Brownian motion and the Feynman-Kac formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
Prof. Dr. Batu Güneysu

Solution hints to Sheet 5:

1. It suffices to prove the $k = 1, 2$ case, as the general k-case then follows by induction.
$k = 1$: Assume f is weakly C^1 and let $x \in \mathbb{R}$. Then the limit

$$\lim_{t\to 0} t^{-1}(f(t+x) - f(x))$$

exists in the weak topology, so that for some $\epsilon > 0$, the familiy $t^{-1}(f(t+x)-f(x))$, $t \in (x-\epsilon, x+\epsilon)$, is weakly bounded and by uniform boundedness in fact bounded in the $\mathscr{H}$-topology, so there exists $C > 0$, such that for all $t \in (x-\epsilon, x+\epsilon)$ one has

$$\left\| t^{-1}(f(t+x) - f(x)) \right\| \leq C,$$

and so f is continuous in the $\mathscr{H}$-topology.
$k = 2$: Assume f is weakly C^2. Fix $x_0 \in \mathbb{R}$. Define a function $g : [x_0, \infty) \to \mathscr{H}$ by

$$\langle g(x), \phi \rangle := \left\langle f'(x), \phi \right\rangle, \quad \phi \in \mathscr{H}.$$

Then g is weakly C^1 and so by the $k = 1$ case in fact continuous in the $\mathscr{H}$-topology. Integrating the latter equation we obtain

$$f(x) = f(x_0) + \int_{x_0}^{x} g(r) dr,$$

where the integral is defined in the $\mathscr{H}$-topology. As g is continuous in the $\mathscr{H}$-topology, its integral (and so f) is C^1 in the $\mathscr{H}$-topology.

2. Assume $x_0 = 0$ is the origin of the chart V and recall that $\overline{V}$ lies in a chart. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and set

$$v(x, t) := e^{-\alpha t} f(|x - \zeta t|^2),$$

where $\alpha > 0$ will be chosen later. To guarantee that $v = 0$ on the lateral surface of $\overline{\Gamma}$ and $v > 0$ elswewhere, note that $(t, x) \in \Gamma$ implies $x - \zeta \in U$, so that we assume

$$f > 0 \text{ on } [0, r^2) \text{ and } f(r^2) = 0. \tag{1}$$

In addition,

$$f' \leq 0, \quad f'' \geq 0 \quad \text{ in } [0, r^2]. \tag{2}$$

Set

$$w(t,x) := |x - \zeta t|^2,$$

so

$$v = e^{-\alpha t} f \circ w.$$

Then for $(t,x) \in \Gamma$ one has by a simply calculation

$$\partial_t v = -e^{\alpha t}(\alpha f \circ w + C f' \circ w),$$

as x, t run through a compact set, where $C > 0$ is a constant. Morever, since the coefficients of Δ are bounded in V (the latter sitting relatively compactly in another chart!), one gets

$$\Delta w \leq C',$$

and

$$|dw|^2 \geq c' \sum_i (\partial_i w)^2 = cw,$$

where c' is a constant that comes from $g^{ij} \geq c' \delta^{ij}$ and $c := c'/4$. Calculating Δv with the chain rule and using (2) one obtains

$$\partial_t v - \Delta v \leq -e^{\alpha t}(\alpha f \circ w + C_1 f' \circ w + C_2 w f'' \circ w).$$

Finally, taking f to be of the form $f(s) := (r^2 - s^2)$ we have (1), (1), and one can pick α large enough such that

$$\alpha f \circ w + C_1 f' \circ w + C_2 w f'' \circ w \geq 0$$

in $[0, r^2]$, which completes the proof.

3. Step 1: $f \leq 1$ implies $(H + \lambda)^{-1} f \leq 1/\lambda$ for all $\lambda > 0$.
Proof of Step 1: We are going to show that $f \leq \lambda$ implies $(H + \lambda)^{-1} f \leq 1$. Set

$$u := (H + \lambda)^{-1} f \in \mathrm{Dom}(H) \subset W_0^{1,2}(M).$$

We are going to show that with

$$v := (u - 1)_+ \in W_0^{1,2}(M)$$

one has $v = 0$. Multiplying

$$-\Delta u + \lambda u = f$$

with v and integrating by parts we get

$$\int (du, dv) d\mu + \lambda \int uv d\mu = \int f v d\mu. \tag{3}$$

In view of $dv = du$ on $\{u > 1\}$ and $dv = 0$ elsewhere, we have

$$\int (du, dv) d\mu = \int_{\{u>1\}} |du|^2 d\mu \geq 0,$$

while

$$\int uv d\mu = \int_{\{v>0\}} (v+1) v d\mu = \int v^2 d\mu + \int v d\mu$$

Thus, using (3) and $f \leq \lambda$ we get

$$\lambda \int v^2 d\mu + \lambda \int v d\mu \leq \int f v d\mu \leq \lambda \int v d\mu,$$

thus

$$\lambda \int v^2 d\mu \leq 0,$$

and so $v = 0$.

Step 2: $f \leq 1$ implies $P_t f \leq 1$ for all $t > 0$.
Proof of Step 2: applying the formula $e^{-tr} = \lim_k \left(\frac{k}{t}\right)^k (r + k/t)^{-k}$ for $r = H$ (spectral calculus), we get the L^2-convergence

$$P_t = e^{-tH} = \lim_k \left(\frac{k}{t}\right)^k (H + k/t)^{-k}.$$

Since by step 1 we have

$$(H + k/t)^{-k} f \leq \left(\frac{k}{t}\right)^{-k},$$

we immediately get $P_t f \leq 1$. Note here that on any measure space, $h_n \to h$ in L^q and $h_n \leq c$ for all n, implies $h \leq c$.