On the Feynman-Kac formula for Schrödinger semigroups on vector bundles

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Abstract

In this thesis we generalize the Feynman-Kac formula to semigroups that correspond to Schrödinger type operators with possibly singular potentials on vector bundles over noncompact Riemannian manifolds.

This probabilistic formula is then used to obtain information about the spectral theory of these operators.

A first class of applications corresponds to semigroup domination: We show how the spectrum can be estimated by usual scalar Schrödinger operators on functions. This includes estimates for the bottom of the spectrum and, from a Brownian bridge version of our Feynman-Kac formula, we also obtain estimates for the integral kernel and the trace of the semigroup.

As another application of the Feynman-Kac formula, we introduce the class of Kato potentials on vector bundles and use probabilistic methods to prove that the semigroups corresponding to Schrödinger type operators with local Kato potentials map square integrable sections to bounded continuous sections. In particular, this implies the boundedness and the continuity of the eigensections of these operators.

We finally specify some of these results to Schrödinger type operators on trivial vector bundles.

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1 Introduction

1.1 Review of path integrals for scalar Schrödinger operators in the Euclidean space

By the predictions of nonrelativistic quantum mechanics the energy of a particle with spin 0 and charge and mass equal to 1, moving in the Euclidean space $\mathbb{R}^m$ under the influence of an electrical potential $v : \mathbb{R}^m \to \mathbb{R}$, is described by the spectrum of a self-adjoint realization $H_0(v)$ of the Schrödinger operator $-\Delta/2 + v$ in the Hilbert space $\mathrm{L}^2(\mathbb{R}^m)$. If the initial state f of this system is in the domain of definition of $H_0(v)$, then the state at the time t is given by $\mathrm{e}^{-\mathrm{i}tH_0(v)}f$. Ever since R. Feynman's seminal paper [31] it has become customary in the physics literature to write $\mathrm{e}^{-\mathrm{i}tH_0(v)}f$ as an ill-defined path integral.

While the time evolution is given by the unitary group $(\mathrm{e}^{-\mathrm{i}tH_0(v)})_{t\in\mathbb{R}}$, it has been demonstrated by B. Simon in [76] [77] that, in case $H_0(v)$ is semibounded from below, the study of the spectrum and the eigenfunctions of $H_0(v)$ is closely related to the *Schrödinger semigroup* $(\mathrm{e}^{-tH_0(v)})_{t\geq 0}$, so that it is natural from this point of view to look for an explicit formula for $\mathrm{e}^{-tH_0(v)}$. Beginning with M. Kac's paper [45] there have been several publications which are concerned with the fact that there is a well-defined "imaginary time" version of Feynman's path integral: If $B(x)$ is a Brownian motion in $\mathbb{R}^m$ which starts in x and which is defined on a filtered probability space with expectation value $\mathbb{E}[\bullet]$, then one has the Feynman-Kac formula,

$$\mathrm{e}^{-tH_0(v)}f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x))\right]. \tag{1}$$

This Feynman-Kac formula is valid for a large class of potentials. For instance, if v is Kato decomposable, which includes all physically relevant cases, then there is a natural quadratic form definition of $H_0(v)$ and (1) holds [83]. If one takes into account a locally integrable magnetic potential β, then $H_0(v)$ has to be replaced by some self-adjoint realization[1] $H(\mathrm{i}\beta, v)$ of the magnetic Schrödinger operator

$$\frac{1}{2}\sum_{j=1}^{m}(-\mathrm{i}\partial_j + \beta_j)^2 + v, \tag{2}$$

and (1) can be generalized as follows:

$$\mathrm{e}^{-tH(\mathrm{i}\beta,v)}f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s + \mathrm{i}\int_0^t \sum_{j=1}^m \beta_j(B_s(x))\underline{\mathrm{d}}B_s^j(x)} f(B_t(x))\right]. \tag{3}$$

[1]The reason for the notation $H(\mathrm{i}\beta, v)$ instead of $H(\beta, v)$ will become clear in section 1.2, in particular in the setting of theorem 1.4.

