Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2022,
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Exercise sheet 4

1. Let M be a Riemannian manifold. Prove the following formulae for all smooth functions $f, f_1, f_2 : M \to \mathbb{C}$, all smooth functions $u : \mathbb{R} \to \mathbb{R}$ and all smooth 1-forms α on M:

$$d^\dagger(f\alpha) = f d^\dagger \alpha - (df, \alpha), \tag{1}$$

$$\Delta(f_1 f_2) = f_1 \Delta f_1 2 + f_2 \Delta f_1 + 2\Re(df_1, df_2), \tag{2}$$

$$\Delta(u \circ f) = (u'' \circ f) \cdot |df|^2 + (u' \circ f) \cdot \Delta f, \quad \text{if } f \text{ is real-valued.} \tag{3}$$

2. Let M be a manifold, let $E, F \to M$ be vector bundles, and let P be a differential operator from $E \to M$ to $F \to M$. Fix $f \in \Gamma_{L^1_{\mathrm{loc}}}(M, E)$ and a subspace $A \subset \Gamma_{L^1_{\mathrm{loc}}}(M, F)$.
Show that there exists $h \in A$ such that for all triples of metrics (g, h_E, h_F) it holds that

$$\int_M h_E \left(P^{g,h_E,h_F} \psi, f \right) \mathrm{d}\mu_g = \int_M h_F (\psi, h) \, \mathrm{d}\mu_g \; \text{ for all } \psi \in \Gamma_{C_c^\infty}(M, F) \;, \tag{4}$$

if and only if there exists $h \in A$ (necesserly the same h as above) such that one has (4) for *some* triple (g, h_E, h_F).

3. Let ϱ be the geodesic distance on the Riemannian manifold M.
a) Show that for the open balls

$$B(x, r) := \{y : \varrho(x, y) < r\} \subset M$$

one has

$$\overline{B(x, r)} = \{y : \varrho(x, y) \leq r\}.$$

b) Show that (M, ϱ) is a complete metric space, if and only if all closed bounded subsets of (M, ϱ) are compact.