Brownian Motion and the Feynman-Kac Formula on Riemannian manifolds, WS 2021/2022,
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Exercise sheet 1

1. Let (X, μ) be a sigma-finite measure space and let $\psi : X \to \mathbb{C}$ be measurable.
a) Show that the maximally defined multiplication operator M_ψ in $L^2(X, \mu)$ is closed.
b) Show that ψ is μ-a.e. real-valued, if and only if M_ψ is self-adjoint. *Hint: Calculate the adjoint explicitly.*
c) Show that $\mathrm{Ker}(M_\psi) = \{0\}$ if and only if $\mu\{\psi = 0\} = 0$ and $\mathrm{Ran}(M_\psi) = L^2(X, \mu)$ if and only if there exists $\epsilon > 0$ with $|\psi| \geq \epsilon$ μ-a.e. What is the spectrum of M_ψ?

2. Let (X, μ) be a sigma-finite measure space and let $\psi : X \to \mathbb{R}$ be measurable.
a) Show that the spectral resolution of the induced multiplication operator is given by $P_{M_\psi}(\lambda) = M_{1_{\{\psi \leq \lambda\}}}$, with 1_A the indicator function of a set A.
b) Show that if $\phi : \mathbb{R} \to \mathbb{C}$ is Borel then $\phi(M_\psi) = M_{\phi \circ \psi}$.

3. Let $\mathscr{H}_1$, $\mathscr{H}_2$ be Hilbert spaces. A bounded map $V : \mathscr{H}_1 \to \mathscr{H}_2$ is called *unitary*, if it is bijective and $V^{-1} = V^*$. Given linear two operators T_j in $\mathscr{H}_j$, the operator T_1 is called *unitarily equivalent to* T_2, if there exists V as above with $T_2 = VT_1V^*$.
a) Show that T_1 is essentially self-adjoint, if and only if T_2 is so.
b) Use exercise 3 a) and exercise 1 b) to show that $-\Delta$ (defined on smooth compactly supported functions) in $L^2(\mathbb{R}^m)$ is essentially self-adjoint. *Hint: consider the operator $T_2 f(x) := |x|^2 f(x)$ defined on smooth compactly supported functions.*

4. Assume T is a self-adjoint operator in a Hilbert space $\mathscr{H}$ with $T \geq c$ for some constant $c \in \mathbb{R}$ and let $f \in \mathscr{H}$.
a) Show that there exists at most one solution $\psi : (0, \infty) \to \mathscr{H}$ of the following Cauchy problem for the abstract heat equation induced by T:

- ψ is continuously differentiable with $\psi(t) \in \mathrm{Dom}(T)$ for all $t > 0$,
- $d/dt\psi(t) = -T\psi(t)$ for all $t > 0$,
- $\lim_{t \to 0+} \psi(t) = f$.

b) Use the spectral calculus to show that $\psi(t) := e^{-tT} f$ *is* a solution of the above Cauchy problem.