r^{\gamma_s})/2\,\mathrm{d}r}\, s&\\ \text{whenever } t\le\min\{\tau^x,\tau^{\gamma_s}\}.&\end{aligned} \tag{4.2}$$

Given any $s\in\big(0,\mathrm{e}^{-1/6}\big]$, define $t_s := -6c\log s$, where $c>0$ is the constant from Lemma 4.4 associated to D. Define the event $A_s := \{\tau^x > t_s,\ \tau^{\gamma_s} > t_s\}$ for $s\in(0,\delta]$. By joint smoothness of the heat semigroup, the time derivative of $|\nabla \mathsf{P}_t f(x)| = (|\nabla \mathsf{P}_t f(x)|^2)^{1/2}$ at $t = 0$ can be written and estimated via

$$\begin{aligned}
&\frac{|\nabla f(x)|^{-1}}{2} \langle \nabla\Delta f(x), \nabla f(x)\rangle \\
&\qquad \leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbb{E}\big[|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|\big] - |\nabla f(x)|\Big] \\
&\qquad = \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbb{E}\big[|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|\,(\mathbb{1}_{A_s} + \mathbb{1}_{A_s^{\mathrm{c}}})\big] - |\nabla f(x)|\Big].
\end{aligned}$$

The contribution of A_s^{c} becomes negligible thanks to

$$\mathbb{E}\big[|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|\,\mathbb{1}_{A_s^{\mathrm{c}}}\big] \leq 2\,\|f\|_{L^\infty}\,\big(\mathbb{P}[\tau^x \leq t_s] + \mathbb{P}[\tau^{\gamma_s} \leq t_s]\big) \leq 4\,\|f\|_{L^\infty}\, s^3$$

by Lemma 4.4 and since $1/t_s$ only grows logarithmically as $s \downarrow 0$. Thus, we concentrate on the behavior of the integrand on A_s, an event which we decompose three further mutually disjoint subsets $V_s := A_s \cap \{\rho(X^1_{t_s}, X^2_{t_s}) \geq s^{1/2}\}$, $W_s := A_s \cap \{\int_0^{t_s} \rho(X^1_r, X^2_r)\,\mathrm{d}r/t_s \geq s^{1/2}\}$, and $U_s := A_s \cap V_s^{\mathrm{c}} \cap W_s^{\mathrm{c}}$. Hence, it remains to estimate these three parts separately.

By (4.1), the contribution of V_s can be bounded via

$$\begin{aligned}
&\mathbb{E}\Big[\frac{|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|}{\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})}\,\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})\,\mathbb{1}_{V_s}\Big] \\
&\qquad \leq \big\||\nabla f|\big\|_{L^\infty}\, s^{-1/2}\,\mathbb{E}\Big[\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})^2\,\mathbb{1}_{A_s}\Big] \\
&\qquad \leq \big\||\nabla f|\big\|_{L^\infty}\, s^{3/2}\,\mathbb{E}\Big[\mathrm{e}^{-\int_0^{t_s} \underline{\ell}(X^x_r, X^{\gamma_s}_r)\,\mathrm{d}r}\,\mathbb{1}_{A_s}\Big] \\
&\qquad \leq \big\||\nabla f|\big\|_{L^\infty}\, s^{3/2}\,\mathrm{e}^{-Kt_s}.
\end{aligned}$$

In a similar way, we can control the influence of W_s by

$$\begin{aligned}
&\mathbb{E}\Big[\frac{|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|}{\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})}\,\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})\,\mathbb{1}_{W_s}\Big] \\
&\qquad \leq \big\||\nabla f|\big\|_{L^\infty}\,\frac{s^{-1/2}}{t_s}\int_0^{t_s}\mathbb{E}\Big[\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})\,\rho(X^x_r, X^{\gamma_s}_r)\,\mathbb{1}_{A_s}\Big]\,\mathrm{d}r \\
&\qquad \leq \big\||\nabla f|\big\|_{L^\infty}\, s^{3/2}\,\mathrm{e}^{-Kt_s}.
\end{aligned}$$

