Brownian motion and the Feynman-Kac formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Solution hints to Sheet 6:

1. If $h_+ \in W_0^{1,2}(M)$, then one has $h \leq v := h_+$.
Conversely, suppose that there exists $v \in W_0^{1,2}(M)$ with $h \leq v$.
Case $v \in C_c^\infty(M)$. Pick $\phi \in C_c^\infty(M)$ with $0 \leq \phi \leq 1$ on M and $\phi = 1$ on $\mathrm{supp}(v)$. Then using $h \leq v$ one easily checks

$$h_+ = ((1-\phi)v + \phi h)_+.$$

Since ϕh is a compactly supported element of $W^{1,2}(M)$, one has $\phi h \in W_0^{1,2}(M)$, and so

$$(1-\phi)v + \phi h \in W_0^{1,2}(M),$$

and thus by Example 3.6 one has

$$h_+ = ((1-\phi)v + \phi h)_+ \in W_0^{1,2}(M).$$

Case $v \in W_0^{1,2}(M)$: Pick a sequence $v_n \in C_c^\infty(M)$ with $v_n \to v$ in $W^{1,2}(M)$. Then, because $h \leq v$, we have

$$h_n := h + (v_n - v) \leq v_n$$

and so by the previous case $(h_n)_+ \in W_0^{1,2}(M)$ for all n. Since $h_n \to h$ in $W^{1,2}(M)$, Example 3.6 gives $(h_n)_+ \to h_+$ in $W^{1,2}(M)$ and $W_0^{1,2}(M) \subset W^{1,2}(M)$ is a closed subspace, and we end up with $h_+ \in W_0^{1,2}(M)$.

2. The map

$$\varphi : H^m \longrightarrow B_e^{\mathbb{R}^m}(0,1) = \{y \in \mathbb{R}^m : |y| < 1\}, \quad (x', x^{m+1}) \longmapsto \frac{x'}{x^{m+1}+1}, \tag{1}$$

where

$$|y|^2 := \sum_{i=1}^m y_i^2,$$

is a global chart with

$$\varphi^{-1} : B_e^{\mathbb{R}^m}(0,1) \longrightarrow H^m, \quad y \longmapsto \left(\frac{2y}{1-|y|^2}, \frac{1+|y|^2}{1-|y|^2} \right). \tag{2}$$

Since for all $(x', x^{m+1}) \in H^m$ one has

$$(x^{m+1})^2 - (x')^2 = 1$$

we get (with the usual abuse of notation denoting the map of the charts and the points with the same symbol)

$$x^{m+1} dx^{m+1} = \sum_{i=1}^{m} (x')^i d(x')^i$$

and (1) implies that for all $i = 1, \ldots, m$,

$$dy^i = \frac{(1 + x^{m+1}) d(x')^i - (x')^i dx^{m+1}}{(1 + x^{m+1})^2}.$$

Using the latter two formulae one calculates

$$\sum_{i=1}^{m} dy^i \otimes dy^i = (1 + x^{m+1})^{-2} \sum_{i=1}^{m} d(x')^i \otimes d(x')^i - dx^{m+1} \otimes dx^{m+1},$$

and so

$$\sum_{i=1}^{m} d(x')^i \otimes d(x')^i - dx^{m+1} \otimes dx^{m+1} = (1 + x^{m+1})^2 \sum_{i=1}^{m} dy^i \otimes dy^i.$$

Since by (2)

$$1 + x^{m+1} = 2/(1 - |y|^2),$$

we end up with the following formula in the above chart (with $g_e^{\mathbb{R}^m}$ the Euclidean metric on $\mathbb{R}^m$):

$$g_{\text{Mink}} = \frac{4}{1 - |\cdot|^2} g_e^{\mathbb{R}^m} \quad \text{on } H^m \cong B_e^{\mathbb{R}^m}. \tag{3}$$

This formula shows that g_{Mink} is positive definite on H^m.
To see that $\mathbb{H}^m := (H^m, g_{\text{Mink}})$ is complete, note that a (connected) Riemannian manifold M is complete, if and only if for every piecewise smooth curve $\gamma : [0, \infty) \to M$ which eventually leaves every compact subset of M one has

$$\int_0^\infty |\dot{\gamma}(t)| dt = \infty.$$

The latter condition can be checked easily on $\mathbb{H}^m$ using (3), noting that a continuous curve $\gamma : [0, \infty) \to H^m$ which eventually leaves every compact subset of H^m, if and only if γ eventually touches a point on $\partial B_e^{\mathbb{R}^m}$ and remains then there for all times.