Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Outlines of solutions to Sheet 3:

1. Existence of μ_g: We need need to show that if $(\phi = (x^1, \dots, x^m), U)$ and $(\psi = (y^1, \dots, y^m), V)$ are two charts on M, and if $N \subset U \cap V$ is Borel set, then (with an obvious notation) one has

$$\int_N \sqrt{\det(g^\phi)} dx = \int_N \sqrt{\det(g^\psi)} dy.$$

We have

$$\det(g^\phi) = \det(J)^2 \det(g^\psi),$$

where $J_{kj} := \partial y^k / \partial x^j$ is the Jacobian of the coordinate change from ϕ to ψ (a diffeomorphism!). The last identity follows from

$$g^\phi = J^\dagger g^\psi J,$$

which in turn follows from the chain rule. Using the transformation formula for integrals

$$\int_{U\cap V} f dy = \int_{U\cap V} f |\det(J)| dx$$

with $f := 1_N \sqrt{\det(g^\psi)}$, the claimed formula follows.
Uniqueness of μ_g: The Borel sigma algebra on M is generated by charts, so this follows from abstract measure theory.

2. Define an endomorphism $A : T^*M \to T^*M$ (that is, a smooth section of the smooth vector bundle $\mathrm{End}(T^*M) \to M$) by

$$g^*(A(x)\alpha(x), \beta(x)) := h^*(\alpha(x), \beta(x)), \quad x \in M,\ \alpha, \beta \in \Omega^1_{C^\infty}(M),$$

where g^*, resp. h^* denote the dual metrics. Then one locally has (it might be a good exercise to check this!)

$$\sqrt{\det(h)} / \sqrt{\det(g)} = 1/\sqrt{\det(A)},$$

so the smooth function $\rho_{g,h} := 1/\sqrt{\det(A)} : M \to (0, \infty)$ satisfies

$$d\mu_h = \rho_{g,h} d\mu_g.$$

A posteriori we get that the quotient $\sqrt{\det(g)}/\sqrt{\det(h)}$ (a priori only defined in charts), is a global definition (that is: independent of the choice of coordinates). This remarkable, since $\sqrt{\det(g)}$ is not a global definition.

3. Both formulae follow from the local formula $df = \sum_{i=1}^m (\partial_i f) dx^i$.