Finally turning to the study of the expectation on U_s, it is not difficult to derive from the Lipschitz continuity of $\underline{\ell}$ and Jensen's inequality that $\int_0^{t_s} \underline{\ell}(X^x_r, X^{\gamma_s}_r)\,\mathrm{d}r \geq \int_0^{t_s} \ell(X^x_r)\,\mathrm{d}r - \mathrm{Lip}(\underline{\ell})\,t_s\,s^{1/2}$ on W_s^{c}, where $\ell \in \mathrm{Lip}_{\mathrm{b}}(\mathrm{M})$ is defined by $\ell(x) := \underline{\ell}(x,x)$. Together with (4.2) and the definition of A_s, we then obtain

$$\begin{aligned}
&\mathbb{E}\Big[\frac{|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|}{\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})}\,\rho(X^x_{t_s}, X^{\gamma_s}_{t_s})\,\mathbb{1}_{U_s}\Big] \\
&\qquad \leq s\,\mathbb{E}\Big[\mathrm{e}^{-\int_0^{t_s} \ell(X^x_r)/2\,\mathrm{d}r}\,\mathrm{e}^{\mathrm{Lip}(\underline{\ell})\,t_s\,s^{1/2}/2}\,\mathrm{G}_s f(X^x_{t_s})\,\mathbb{1}_{A_s}\Big] \\
&\qquad \leq s\,\mathbb{E}\Big[\mathrm{e}^{-\int_0^{t_s} \ell(Y^x_r)/2\,\mathrm{d}r}\,\mathrm{e}^{\mathrm{Lip}(\underline{\ell})\,t_s\,s^{1/2}/2}\,\mathrm{G}_s f(Y^x_{t_s})\Big],
\end{aligned}$$

where $\mathrm{G}_s f(y) := \sup\{|f(y)-f(z)|/\rho(y,z) : z \in B_{s^{1/2}}(y)\}$. In the last step, we switched to a Brownian motion Y^x starting in x which is independent of s.

Now we paste these three estimates together. Using smoothness and uniform continuity of f (and of $|\nabla f|$ near x) as well as the fact that the Schrödinger semigroup with generator $-(\Delta-\ell)/2$ is well-defined, see Subsection A.2, we then finally arrive at

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbb{E}\big[|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|\,\mathbb{1}_{A_s}\big] - |\nabla f(x)|\Big]\\
&\qquad \leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbb{E}\big[|f(X^x_{t_s}) - f(X^{\gamma_s}_{t_s})|\,\mathbb{1}_{U_s}\big] - |\nabla f(x)|\Big]\\
&\qquad \leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\mathbb{E}\Big[\mathrm{e}^{-\int_0^{t_s} \ell(Y^x_r)/2\,\mathrm{d}r}\,\mathrm{e}^{\mathrm{Lip}(\underline{\ell})\,t_s\,s^{1/2}/2}\,\mathrm{G}_s f(Y^x_{t_s})\Big] - |\nabla f(x)|\Big]\\
&\qquad = \frac{1}{2}(\Delta-\ell)|\nabla f(x)|.
\end{aligned}$$

Since $\underline{\ell}$ was arbitrary, we conclude the inequality (1.5) by Lemma 4.3. □

A Kato decomposable lower Ricci bounds and their Schrödinger semigroups

A.1 The L^1-gradient estimate

In this section, we present a last equivalent characterization of the condition $\mathrm{Ric} \geq k$ on M for the class of Kato decomposable k in terms of gradient estimates for $(\mathsf{P}_t)_{t\geq 0}$. A similar result can be found in [Wu20, Corollary 2.2]. See also [Wan14, Theorem 2.3.1] for more geometric growth conditions on k^-, and [BHS19, Theorem 1.1] for the nonsmooth case under boundedness of k^-, the condition $\mathrm{Ric} \geq k$ on M interpreted in a synthetic sense [Stu15].

Theorem A.1. *Assume that $k\colon \mathrm{M} \to \mathbb{R}$ is a continuous and Kato decomposable function. Then any of the equivalent conditions in Theorem 1.6 is equivalent to the L^1-*gradient estimate w.r.t. *k, i.e. for every $f \in C^\infty_{\mathrm{c}}(\mathrm{M})$,*

$$|\nabla \mathsf{P}_t f(x)| \leq \mathbb{E}\Big[\mathrm{e}^{-\int_0^t k(X^x_r)/2\,\mathrm{d}r}\,|\nabla f|(X^x_t)\,\mathbb{1}_{\{t<\zeta^x\}}\Big] \quad \textit{for every } x \in \mathrm{M},\ t>0. \tag{A.1}$$

Proof. If k obeys $\mathrm{Ric} \geq k$ on M, then the claimed L^1-gradient estimate is just a restatement of Theorem 2.1 for exact 1-forms together with (2.4).