Formula (3) is known as Feynman-Kac-Itô formula and it holds for the natural quadratic form realization of (2), if $\|\beta(\bullet)\|_{\mathbb{R}^m}$, $\mathrm{div}\beta$ are in the local Kato class and v is Kato decomposable. This formula (and a natural extension of it to arbitrary open subsets of $\mathbb{R}^m$) has been proved in [13] by K. Broderix, D. Hundertmark and H. Leschke. Their paper seems to contain the state of the art in the Euclidean setting. We would also like to mention [14]. There, in contrast to all the papers cited so far, the authors have extended ideas from [78] and proved a Feynman-Kac-Itô formula under assumptions on the pair (β, v), under which the considered operator $H(\mathrm{i}\beta, v)$ need not be semibounded from below. As a consequence, the self-adjoint nonnegative operator $\mathrm{e}^{-tH(\mathrm{i}\beta,v)}$ is in general not bounded, but formula (3) remains true for all f in the domain of definition of $\mathrm{e}^{-tH(\mathrm{i}\beta,v)}$.

1.2 Main results and organization of this work

In terms of theoretical physics, we are interested in this work to extend the above path integral formulae and their applications to particles that live on Riemannian manifolds and that are subject to certain abstract internal symmetries. In order to motivate the form of these generalized vector valued path integral formulae on manifolds, let us continue our review of the scalar Euclidean case with a geometric interpretation of (3). We consider $\mathbb{R}^m$ as a smooth Riemannian manifold with its Euclidean metric and assume that the magnetic potential β is smooth (this is a satisfactory assumption for applications in theoretical physics), so that it can be considered as a smooth 1-form in $\mathbb{R}^m$. With $\alpha := \mathrm{i}\beta$, $\mathrm{d}+\alpha$ can be considered as a covariant derivative on the trivial line bundle $\mathbb{R}^m \times \mathbb{C}$, and (2) is nothing but $1/2$ times the Bochner Laplacian corresponding to this covariant derivative. We define the Stratonovic line integral of α along $B(x)$ as

$$\int_0^t \alpha(\underline{\mathrm{d}}B_s(x)) := \int_0^t \sum_{j=1}^m \alpha_j(B_s(x))\underline{\mathrm{d}}B_s^j(x), \tag{4}$$

and remark that

$$//_{\alpha,t}^x := \mathrm{e}^{-\int_0^t \alpha(\underline{\mathrm{d}}B_s(x))}$$

satisfies the linear $\mathrm{U}(1)$-valued Stratonovic equation

$$//_{\alpha,t}^x = 1 - \int_0^t //_{\alpha,s}^x \alpha(\underline{\mathrm{d}}B_s(x)), \tag{5}$$

where $\mathrm{U}(d)$ stands for the Lie group of unitary $d \times d$ matrices in the following. By the analogy to the usual parallel transport along smooth paths, $//_\alpha^x$ can be considered as the stochastic parallel transport with respect to the covariant derivative determined by α, along the paths of $B(x)$. If the potential v is sufficiently regular (for instance Kato decomposable), then, by the results cited above, the process

$$\mathscr{V}^x_{\alpha,t} := \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s}$$

is well-defined and it satisfies the complex valued linear ordinary initial value problem

$$\mathscr{V}^x_{\alpha,t} = 1 - \int_0^t \mathscr{V}^x_{\alpha,s} v(B_s(x))\mathrm{d}s. \tag{6}$$

