Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Solution hints to Sheet 4:

1. We can assume WLOG that all data is real-valued.
Let h be an arbitrary smooth function $M :\to \mathbb{R}$ with compact support. Then, using (Ex. sheet 3) he product rule for d,

$$fdh = d(fh) - hdf,$$

we get

$$\begin{align} &\int_M d^\dagger(f\alpha)hd\mu \tag{1}\\ &= \int_M (f\alpha, dh)d\mu \tag{2}\\ &= \int_M (\alpha, fdh)d\mu \tag{3}\\ &= \int_M (\alpha, d(fh))d\mu - \int_M (\alpha, hdf)d\mu \tag{4}\\ &= \int_M (d^\dagger\alpha)fhd\mu - \int_M (df,\alpha)hd\mu, \tag{5} \end{align}$$

so the product rule for follows from the fundamental theorem of distribution theory.
The product rule for $-\Delta = d^\dagger d$ follows from the one for d and the one for $d^\dagger$.
The chain rule for $-\Delta = d^\dagger d$ follows from the one for d and the product rule for $d^\dagger$.

2. i) The asserted property does not depend on g: indeed, let g' be another Riemannian metric on g. For all smooth sections Ψ in $F \to M$ an integration by parts shows the formula

$$P^{g',h_E,h_F}\Psi = \frac{d\mu_g}{d\mu'_g}P^{g,h_E,h_F}\left(\frac{d\mu'_g}{d\mu_g}\Psi\right),$$

where

$$0 < \frac{d\mu_g}{d\mu'_g}, \frac{d\mu'_g}{d\mu_g} \in C^\infty(M)$$

denote the Radon-Nikodym densities. Thus for all smooth sections Ψ in $F \to M$ with compact support one has

$$\begin{aligned}
&\int h_E(h, P^{g',h_E,h_F}\Psi)d\mu'_g \\
&= \int h_E\left(h, \frac{d\mu_g}{d\mu'_g}P^{g,h_E,h_F}\left(\frac{d\mu'_g}{d\mu_g}\Psi\right)\right)\frac{d\mu'_g}{d\mu_g}d\mu_g \\
&= \int h_E\left(h, P^{g,h_E,h_F}\left(\frac{d\mu'_g}{d\mu_g}\Psi\right)\right)d\mu_g.
\end{aligned}$$

Since $\frac{d\mu'_g}{d\mu_g}\Psi$ is again smooth and compactly supported, the latter expression is (by definition of h)

$$= \int h_E\left(Ph, \left(\frac{d\mu'_g}{d\mu_g}\Psi\right)\right)d\mu_g \tag{6}$$

$$= \int h_E\left(Ph, \Psi\right)d\mu'_g. \tag{7}$$

ii) The asserted property does not depend on h_E, h_F: let h'_E and h'_F be other smooth metrics on $E \to X$ and on $F \to X$, respectively. Define isomorphisms of smooth vector bundles by

$$\begin{aligned}
&S_E : E \longrightarrow E,\ h'_E(S_E\phi_1, \phi_2) := h_E(\phi_1, \phi_2), \\
&S_F : F \longrightarrow F,\ \ h'_F(S_F\psi_1, \psi_2) := h_F(\psi_1, \psi_2).
\end{aligned}$$

Note that $h_E(S_E^{-1}\phi_1, \phi_2) = h'_E(\phi_1, \phi_2)$, and $h_F(S_F^{-1}\psi_1, \psi_2) = h'_F(\psi_1, \psi_2)$. Integrating by parts one finds

$$P^{g,h'_E,h'_F} = S_E^{-1}P^{g,h_E,h_F}S_F,$$

which easily entails the asserted independence by the same reasoning as above.