Conversely, assume the L^1-gradient estimate. A similar argument as in the proof of (i) in Theorem 1.1 in Section 3.1 – directly employing (2.4) and (A.1) instead of Theorem 2.1 – shows that M is stochastically complete. Let $f \in C^\infty_{\mathrm{c}}(\mathrm{M})$ and $x \in \mathrm{M}$ with $|\nabla f(x)| \neq 0$. Then

$$\begin{aligned}
&\frac{1}{|\nabla f(x)|}\,\langle \nabla f(x), \nabla\Delta f(x)\rangle\\
&\qquad = \lim_{t\downarrow 0}\frac{1}{t}\big[|\nabla \mathsf{P}_t f(x)| - |\nabla f(x)|\big]\\
&\qquad \leq \limsup_{t\downarrow 0}\frac{1}{t}\Big[\mathbb{E}\Big[\mathrm{e}^{-\int_0^t k(X^x_r)/2\,\mathrm{d}r}\,|\nabla f|(X^x_t)\Big] - |\nabla f(x)|\Big]\\
&\qquad \leq \Delta|\nabla f(x)| + \limsup_{t\downarrow 0}\frac{1}{t}\,\mathbb{E}\Big[\big[\mathrm{e}^{-\int_0^t k(X^x_r)/2\,\mathrm{d}r} - 1\big]\,|\nabla f|(X^x_t)\Big].
\end{aligned}$$

It remains to estimate the latter limit. Let τ^x be the first exit time of X^x from $B_1(x)$. Since k is bounded on the compact set $\overline{B}_1(x)$, and by continuity of Brownian sample paths, the dominated convergence theorem gives

$$\limsup_{t\downarrow 0} \frac{1}{t}\,\mathbb{E}\Big[\big[\mathrm{e}^{-\int_0^t k(X_r^x)/2\,\mathrm{d}r} - 1\big]\,|\nabla f|(X_t^x)\,\mathbb{1}_{\{t<\tau^x\}}\Big] = -k(x)\,|\nabla f(x)|.$$

Given any $\varepsilon > 0$, the uniform integrability condition (1.1) holds for $(1+\varepsilon)k^-$ in place of k^-, see Lemma 1.4. Hölder's inequality and Lemma 4.4 yield, for arbitrary $T > 0$,

$$\begin{aligned}\limsup_{t\downarrow 0} \frac{1}{t}\,\mathbb{E}\Big[\big[\mathrm{e}^{-\int_0^t k(X_r^x)/2\,\mathrm{d}r} - 1\big]\,|\nabla f|(X_t^x)\,\mathbb{1}_{\{t\geq\tau^x\}}\Big]&\\ \leq 2\,\big\||\nabla f|\big\|_{L^\infty}\left[\sup_{y\in\mathrm{M}}\mathbb{E}\Big[\mathrm{e}^{(1+\varepsilon)\int_0^T k^-(X_r^y)/2\,\mathrm{d}r}\Big]^{1/(1+\varepsilon)} + 1\right]&\\ \times \limsup_{t\downarrow 0}\frac{1}{t}\,\mathbb{P}\big[\tau^x \leq t\big]^{\varepsilon/(1+\varepsilon)} = 0,&\end{aligned}$$

and the L^1-Bochner inequality (1.5) readily follows. □

Remark A.2. One can replace $C_c^\infty(\mathrm{M})$ by $W^{1,2}(\mathrm{M})$ in Theorem A.1. This follows from semigroup domination [Gün17a, Theorem VII.8], and in turn relies on the fact that the Feynman–Kac formula for the heat semigroup on 1-forms, Theorem 2.1, holds for all square integrable 1-forms under Kato decomposability [Gün12], while this Feynman–Kac formula only holds for smooth compactly supported 1-forms under (1.1). ■

Remark A.3. Assume that k satisfies the more general condition (1.1) instead of Kato decomposability. Of course, if $\mathrm{Ric} \geq k$ on M, the L^1-gradient estimate from Theorem A.1 then still holds by virtue of Theorem 2.1. However, as it becomes apparent from the above proof, the converse implication seems to be more involved and to require at least some higher order exponential integrability of k^-. ■

A.2 Schrödinger semigroups

For Kato decomposable k, the right-hand side of (A.1) has a more analytic interpretation in terms of the *Schrödinger semigroup* associated to k, which is briefly discussed now.

