1) One has $\mathrm{Dom}(M_\psi) := \{f \in L^2(X,\mu) : \psi f \in L^2(X,\mu)\}$

$$M_\psi f := \psi \cdot f$$

~~m~~ Remark: $M_\psi \in \mathcal{L}(L^2(X,\mu)) \iff \|\psi\|_{\mu,\infty} < \infty$

However, we will mainly deal with unbounded ψ's.

(1a) Let $(f_n) \subset \mathrm{Dom}(M_\psi)$ with $f_n \to f$ in $L^2(X,\mu)$ and $M_\psi f_n = \psi f_n \to g$ in $L^2(X,\mu)$. Pick a subsequence (f_{n_k}) of (f_n) with $f_{n_k}(x) \to f(x)$ and $\psi(x) f_{n_k}(x) \to \psi(x) f(x)$ for μ-a.e $x \in X$.

Then $g(x) = \psi(x) f(x)$ for μ-a.e $x \in X$, so $\psi f \in L^2(X,\mu)$, thus $f \in \mathrm{Dom}(M_\psi)$ and $g = \psi f = M_\psi f$.

(1b) Claim: $\mathrm{Dom}(M_\psi)$ is always dense (so M_ψ^* exists) and $M_\psi^* = M_{\overline{\psi}}$ ($\overline{z}$ = compl. conj. of $z \in \mathbb{C}$).

Proof: Let $f \in L^2(X,\mu)$, $X_n := \{|\psi| \le n\} \subset X$,

$$f_n(x) := \begin{cases} f(x), & x \in X_n \\ 0, & x \in X \setminus X_n \end{cases}$$

Then $(f_n) \subset \mathrm{Dom}(M_\psi)$ and $f_n \to f$ in $L^2(X,\mu)$ by dom. convergence, so $\mathrm{Dom}(M_\psi)$ is dense.

Obviously $\mathrm{Dom}(M_{\overline{\psi}}) = \mathrm{Dom}(M_\psi)$.

We have $\mathrm{Dom}(M_\psi^*)$

$$:= \{ g \in L^2(X,\mu) : \exists \tilde{g} \in L^2(X,\mu)\ \forall h \in \mathrm{Dom}(M_\psi)\ \int_X \bar{\tilde{g}}\, h \, d\mu = \int \bar{g} \cdot \psi \cdot h \, d\mu \}$$

$$M_\psi^* g = \tilde{g}.$$

Let $g \in \mathrm{Dom}(M_{\overline{\psi}})$, set $\tilde{g} := \overline{\psi} g$. Then for all $h \in \mathrm{Dom}(M_\psi)$ we have $\int \bar{\tilde{g}} h = \int \psi \bar{g} h$

$= \int \bar{g} \psi h \implies g \in \mathrm{Dom}(M_\psi^*)$ and $M_\psi^* g = \overline{\psi} g$

$\Rightarrow M_{\overline{\psi}} \subset M_\psi^*$.

Thus for $M_{\overline{\psi}} = M_\psi^*$ it remains to prove $\mathrm{Dom}(M_\psi^*) \subset \mathrm{Dom}(M_{\overline{\psi}})$. Let $g \in \mathrm{Dom}(M_\psi^*)$, let $n \in \mathbb{N}$ and let $f \in L^2(X,\mu)$ be supported in $X_n = \{ |\psi| \le n \}$.

Then $$\int_{X_n} (M_\psi^* g) \cdot f = \int_X (M_\psi^* g) \cdot f \overset{\#}{=} \int_X g (M_{\overline{\psi}} f) = \int_{X_n} g \cdot \overline{\psi} \cdot f$$

where $\#$ follows from $f \in \mathrm{Dom}(M_{\overline{\psi}})$ and $M_{\overline{\psi}} \subset M_\psi^*$.

So $\int_{X_n} (M_\psi^* g - \overline{\psi} g) f = 0$. Because $M_\psi^* g - \overline{\psi} g \in L^2(X_n,\mu)$ ($M_\psi^* g \in L^2(X,\mu)$, $g \in L^2(X,\mu)$, $\overline{\psi}$ bounded on X_n) this implies $M_\psi^* g = \overline{\psi} g$ in X_n and as n is arbitrary

$\underbrace{M_\psi^* g}_{\in L^2(X,\mu)} = \overline{\psi} g$ and so $\overline{\psi} g \in L^2(X,\mu)$ and so $g \in \mathrm{Dom}(M_{\overline{\psi}})$.

