Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, TU Chemnitz, WS 2021/2021,
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Exercise sheet 3

1. Let (M, g) be a Riemannian m-manifold. Show that there exists precisely one Borel measure μ_g on M such that for every chart $((x^1, \dots, x^m), U)$ for M and any Borel set $N \subset U$, one has

$$\mu_g(N) = \int_N \sqrt{\det(g(x))}dx,$$

where $\det(g(x))$ is the determinant of the matrix $g_{ij}(x) := g(\partial_i, \partial_j)(x)$ and where $dx = dx^1 \cdots dx^m$ stands for the Lebesgue integration.

2. Given two Riemannian metrics g, h on a manifold M, calculate the Radon-Nikodym density

$$\rho_{h,g} := \frac{d\mu_g}{d\mu_h} : M \to (0, \infty)$$

explicitly (it follows in particular that any two Riemannian volume measures on the same manifold are absolutely continuous with respect to each other).

3. Let M be a smooth manifold.
a) Show that for all smooth functions u, v on M one has the product rule $d(uv) = u(dv) + v(du)$.
b) Show that for all smooth functions $f : \mathbb{R} \to \mathbb{R}$ and all smooth functions $u : M \to \mathbb{R}$ one has the chain rule $d(f \circ u) = f'(u)du$.