Assume in this section that k is – for simplicity [SV96] – a function in $L^2_{\mathrm{loc}}(\mathrm{M})$, not necessarily continuous, which is Kato decomposable. Then $\Delta - k$ is essentially self-adjont in $L^2(\mathrm{M})$ [Gün17b], and the *Schrödinger semigroup* $(\mathsf{P}_t^k)_{t\geq 0}$ is defined to be $\mathsf{P}_t^k := \mathrm{e}^{t(\Delta-k)/2}$ via spectral calculus. This is a strongly continuous semigroup of bounded linear operators in $L^2(\mathrm{M})$. As k is Kato decomposable [SV96, Gün17a], $(\mathsf{P}_t^k)_{t\geq 0}$ has a pointwise well-defined version which, for every $f \in L^2(\mathrm{M})$, can be expressed via Brownian motion X^x on M in terms of the Feynman–Kac formula

$$\mathsf{P}_t^k f(x) = \mathbb{E}\Big[\mathrm{e}^{-\int_0^t k(X_r^x)/2\,\mathrm{d}r}\,f(X_t^x)\,\mathbb{1}_{\{t<\zeta^x\}}\Big] \quad \text{for every } x\in\mathrm{M},\ t\geq 0. \tag{A.2}$$

We are going to show that this semigroup extends to a strongly continuous semigroup of bounded operators in $L^p(\mathrm{M})$ for all $p \in [1,\infty)$, see Theorem A.5. To this end, we record *Khasminskii's lemma* (which relies on the Markov property of Brownian motion on M).

Lemma A.4 [Gün17a, Lemma VI.8]**.** *Let $v \in \mathcal{K}(\mathrm{M})$. Then for every $\delta > 1$ there exists a finite constant $C \geq 0$ depending only on $|v|$ and δ such that*

$$\sup_{x\in\mathrm{M}} \mathbb{E}\Big[\mathrm{e}^{\int_0^t |v(X_r^x)|\,\mathrm{d}r}\, \mathbb{1}_{\{t<\zeta^x\}}\Big] \leq \delta\, \mathrm{e}^{Ct} \quad \textit{for every } t \geq 0.$$

Theorem A.5. *Let $k\colon \mathrm{M} \to \mathbb{R}$ be a Kato decomposable function in $L^2_{\mathrm{loc}}(\mathrm{M})$. Then there exist finite constants $C_1, C_2 \geq 0$ depending only on k^- such that, for every $p \in [1,\infty]$ and $f \in L^2(\mathrm{M}) \cap L^p(\mathrm{M})$, we have*

$$\big\|\mathsf{P}_t^k f\big\|_{L^p} \leq C_1\, \mathrm{e}^{C_2 t}\, \|f\|_{L^p} \quad \textit{for every } t \geq 0. \tag{A.3}$$

In particular, for every $p \in [1,\infty]$, $(\mathsf{P}_t^k)_{t\geq 0}$ extends to a semigroup of bounded operators from $L^p(\mathrm{M})$ into $L^p(\mathrm{M})$ which indeed satisfies (A.3) *for every $f \in L^p(\mathrm{M})$ and, if $p < \infty$, is strongly continuous.*

Proof. The idea to prove (A.3) is to use Feynman–Kac's formula (A.2) together with Lemma A.4 to show the desired inequality in the cases $p = \infty$ and $p = 1$ (which needs an additional, but elementary exhaustion argument) and to apply Riesz–Thorin's theorem to extend it to all exponents $p \in [1,\infty]$. See [Gün17a, Theorem IX.2, Corollary IX.4] for details.