As the final step of our geometric interpretation, we consider (6) as a "covariant equation" by writing the right-hand side as

$$\mathscr{V}^x_{\alpha,s} v(B_s(x)) = \mathscr{V}^x_{\alpha,s} //^{x,-1}_{\alpha,s} v(B_s(x)) //^x_{\alpha,s},$$

which explains the artificial notational dependence of $\mathscr{V}^x_\alpha$ on α, and the Feynman-Kac-Itô formula takes the form

$$\mathrm{e}^{-tH(\alpha,v)} f(x) = \mathbb{E}\left[\mathscr{V}^x_{\alpha,t} //^{x,-1}_{\alpha,t} f(B_t(x))\right]. \tag{7}$$

The aim of this thesis is to generalize formula (7) and its applications to the spectral theory of $H(\alpha, v)$ in the spirit of [76] [77] to the setting of arbitrary vector bundles over Riemannian manifolds, allowing possibly singular generalized potentials. To this end, we fix some notation.

Let $M = (M, g)$ be a geodesically and stochastically complete smooth connected Riemannian manifold. For example, stochastic completeness is implied by geodesic completeness, if the Ricci curvature is bounded from below by a constant, or more generally, if the Ricci curvature is bounded from below in radial direction by some quadratic function of the geodesic distance function (for some fixed reference point). Furthermore, let $E \to M$ be a smooth Hermitian vector bundle with a fixed Hermitian covariant derivative ∇, and let V be a potential, in the sense that V is a measurable, pointwise Hermitian section in $\mathrm{End}(E)$. The class of potentials under consideration in this thesis is the one of locally square integrable potentials that are bounded from below, so let

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \text{for some } C_V \in \mathbb{R} \tag{8}$$

for the rest of this introduction. In analogy to a classical result of T. Kato, it has been proved by M. Braverman, O. Milatovic and M. Shubin in [11] that the Schrödinger type operator $\nabla^*\nabla/2 + V$ in the Hilbert space $\Gamma_{\mathrm{L}^2}(M, E)$ of square integrable sections in E is essentially self-adjoint on the domain of smooth sections with compact support. The corresponding operator closure will be denoted with $H(V)$ and the semigroup

$$(\mathrm{e}^{-tH(V)})_{t\geq 0} \subset \mathscr{L}(\Gamma_{\mathrm{L}^2}(M, E))$$

will be called the *Schrödinger semigroup* corresponding to $H(V)$ in the following. We remark that in analogy to the considerations of section 1.1, the energy of a nonrelativistic particle with mass 1 which has internal symmetries (that are modelled by a subgroup of $\mathrm{U}(d)$) and which lives on M under the influence of an "electrical" potential V is described [20] by an operator of the type $H(V)$.

As we have already stated, the path integral formula for $\mathrm{e}^{-tH(V)}$ is of the form (7), so let us explain our approach for constructing Brownian motion in this general setting, which is the one initiated by L. Schwartz: By embedding M into some Euclidean $\mathbb{R}^l$ in an isometric way, we define a Brownian motion $B(x)$ on M with initial value x as the unique maximally defined solution of a Stratonovic equation on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_*, \mathbb{P})$. Now the stochastic parallel transport $//^x$ with respect to the data $(\nabla, B(x))$ can be defined conveniently by solving the lift of the defining Stratonovic equation of $B(x)$ to the $\mathrm{U}(d)$-principal bundle of unitary frames in E. The process $//^x$ can be read as an isometry along the paths of $B(x)$,

$$//_t^x : E_x \longrightarrow E_{B_t(x)}.$$

With these preparations, we can state the central result of this thesis:

Theorem 1.1 *For almost every $x \in M$ there is a unique process*

$$\mathscr{V}^x : [0, \infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

which satisfies the initial value problem

$$\mathrm{d}\mathscr{V}_t^x = -\mathscr{V}_t^x //_t^{x,-1} V(B_t(x)) //_t^x \mathrm{d}t, \;\; \mathscr{V}_0^x = \mathbf{1} \tag{9}$$

pathwise in the weak sense, and for any $t \geq 0$, $f \in \Gamma_{\mathrm{L}^2}(M, E)$ and almost every $x \in M$ one has the following identity,

$$\mathrm{e}^{-tH(V)} f(x) = \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right]. \tag{10}$$