1 c) $\Leftarrow$ If $\varphi \neq 0$ μ a.e, then $M_\varphi f \neq 0$ for all $f \in Dom(M_\varphi) \setminus \{0\}$, so M_φ is inj.

If $\mu\{\varphi = 0\} > 0$, then $M_\varphi f = 0$ for all $f \in L^2(X,\mu)$ which are supported in $\{\varphi = 0\}$, so M_φ is not injective.

$\Leftarrow$ If $|\varphi| \geq \varepsilon$ μ.a., then $|1/\varphi| \leq \frac{1}{\varepsilon}$ μ.a.e, so for all $f \in L^2(X,\mu)$ one has $f/\varphi \in Dom(M_\varphi)$, and so $f \in Ran(M_\varphi)$ and so M_φ is surjective.

If there exists no ε with $|\varphi| \geq \varepsilon$ μ.a., then $\|\frac{1}{\varphi}\|_{\infty,\mu} = \infty$, and M_φ is unbounded. But M_φ is closed, so we have $Dom(M_\varphi) \neq L^2(X,\mu)$

$\Rightarrow$ There exists $f \in L^2(X,\mu)$ with $f/\varphi \notin L^2(X,\mu)$, and so $f \notin Ran(M_\varphi)$ and M_φ is not surjective

$\ominus$ By what we have ~~shown~~ shown:

M_φ is surjective $\Leftrightarrow$ $M_{1/\varphi}$ is bounded;

this implies $\lambda \in \mathbb{C}$ is in $\sigma(M_\varphi) \Leftrightarrow$ for all $\varepsilon > 0$ one has $\mu\{|\varphi - \lambda| < \varepsilon\}$.

(2a) Given a self-adj op T in a Hilbert space $\mathcal{H}$, its spectral resolution is given as follows (this can be seen as the existence part of the spectral theorem)

$$\langle f_1, P_T(\lambda) f_2 \rangle = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{-\infty}^{\lambda+\delta} \langle f_1,$$

$$\left((T-s-i\varepsilon)^{-1} - (T-s+i\varepsilon)^{-1} \right) f_2 \rangle \, ds$$

For $T = M_\varphi$ ~~can~~ this can be evaluated explicitly and gives $P_{M_\varphi}(\lambda) = M_{1_{\{\varphi \le \lambda\}}}$

Details: Bsp 8.13 in "Lin Op. in Hilberträume", Teil 1" by J. Weidman.

(2b) By a density argument (see page 303 in Weidman) W.l.o.g $\phi = \sum_j c_j 1_{I_j}$ is a simple ~~then~~ function.

Then by (2a) we have

$$\phi(M_\varphi) f = \sum_j c_j P_{M_\lambda}(I_j) f = \sum_j c_j 1_{\{\varphi \in I_j\}} f$$

$$= (\phi \circ \varphi) \cdot f .$$

(3) Convention: Given lin operators A, B one defines (whenever it makes sense!)

$$Dom(A+B) := Dom(A) \cap Dom(B), \quad (A+B)f := Af + Bf$$

$$Dom(AB) := \{f \in Dom(B) : Bf \in Dom(A)\}, \quad (AB)f := A(Bf)$$

(3a) Assume $T_2 = V T_1 V^*$ ($\Leftrightarrow V^* T_2 V = T_1$)

Then i) T_2 is densely def. $\Leftrightarrow T_1$ is so (and then $T_2^* = V T_1^* V^*$)

ii) T_2 is sym $\Leftrightarrow T_1$ is sym ($\langle T_2 f, g\rangle$

$= \langle V T_1 V^* f, g\rangle = \langle T_1 V^* f, V^* g\rangle = \langle V^* f, T_1 V^* g\rangle$

$= \langle f, V T_1 V^* g\rangle = \langle f, T_2 g\rangle$)

iii) T_2 is closable $\Leftrightarrow T_1$ is so

(and then $\overline{T_2} = V \overline{T_1} V^*$)

$\Rightarrow$ iv) T_2 essentially self-adj. $\Leftrightarrow T_1$ is so

(T "ess. s.a": $\Leftrightarrow$ T sym and $\overline{T}$ is s.a.)

(3b) Let $F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the Fourier transform

Then $-\Delta = F\, T_2\, F^*$, where

$$Dom(T_2) = C_c^\infty(\mathbb{R}^n), \quad T_2 f(x) := |x|^2 f(x)$$

$\overline{T_2} = M_\psi$, where $\psi : \mathbb{R}^n \to \mathbb{R}$, $\psi(x) := |x|^2 f(x)$

is s.a by 1b).

