Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, WS 2021/2022,
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Exercise sheet 2

Recall that a symmetric operator is called essentially self-adjoint, if its closure (which always exists) is self-adjoint. It is easy to show that a symmetric operator is essentially self-ajoint, if and only if it has precisely one self-adjoint extension (which is then the closure).

Let $U \subset \mathbb{R}^m$ be an open connected subset. We consider $-\Delta$ as a symmetric nonnegative operator T_U in $L^2(U)$ with dense domain of definition $C_c^\infty(U)$.

1. Show that T_U is closable but not closed (and thus not bounded and not self-adjoint). Hint: Show closability by writing down a closed extension, and then calculate the closure explicitly. Remark: this fact is true for any possibly nonsymmetric differential operator with sufficiently smooth coefficients!

2. a) Calculate T_U^*.
b) Show that there exist self-adjoint extensions of T_U.

3 a) Show that if $U = \mathbb{R}^m$, then $T_{\mathbb{R}^m}$ has precisely one self-adjoint extension T; show that $\sigma(T) = \sigma_{\mathrm{ess}}(T) = [0, \infty)$ (in particular, T has no discrete spectrum).
b) Give an example of a U such that T_U has at least two self-adjoint extensions.
c) Give an example of a bounded U and a self-adjoint extension of T_U which has a purely discrete spectrum. Remark: We will see later that this is a somewhat generic phanomenon, in the sense that for any bounded U the Dirichlet realization of T_U has a purely discrete spectrum.

Upshot: $T_{\mathbb{R}^m}$ has precisely one self-ajoint extension (so essential self - adjointness corresponds[1] to completeness of U) which has a purely essential spectrum. If $U \neq \mathbb{R}^m$, then T_U has several self-adjoint extensions (in general: infinitely many!). If U is bounded, then self-adjoint extensions of T_U typically have a purely discrete spectrum[2].

[1] There exist U's which are not complete ($\Leftrightarrow U \neq \mathbb{R}^m$) for which T_U is essentially self-adjoint, but this should be condered as pathological!

[2] There exist self-adjoint extensions of T_U with U bounded whose spectrum is not purely discrete; but this should be considered as pathological.