The existence of an extension of $(\mathsf{P}_t^k)_{t\geq 0}$ to a semigroup of bounded operators from $L^p(\mathrm{M})$ into $L^p(\mathrm{M})$ for every $p \in [1,\infty]$ still satisfying (A.3) is standard by approximation, but we include the argument for the convenience of the reader since similar arguments will appear later (see the proofs of Lemma 3.3 and of (ii) in Theorem 1.1). For $p < \infty$ and $f \in L^p(\mathrm{M})$, for any sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(\mathrm{M}) \cap L^p(\mathrm{M})$ converging to f in $L^p(\mathrm{M})$, (A.3) implies that $(\mathsf{P}_t^k f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathrm{M})$. Thus, we define $\mathsf{P}_t^k f$ as the L^p-limit of the latter sequence as $n \to \infty$, and (A.3) again shows that this definition is independent of the choice of $(f_n)_{n\in\mathbb{N}}$. In the case $p = \infty$, given any reference point $o \in \mathrm{M}$, the sequence $(f_n)_{n\in\mathbb{N}}$ defined by $f_n := f\, \mathbb{1}_{B_n(o)}$ converges pointwise to f. By (A.2) and Lemma A.4, the dominated convergence theorem shows that the pointwise limit $\mathsf{P}_t^k f$ of $(\mathsf{P}_t^k f_n)_{n\in\mathbb{N}}$ as $n \to \infty$ is well-defined, and again this definition does not depend on the choice of $(f_n)_{n\in\mathbb{N}}$ once it is demanded that $\|f_n\|_{L^\infty} \leq \|f\|_{L^\infty}$. It is clear that both approximation procedures preserve (A.3).

To show strong continuity of $(\mathsf{P}_t^k)_{t\geq 0}$ in $L^p(\mathrm{M})$ for $p < \infty$, by approximation and (A.3), it suffices to show continuity of $t \mapsto \mathsf{P}_t^k f$ on $[0,\infty)$ in $L^p(\mathrm{M})$ for $f \in L^2(\mathrm{M}) \cap L^p(\mathrm{M}) \cap L^\infty(\mathrm{M})$. By the semigroup property, we may restrict to continuity at $t = 0$. Given any $x \in \mathrm{M}$, note that a.s., we have $\int_0^t k(X_r^x)\,\mathrm{d}r \to 0$ as $t \downarrow 0$ since $k \in L^1_{\mathrm{loc}}(\mathrm{M})$, and that

$$\Big|\mathrm{e}^{-\int_0^t k(X_r^x)/2\,\mathrm{d}r} - 1\Big| \leq \mathrm{e}^{\int_0^T k^-(X_r^x)/2\,\mathrm{d}r} + 1 \tag{A.4}$$

for every $t \in [0,T]$ is satisfied a.s. for fixed $T > 0$. Since

$$\int_{\mathrm{M}} \big|\mathsf{P}_t^k f - \mathsf{P}_t f\big|^p \,\mathrm{d}\mathfrak{m} \leq \int_{\mathrm{M}} \mathbb{E}\Big[\Big|\mathrm{e}^{-\int_0^t k(X_r^x)/2\,\mathrm{d}r} - 1\Big|\, |f(X_t^x)|\, \mathbb{1}_{\{t<\zeta^x\}}\Big]^p \,\mathrm{d}\mathfrak{m}(x),$$

applying the dominated convergence theorem twice using (A.4) as well as Lemma A.4, we obtain $\mathsf{P}_t^k f - \mathsf{P}_t f \to 0$ in $L^p(\mathrm{M})$ as $t \downarrow 0$. The result follows immediately by strong continuity of the heat flow $(\mathsf{P}_t)_{t\geq 0}$ in $L^p(\mathrm{M})$. □

A.3 Proof of Theorem 1.3

Now, we present one possible step-by-step analysis in order to check the existence of (continuous) Kato decomposable lower Ricci bounds for M, along with proving Theorem 1.3. Let us abbreviate $d := \dim(\mathrm{M})$.