Note that the existence of $\mathscr{V}^x$ is not trivial, since V is not assumed to be continuous in general. Versions of formula (10) have been known for some

time for certain *smooth* potentials: For example in [9][60], the authors have proved a similar formula for closed M, and in [24] a formula is worked out for the Friedrichs realization of a Schrödinger type operator with some growth restriction on the potential.
After having defined the stochastic parallel transport conveniently, it is rather straightforward to establish (10) for continuous bounded potentials. The proof for the general case is quite technical and uses a chain of approximation arguments.

Besides of being a generalization to singular potentials, formula (10) has several applications in the spectral theory of $H(V)$. As we have already stated, these applications mainly represent extensions of [76] [77] to our geometric setting, but due to the vector valued character of our calculus, we also obtain some new results for $M = \mathbb{R}^m$ with its Euclidean metric. We also remark that the results below are all valid without any kind of boundedness assumptions on the geometry of E.
In the following, let $v : M \to \mathbb{R}$ be a locally square integrable potential which is bounded from below, and let $H_0(v)$ be the self-adjoint realization of $-\Delta/2 + v$ in $\mathrm{L}^2(M)$, where Δ stands for the Laplace-Beltrami operator on M. The Feynman-Kac formula can easily be brought into the following form in this situation,

$$\mathrm{e}^{-tH_0(v)} f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x))\right], \tag{11}$$

and this shows that this operator is positivity improving. In particular, if the ground state energy $\lambda := \inf \sigma(H_0(v))$ is an eigenvalue of $H_0(v)$, then λ is simple and the corresponding ground state eigenfunction can be chosen strictly positive. Here, $\sigma(\bullet)$ stands for the spectrum.
Furthermore, the combination of (10) and (11) leads to an important fact, namely semigroup domination: If $V \geq v\mathbf{1}$, then one has the inequality

$$\left\|\mathrm{e}^{-tH(V)} f(x)\right\|_x \leq \mathrm{e}^{-tH_0(v)} \left|f\right|(x) \quad \text{for any } f \in \Gamma_{\mathrm{L}^2}(M, E),$$

where the function $|f| \in \mathrm{L}^2(M)$ is defined by $|f|(x) := \|f(x)\|_x$. This has an important consequence: If a section $x \mapsto f(x)$ is in the quadratic form domain of $H(V)$, then the function $x \mapsto \|f(x)\|_x$ is in the quadratic form domain of $H_0(v)$ and one has

$$\inf \sigma(H(V)) \geq \inf \sigma(H_0(v)), \tag{12}$$

a remarkable fact, since both operators act in different Hilbert spaces.

Another consequence of formula (10) is that, with some control on the Riemannian structure of M, the Schrödinger semigroup can also be considered as acting in the spaces of L^p-sections in E, $\Gamma_{\mathrm{L}^p}(M,E)$: To this end, we use the right-hand side of the Feynman-Kac formula to *define* the expression $\mathrm{e}^{-tH(V)}f(x)$ for any section f in E. Let $\|\bullet\|_{p,q}$ denote the operator norm for linear operators mapping $\Gamma_{\mathrm{L}^p}(M,E)$ to $\Gamma_{\mathrm{L}^q}(M,E)$. We use semigroup domination to prove the following result: If M has a bounded geometry, then for any $1 \leq p \leq q \leq \infty$ one has

$$\mathrm{e}^{-tH(V)} \in \mathscr{L}\Big(\Gamma_{\mathrm{L}^p}(M,E), \Gamma_{\mathrm{L}^q}(M,E)\Big) \tag{13}$$

and there is a $C > 0$, which only depends on the Riemannian structure of M, such that for all $1 \leq p \leq q \leq \infty$ one has

$$\left\|\mathrm{e}^{-tH(V)}\right\|_{p,q} \leq C^{\frac{1}{p}-\frac{1}{q}} \min\left\{t^{\frac{m}{2}}, 1\right\}^{-\frac{1}{p}+\frac{1}{q}} \mathrm{e}^{-tC_V}. \tag{14}$$

The importance of (13) is discussed in remark 8.5.

