Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Exercise sheet 6

1. Let $h \in W^{1,2}(M)$. Show that there exists $v \in W_0^{1,2}(M)$ with $h \leq v$, if and only if one has $h_+ \in W_0^{1,2}(M)$.

2. Define the hyperboloid

$$H^m := \big\{(x', x^{m+1}) : x^{m+1} > 0, (x^{m+1})^2 - (((x')^1)^2 + \dots ((x')^m)^2) = 1\big\} \subset \mathbb{R}^{m+1},$$

where we have writen the points of $\mathbb{R}^{m+1}$ as (x', x^{m+1}) with $x' \in \mathbb{R}^m$. Note that H^m is an m-dimensional submanifold.
Define a smooth section of $T^*\mathbb{R}^{m+1} \otimes T^*\mathbb{R}^{m+1}$ by

$$g_{\mathrm{Mink}} := dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m - dx^{m+1} \otimes dx^{m+1}.$$

Show that the restriction of g_{Mink} of H^m *is* a complete Riemannian metric on H.
*Remark 1:*The Riemannian manifold $\mathbb{H}^m := (H^m, g_{\mathrm{Mink}})$ is called the *Hyperbolic space of dimension m*.
Remark 2: g_{Mink} is not a Riemannian metric on $\mathbb{R}^{m+1}$ (as it is not positive definite), but it is a so called semi-Riemannian metric on $\mathbb{R}^{m+1}$, called the *Minkowski metric*. The semi-Riemannian manifold $(\mathbb{R}^{m+1}, g_{\mathrm{Mink}})$ plays a crucial role in special relativity.