Proof of Theorem 1.3. Let $\Xi : M \to (0, \infty)$ be a Borel function such that

$$\sup_{y \in M} \mathsf{p}_t(x, y) \leq \Xi(x)\, t^{-d/2} \quad \text{for every } x \in \mathrm{M},\ t \in (0, 1]. \tag{A.5}$$

It has been shown in [Gün17b], using a parabolic L^2-mean value inequality, that every Riemannian manifold admits a canonical choice of a function Ξ as above. [Gün17a, Proposition VI.10] states that for every $p \in [1, \infty)$, if $d = 1$, and every $p \in (d/2, \infty)$, if $d \geq 2$, we have $L^p_\Xi(\mathrm{M}) + L^\infty(\mathrm{M}) \subset \mathcal{K}(\mathrm{M})$. Thus, any locally $\mathfrak{m}$-integrable function $k\colon \mathrm{M} \to \mathbb{R}$ such that

$$k^- \in L^p_\Xi(\mathrm{M}) + L^\infty(\mathrm{M})$$

for some Ξ and p as above is Kato decomposable.

Now let $\langle \cdot, \cdot \rangle$ be quasi-isometric to a complete metric on M whose Ricci curvature is bounded from below by constant. Then, as the Li–Yau heat kernel estimate, the Cheeger–Gromov volume estimate and the local volume doubling property are qualitatively stable under quasi-isometry, it follows from the considerations in [Gün17a, Example IV.18] that there exists a constant $C_1 > 0$ such that

$$\mathsf{p}_t(x, y) \leq C_1 \operatorname{vol}\big[B_1(x)\big]^{-1} t^{-d/2} \quad \text{for every } x, y \in \mathrm{M},\ t \in (0, 1].$$

Thus every $k : \mathrm{M} \to \mathbb{R}$ such that, choosing $\Xi := \operatorname{vol}\big[B_1(\cdot)\big]^{-1}$, one has

$$k^- \in L^p_\Xi(\mathrm{M}) + L^\infty(\mathrm{M})$$

for some p as in the previous step is Kato decomposable. □

Remark A.6. The previous proof shows that the assertion of Theorem 1.3 remains valid if the inverse volume function used therein is replaced by any function obeying (A.5). ■

Example A.7. Assume that M is a model manifold in the sense of [Gri09], meaning that $\mathrm{M} = \mathbb{R}^d$ as a manifold with $d \geq 2$, and that the Riemannian metric $\langle \cdot, \cdot \rangle$ is given in polar coordinates as $\mathrm{d}r^2 + \psi(r)\,\mathrm{d}\theta^2$, where ψ is a smooth positive function on $(0, \infty)$. The volume of balls on such manifolds does not depend on the center, and the Ricci curvature behaves in the radial direction like $\psi''/\psi - (d-1)(\psi')^2/\psi^2$, see e.g. page 266 in [Bes87]. Assume now

$$\big(\psi''/\psi - (d-1)(\psi')^2/\psi^2\big)^- \in L^p_{\psi^{d-1}}(\mathbb{R}) + L^\infty(\mathbb{R}) \quad \text{for some } p > d/2.$$

Since the volume measure behaves in the radial direction as $\psi^{d-1}\,\mathrm{d}r$, it follows that the Ricci curvature is bounded from below by a function with negative part in $L^p(\mathrm{M}) + L^\infty(\mathrm{M})$.

To make sure that the latter function space is included in $\mathcal{K}(\mathrm{M})$ it suffices from the above considerations to assume that there exists a smooth positive function ψ_0 on $(0, \infty)$ such that

a. $\psi_0(0) = 0$, $\psi_0'(0) = 1$ and $\psi_0''(0) = 0$,

b. $\psi_0''/\psi_0 - (d-1)(\psi_0')^2/\psi_0^2$ is uniformly bounded from below by a constant, and

c. $\psi_0/C \leq \psi \leq C\psi_0$ for some constant $C > 1$.

Indeed, a. guarantees that there exists a complete metric g_0 on M which – in polar coordinates – is written as $g_0 = \mathrm{d}r^2 + \psi_0(r)\,\mathrm{d}\theta^2$. Assumption b. guarantees that the Ricci curvature associated to g_0 is bounded from below by a constant, and c. implies that g is quasi-isometric to g_0. For instance, one can take the Euclidean metric corresponding to $\psi_0(r) := r$ or the hyperbolic metric corresponding to $\psi_0(r) = \sinh(r)$ as reference metrics. ■

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