(1) $T_U$ is closable: The operator

$T_U^{max}$ def. by $\mathrm{Dom}(T_U^{max}) := \{ f \in L^2(U) :$

$\underbrace{-\Delta f}_{\text{as a distrib.}} \in L^2(U) \}$, $T_U^{max} f := -\Delta f$

is easily checked to be closed (distrib. theory; see also solution to exe. sheet 2, ex. 2)

and clearly $T_U \subset T_U^{max}$.

$T_U$ is not closed (I could only prove it for $U = \mathbb{R}^m$ !):

Pick $f \in L^2(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m)$ with $-\Delta f \in L^2(\mathbb{R}^m)$
and $f \notin C_c^\infty(\mathbb{R}^m)$ (such $f$ should also exist for $U \neq \mathbb{R}^m$).

Pick $(\varphi_n) \subset C_c^\infty(\mathbb{R}^m)$ with $\varphi_n \to 1$ pointwise,

$|\nabla \varphi_n| \to 0$, $\Delta \varphi_n \to 0$ (e.g. $\varphi_n(x) := f\left(\frac{|x|}{n}\right)$
where $f \in C_c^\infty(\mathbb{R}^m)$, $f(0) = 1$) (Such a sequence $\varphi_n$
only exists for $U = \mathbb{R}^m$ !).

Set $f_n := \varphi_n f$. Then $f_n \in C_c^\infty(\mathbb{R})$,

$f_n \to f$ in $L^2(\mathbb{R}^m)$, $\Delta f_n \to \Delta f$ in $L^2(\mathbb{R}^m)$

But $f \notin C_c^\infty(\mathbb{R}^m)$

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2a) $T_u^*$ is given (distribution theory!) as follows:

$$\mathrm{Dom}(T_u^*) = \{ f \in L^2(U) : \Delta f \in L^2(U) \}$$

(as $L^2(U) \subset L^1_{loc}(U) \subset D'(U) = \{\text{distributions on } U\}$

$\Delta f$ is a priori def. as an element of $D'(U)$, for all $f \in L^2(U)$; the requirement $\Delta f \in L^2(U)$ means that $\Delta f$ is a regular distrib. which ~~stems~~ stems ~~fro~~ from an $L^2(U)$ function)

2b) The Dirichlet realization of $T_u$ is a self-adj extension of $T_u$!

(3a) $T_{\mathbb{R}^m}$ is unitarily equiv. to $\tilde{T}_{\mathbb{R}^m}$,

where $\mathrm{Dom}(\tilde{T}_{\mathbb{R}^m}) = C_c^\infty(\mathbb{R}^m)$, $\tilde{T}_{\mathbb{R}^m} f(x) = |x|^2 f(x)$

(Fourier transform), $\tilde{T}_{\mathbb{R}^m}$ is ess. s.a:

$\overline{\tilde{T}_{\mathbb{R}^m}}$ is $M_\varphi$, where $\varphi(x) := |x|^2$, and $M_\varphi$

is self-adj.

We have $\overline{T_{\mathbb{R}^m}}$ = unique s.a. extension of

$T_{\mathbb{R}^m}$, and by unit. equiv (which extends to closures)

$$\sigma(\overline{T_{\mathbb{R}^m}}) = \sigma(M_\varphi) \overset{(\alpha)}{=} [0,\infty) \text{ and}$$

$$\sigma_d(\overline{T_{\mathbb{R}^m}}) = \sigma_d(M_\varphi) \overset{(\beta)}{=} \emptyset \text{ where}$$

($\alpha$) follows from exe 1c) on sheet 1 and ($\beta$)

from

$\lambda \in \sigma_d(M_\varphi) \Leftrightarrow \lambda$ is isolated eigenvalue of $M_\varphi$ with finite multiplicity, but $M_\varphi$ has no eigenvalues at all: $\mathrm{Ker}(M_\varphi - \lambda) = \{0\} \Leftrightarrow \{x \in \mathbb{R}^m : |x|^2 = \lambda\}$

has Lebesgue measure 0 (exe. 1c) on sheet 1),

and $\{x \in \mathbb{R}^m : |x|^2 = \lambda\}$ has measure $0$.

(3b) Given an ~~arbitrary~~ arbitrary $\mathcal{U}$,
we have with $-\Delta_{\mathcal{U}} :=$ ~~the~~ Dirichlet realization
of $T_{\mathcal{U}}$, that $\mathrm{Dom}(Q_{-\Delta_{\mathcal{U}}})$
$:=$ Dom of def of form induced by $-\Delta_{\mathcal{U}}$
$= W_0^{1,2}(\mathcal{U}) =$ Closure of $C_c^\infty(\mathcal{U})$ w.r.t
$$\|f\|_{W^{1,2}(\mathcal{U})} := \|f\|_{L^2(\mathcal{U})} + \|\nabla f\|_{L^2(\mathcal{U},\mathbb{C}^n)}$$

Define ~~a~~ another form by
$$\mathrm{Dom}(Q'_{\mathcal{U}}) := \{ f \in L^2(\mathcal{U}) : \nabla f \in L^2(\mathcal{U},\mathbb{C}^n) \} =: W^{1,2}(\mathcal{U})$$
as a distrib.

$$Q'_{\mathcal{U}}(f,g) := \int_{\mathcal{U}} (\nabla f, \nabla g)\, dx$$

Then $Q'_{\mathcal{U}}$ is densely def., closed, $\geq 0$

$\Longrightarrow \exists!$ s.a. op $T'_{\mathcal{U}}$ in $L^2(\mathcal{U})$ with $Q_{T'_{\mathcal{U}}} = Q'_{\mathcal{U}}$
(This is called the Neumann real. of $T_{\mathcal{U}}$)

The spaces ~~the~~ $W^{1,2}(\mathcal{U})$ and $W_0^{1,2}(\mathcal{U})$ are in general not equal, and so in general we have $T'_{\mathcal{U}} \neq$ ~~the~~ $-\Delta_{\mathcal{U}}$

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For example: Assume

$U$ has finite Lebesgue measure and $\mathbb{R}^m \setminus \overline{U} \neq \emptyset$.

Then $1_U \in W^{1,2}(U)$, because $1_U \in L^2(U)$ and $\nabla 1_U = 0 \in L^2(U, \mathbb{R}^m)$, But $1_U \notin W_0^{1,2}(U)$ (this is a bit harder to prove).

(3c) Let again $-\Delta_U :=$ Dirichlet realiz. Then if $U$ is bounded with suff. nice boundary, the embedding $W_0^{1,2}(U) \overset{c}{\longrightarrow} L^2(U)$ is a compact op by the Rellich-Kondrachov theorem

Then the Resolvent $(-\Delta_U + \alpha)^{-1} \in \mathcal{L}(L^2(U))$ is compact for all $\alpha > 0$, because it factors as follows:

$$\begin{tikzcd} L^2(U) \arrow[r, "A"] \arrow[rr, bend right, "(-\Delta_U+\alpha)^{-1}"'] & W_0^{1,2}(U) \arrow[r, "c"] & L^2(U) \end{tikzcd}$$

, where $Af := (-\Delta_U + \alpha)^{-1} f$ (resolvent, but with diff. target space!)

Since the spectrum of compact operators consists of eigenvalues with fin. multipl, which converge to ~~zero~~ (we apply this to $(-\Delta_U+\alpha)^{-1}$), it follows easily that spectr of $-\Delta_U$ const. of eigenvalues of fin. multipl. with no finite acc. point.