(4) Uniqueness: Assume $\psi_1, \psi_2 : [0,\infty) \to \mathcal{H}$ cont.

$\psi_j : (0,\infty) \to \mathcal{H}$ cont. diff, $\psi_j(0) = f$, $\dot{\psi}_j(t) = -T\psi(t)$

for all $t > 0$ (which implicitly means $\psi_j(t) \in \mathrm{Dom}(T)$ for all $t > 0$). Then for all $t > 0$, $\varepsilon \in (0,t)$

$$\frac{d}{dt}\|\psi_1(t) - \psi_2(t)\|^2 = \frac{d}{dt}\langle \psi_1(t) - \psi_2(t), \psi_1(t) - \psi_2(t)\rangle$$

$$= 2\,\mathrm{Re}\,\langle \dot{\psi}_1(t) - \dot{\psi}_2(t), \psi_1(t) - \psi_2(t)\rangle$$

$$= 2\,\mathrm{Re}\,\underbrace{\langle -T(\psi_1(t) - \psi_2(t)), \psi_1(t) - \psi_2(t)\rangle}_{\in \mathbb{R},\ \text{because } T = T^*,\ \langle Tv, v\rangle \geq c}$$

$$\leq 2c\,\|\psi_1(t) - \psi_2(t)\|^2$$

"Gronwall's inequ."

$$\Rightarrow \|\psi_1(t) - \psi_2(t)\|^2 \leq \|\psi_1(\varepsilon) - \psi_2(\varepsilon)\|^2 e^{(t-\varepsilon)\cdot 2\cdot c}$$

$$\xrightarrow{\varepsilon \to 0+} \|\psi_1(0) - \psi_2(0)\|^2 e^{t2c} = 0.$$

Existence: With $\phi(\lambda) := e^{-t\lambda}$, $t \geq 0$, spec. calc. implies $\| e^{-tT} \| \leq \sup_{\underbrace{\lambda \in \sigma(T)}_{\subset [c,\infty), \text{ as } T \geq c}} |e^{-t\lambda}| \leq e^{-tc}$

α) For all $t, s \geq 0$ one has (spec. calc.)

$$e^{-tT} e^{-sT} = \left(e^{-t(\cdot)} \cdot e^{-s(\cdot)}\right)(T) = e^{-(t+s)T}$$

β) $e^{-t\lambda} \xrightarrow{t \to 0+} 1$ for all $\lambda \in [c, \infty)$, so

$$\| e^{-tT} f - f \|^2 = \| (e^{-t(\cdot)} - 1)(T) f \|^2$$

$$= \int_{\sigma(T)} |e^{-t\lambda} - 1|^2 \underbrace{\| P_T(d\lambda) f \|^2}_{\text{Borel measure on } \mathbb{R} \text{ with finite mass}} \xrightarrow{t \to 0+} 0$$

by bounded conv. theorem.

γ) $Te^{-tT} = (\varphi \cdot \psi)(T)$, where $\varphi(\lambda) = \lambda$, $\underbrace{\psi(\lambda) = e^{-t\lambda}}_{\text{bounded}}$

so Te^{-tT} is bounded and everywhere def. by spect. calc, so

$\mathcal{H} = \mathrm{Dom}(Te^{-tT}) = \{\tilde{f} \in \mathcal{H} : e^{-tT}\tilde{f} \in \mathrm{Dom}(T)\}$

and so $\mathrm{Ran}(e^{-tT}) \subset \mathrm{Dom}(T)$ for all $t > 0$, and

We have $\left\| \frac{d}{dt} e^{-tT} f + T e^{-tT} f \right\|^2$

$$= \lim_{s \to 0} \left\| \frac{e^{-(s+t)T} f - e^{-tT} f}{s} + T e^{-tT} f \right\|^2$$

$\underline{\underline{\text{spec. calc.}}}$

$$\lim_{s \to 0} \int_{\sigma(T)} \left| \frac{e^{-s\lambda} - 1}{s} e^{-t\lambda} + \lambda e^{-t\lambda} \right|^2 \underbrace{\|P_T(d\lambda) f\|^2}_{\text{Borel meas on } \mathbb{R}}$$

$= 0$ (by bounded conv. Theorem) (This requires some Analysis 1 estimates).