Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Exercise sheet 5

From here on (as in the manuscript), in the exercises, M will always denote a connected Riemannian manifold of dimension m.

1. Let $\mathscr{H}$ be a complex Hilbert space, let $f : \mathbb{R} \to \mathscr{H}$ and let $k \in \mathbb{N}_{\geq 1}$. Show that f is weakly[1] C^k, if and only f is C^{k-1}. *Remark: In view of the characterization of holomorphy in terms of the validity of the Cauchy-Riemann differential equations, this result implies that a weakly holomorphic function is actually holomorphic.*

2. Show the existence of a function v as in the proof of Step 2 of strong parabolic minimum principle.

3. Given $f \in L^2(M)$, show that $f \leq 1$ μ-a.e. on M implies $P_t f \leq 1$ μ-a.e. on M (and in fact everywhere on M, as $P_t f$ is smooth) for all $t > 0$.

[1] that is, for all fixed $\phi \in \mathscr{H}$, the map $\mathbb{R} \ni x \mapsto \langle f(x), \phi \rangle \in \mathbb{C}$ is C^k