Next, we present some of our results concerning the integral kernel of the Schrödinger semigroup. To this end, we first prove that if M is geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, then one can define the Brownian bridge measures $\mathbb{P}^{x,y}_t$ in a way that the expectation values $\mathbb{E}^{x,y}_t[\bullet]$ are a rigorous version of the conditional expectation values $\mathbb{E}[\bullet | B_t(x) = y]$. We believe that this construction of $\mathbb{E}^{x,y}_t[\bullet]$ is possibly not well-known for noncompact manifolds. With this disintegration, we prove:

Theorem 1.2 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. Then for any $t > 0$, the section*

$$M \times M \ni (x,y) \longmapsto \mathrm{e}^{-tH(V)}(x,y) \in \mathrm{Hom}(E_y, E_x),$$
$$\mathrm{e}^{-tH(V)}(x,y) := p_t(x,y)\mathbb{E}^{x,y}_t\left[\mathscr{V}^x_t //^{x,-1}_t\right]$$

in $E \boxtimes E^*$ is well-defined for a.e. $(x,y) \in M \times M$ and it defines an essentially bounded integral kernel for the operator $\mathrm{e}^{-tH(V)}$.

Theorem 1.2 has several consequences. Firstly, one gets another aspect of semigroup domination: $V \geq v\mathbf{1}$ implies

$$\left\|\mathrm{e}^{-tH(V)}(x,y)\right\|_{y,x} \leq \mathrm{e}^{-tH_0(v)}(x,y). \tag{15}$$

Secondly, standard arguments imply the path integral formula

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) = \int_M \int_M \mathrm{tr}_{E_y}\left(\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)^* \mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right) \mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y). \tag{16}$$

Here, we remark that whenever $\mathrm{e}^{-tH(V)}(\bullet,\bullet)$ is pointwise well-defined and continuous, it is possible to derive the more familiar formula

$$\operatorname{tr}\left(\mathrm{e}^{-tH(V)}\right) = \int_M \operatorname{tr}_{E_x}\left(\mathrm{e}^{-tH(V)}(x,x)\right)\operatorname{vol}(\mathrm{d}x) \tag{17}$$

from (16) (the short time asymptotics of a supertrace variant of this formula for M closed and V smooth has been used by J.-M. Bismut [9] for his probabilistic proof of the Atiyah-Singer index theorem). Unfortunately, we believe that under our weak assumptions on M and V this continuity need not be true for M high dimensional. However, our substitute for (17) will turn out to work equally well for the applications that we have in mind. For example, formula (16) can be combined with (15) to give

$$\operatorname{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq d \operatorname{tr}\left(\mathrm{e}^{-tH_0(v)}\right),$$

where the number d stands for the dimension of the fibers of E.

As another consequence, we get a generalized Goldon-Thompson-Symanzik inequality[2]: If M has a bounded geometry, then there is a constant $C > 0$, which only depends on the Riemannian structure of M, such that

$$\operatorname{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq \frac{Cd}{\min\{t^{\frac{m}{2}},1\}} \int_M \mathrm{e}^{-t\underline{V}(y)}\operatorname{vol}(\mathrm{d}y),$$

where the scalar potential $\underline{V} : M \to \mathbb{R}$ is given by

$$\underline{V}(y) := \text{ the smallest eigenvalue of } V(y).$$

This in turn implies a generalized phase space bound for small times,

$$\operatorname{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq Cd \int_M \int_{\mathrm{T}^*_y M} \mathrm{e}^{-t\left(\frac{1}{2}\|x\|^2_{\mathrm{T}^*_x M} + \underline{V}(y)\right)} \operatorname{vol}_{\mathrm{T}^*_y M}(\mathrm{d}x)\operatorname{vol}(\mathrm{d}y).$$