3. i) It suffices to show that $\overline{B(x,r)} \supset C(x,r) := \overline{B}(x,r)$. Let $y \in C(x,r)$. The only interesting case occurs when $\varrho(x,y) = r$. For any $\epsilon > 0$ we find a piecewise smooth path $\alpha : [0,1] \to M$ such that $\alpha(0) = x$, $\alpha(1) = y$, $\ell(\alpha) < r + \epsilon$. Since $\ell(\alpha|_{[t,1]})$ is continuous, there exists $0 < t_\epsilon < 1$ such that

$$\ell(\alpha|_{[t_\epsilon,1]}) = 3\epsilon.$$

Set

$$z_\epsilon = \alpha(t_\epsilon).$$

Then

$$\varrho(y, z_\epsilon) \leq \ell(\alpha|_{[t_\epsilon,1]}) = 3\epsilon.$$

On the other hand,

$$\varrho(x, z_\epsilon) \leq \ell(\alpha|_{[0,t_\epsilon]}) = \ell(\alpha) - \ell(\alpha|_{[t_\epsilon,1]}) \leq r + \epsilon - 3\epsilon < r$$

showing that

$$z_\epsilon \in B(x, r).$$

We have obtained that: given y with $\varrho(x, y) = r$, for any $\epsilon > 0$ there exists $z_\epsilon \in B(x, r)$ such that $\varrho(z_\epsilon, y) = 3\epsilon$. Choosing $\epsilon = 1/n$ and letting $n \to \infty$ we get that the there exists a sequence $z_n \in B(x, r)$ converging to y. Therefore, $y \in \overline{B(x, r)}$.

ii) For an arbitrary (that is, possibly incomplete) M, all bounded closed subsets of M are compact, if and only if all closed balls are compact (this is true on any metric space), and by i) this is equivalent to all open balls being relatively compact, which is equivalent to all bounded open subsets being relatively compact. If any of these equivalent statements are satisfied, then M is complete: indeed, any metric space whose bounded closed subsets are compact is complete.

Lemma 0.1. *For an arbitrary M given, given $\epsilon > 0$, $a, b \in M$ there exists $c \in M$ with*

$$\max(\varrho(a, c), \varrho(b, c)) \leq \frac{1}{2}\varrho(a, b) + \epsilon.$$

Proof. Pick a piecewise smooth path $\alpha : [0, 1] \to M$ from x to y with $\ell(\gamma) < \varrho(a, b) + \epsilon$. Pick $T \in (0, 1]$ with $\ell(\alpha|_{[0,T]}) = (1/2)\ell(\alpha)$. Then $c := \alpha(T)$ does the job. ■

Now assume M is complete. Fix $z \in M$. We are going to show that for all $r \geq 0$ the set $C(z, r) = \overline{B}(z, r)$ is compact. Let

$$I := \{r \geq 0 : C(z, r) \text{ is compact}\}.$$

Then I is an interval which contains 0. We are going to show that $I \subset [0, \infty)$ is open and closed.
I is open: Given $r \in I$, cover $C(z, r)$ with finitely many relatively compact $B(x_i, s_i)$ (here we use that M is locally compct). Then there exists $\delta > 0$ such that

$$C(z, r + \delta) \subset \bigcup_i C(x_i, s_i),$$

the latter being compact, and so $r + \delta \in I$.

I is closed: Let $R > 0$ with $[0, R) \subset I$. We are going to show that every sequence y_j in $C(z, R)$ has a convergent subsequence. As a consequence $C(z, R)$ is compact and so $[0, R] \subset I$.
To this end, let s_i be a decreasing sequence in $(0, R)$ with $s_i \to 0$. By the above Lemma, for all i, j we can pick

$$x_j^i \in C(z, R - s_i/2)$$

such that

$$\varrho(x_j^i, y_j) \leq s_i.$$

Since $C(z, R - s_1/2)$ is compact, The sequence x_j^1 has a convergent subsequence $x_{j(1,k)}^1$. The induced sequence $x_{j(1,k)}^2$ has a convergent subsequence $x_{j(2,k)}^2$, and so on. Put $j(k) := j(k, k)$. Then $(x_{j(k)}^i)_k$ converges for all i, and using this one easily proves that the subsequence $(y_{j(k)})_k$ of $(y_j)_j$ is Cauchy and, since X is complete, converges.