The latter inequality can be interpreted as follows: Even in the setting of curved configuration spaces and particles with abstract internal symmetries, the corresponding quantum mechanical partition function is bounded from above by the corresponding classical partition function, an assertion which has been known since the 1960's [82] for the scalar Euclidean Schrödinger operators from section 1.1. These results generalize some results of C. Bär and F. Pfäffle to noncompact and not necessarily smooth potentials: In [4] and [5], the authors derive similar estimates by approximating a variant of the path integral formula (10) by finite dimensional integrals.

[2] The name goes back to [32][85][82]; see in particular also [77].

As a next application, we would like to explain how the Feynman-Kac formula can be used to derive a pointwise result for $H(V)$: The continuity of the eigensections of $H(V)$. To this end, we assume that M is geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, and that V is (in addition to our standing assumption (8)) in the local Kato class. Here, we have extended the definition of real-valued local Kato functions [76] to potentials that are sections in $\mathrm{End}(E)$ as follows: A potential W is said to be in the Kato class, if

$$\lim_{t\searrow 0}\sup_{x\in M}\mathbb{E}\left[\int_0^t \|W(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right]=0,$$

and W is said to be in the local Kato class, if $1_K W$ is in the Kato class for any compact subset $K\subset M$. By a result of K. Kuwae and M. Takahashi [53], one finds that under the above assumptions on M, being locally Kato is not very restrictive[3] for V. In this situation, one can show that $\mathscr{V}^x$ can be defined *for all* $x\in M$, so that in particular the right-hand side

$$Q^V_t f(x)=\mathbb{E}\left[\mathscr{V}^x_t //^{x,-1}_t f(B_t(x))\right]$$

of the Feynman-Kac formula is well-defined *for all* $x\in M$. We show the pointwise perturbation formula

$$Q^0_s Q^V_{t-s} f(x)=\mathbb{E}\left[\mathscr{V}^{x,-1}_s \mathscr{V}^x_t //^{x,-1}_t f(B_t(x))\right],\quad t\geq s\geq 0,\, x\in M, \tag{18}$$

and use this formula to approximate Q^V in some locally uniform way by the semigroup Q^0, so that, using additionally the local elliptic regularity for $H(0)$, we can prove the following result:

Theorem 1.3 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, and let V be in the local Kato class. Then for any $t>0$ and $f\in\Gamma_{\mathrm{L}^2}(M,E)$, the section*

$$M\longrightarrow E,\;\; x\longmapsto Q^V_t f(x)=\mathbb{E}\left[\mathscr{V}^x_t //^{x,-1}_t f(B_t(x))\right]\in E_x$$

is continuous and bounded. In particular, the eigensections of $H(V)$ can be chosen continuous and bounded.

Theorem 1.3 generalizes the corresponding result [13] for scalar Schrödinger operators with magnetic fields in the Euclidean $\mathbb{R}^m$ to our setting.

[3] For example, if $\dim M\leq 3$, then any locally square integrable potential is in the local Kato class (under the stated assumptions on the Riemanian structure of M).

Finally, we specify some of the previous results to trivial vector bundles, that is, $E = M \times \mathbb{C}^d$ with its standard Hermitian structure. Let $\mathscr{U}(d)$ denote the Lie algebra corresponding to $\mathrm{U}(d)$, let

$$\alpha \in \Omega^1(M, \mathscr{U}(d)),$$

and let $V : M \to \mathrm{Mat}(\mathbb{C}^d)$ be a locally square integrable potential which is bounded from below. The self-adjoint realization of $\frac{1}{2}(\mathrm{d}+\alpha)^*(\mathrm{d}+\alpha)+V$ in $\mathrm{L}^2(M, \mathbb{C}^d)$ shall be denoted with $H(\alpha, V)$. The Feynman-Kac formula can brought into the following form in this situation:

Theorem 1.4 *Let α, V and $H(\alpha, V)$ be as above. Then for almost every $x \in M$ there is a unique solution*

$$\mathscr{A}^{\alpha,V}(x) : [0, \infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d)$$

of the Stratonovic equation

$$\mathrm{d}\mathscr{A}_t^{\alpha,V}(x) = \mathscr{A}_t^{\alpha,V}(x)\Big(\alpha(\underline{\mathrm{d}}B_t(x)) - V(B_t(x))\mathrm{d}t\Big), \ \ \mathscr{A}_0^{\alpha,V}(x) = \mathbf{1},$$

and the following formula holds for any $t \geq 0$, $f \in \mathrm{L}^2(M, \mathbb{C}^d)$ and almost every $x \in M$,

$$\mathrm{e}^{-tH(\alpha,V)} f(x) = \mathbb{E}\left[\mathscr{A}_t^{\alpha,V}(x) f(B_t(x))\right]. \tag{19}$$

Here, $\int_0^t \alpha(\underline{\mathrm{d}}B_s(x))$ stands for the Stratonovic line integral of α along $B(x)$, a semi-martingale with values in $\mathrm{Mat}(\mathbb{C}^d)$ which can be defined in anology to (4) with our embedding approach. In particular, we obtain a Feynman-Kac-Itô type formula for manifolds: Let $\beta \in \Omega_{\mathbb{R}}(M)$, let $v : M \to \mathbb{R}$ be a locally square integrable potential which is bounded from below, with $H(\mathrm{i}\beta, v)$ the self-adjoint realization in $\mathrm{L}^2(M)$ corresponding to $\frac{1}{2}(\mathrm{d}+\mathrm{i}\beta)^*(\mathrm{d}+\mathrm{i}\beta)+v$, so that $H_0(v) = H(0, v)$. Analogously to section 1.1, the operator $H(\mathrm{i}\beta, v)$ describes the energy of a particle with spin 0 and charge and mass equal to 1, moving in M under the influence of the electrical potential v and the magnetic potential β. In this situation, formula (19) reads

$$\mathrm{e}^{-tH(\mathrm{i}\beta,v)} f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s + \mathrm{i}\int_0^t \beta(\underline{\mathrm{d}}B_s(x))} f(B_t(x))\right], \tag{20}$$

which extends formula (3) to Riemannian manifolds. Using the formulae (11) and (20), one obviously has the semigroup domination

$$\left|\mathrm{e}^{-tH(\mathrm{i}\beta,v)} f(x)\right| \leq \mathrm{e}^{-tH_0(v)} \left|f\right|(x),$$

which directly implies the adaption of (12) to this simple situation:

$$\sigma(H(\mathrm{i}\beta, v)) \geq \inf \sigma(H_0(v)).$$

The latter inequality can be interpreted as "switching on a magnetic field leads to an increase of the lower energy of charged quantum particles without spin".

This thesis is organized as follows:

In section 2, we first review the concepts of stochastic differential equations on manifolds and stochastic horizontal lifts (to arbitrary principal bundles) and recall the corresponding standard existence and uniqueness theorems. Then we briefly explain possible constructions of Brownian motions on Riemannian manifolds. The main goal of this section is to calculate the Stratonovic differential of processes of the form $//^{x,-1}\Psi(B(x))$, with Ψ a smooth section and $B(x)$ a Brownian motion that is constructed by the Nash embedding theorem. As far as we know, this result has not appeared in the literature in this generality, although we believe that it is known among probabilists.

In section 3, we explain the proof of the above essential self-adjointness result for $H(V)$, and we present a new proof for M with bounded geometry.

After having fixed some notation in section 4, section 5 is completely devoted to the proof of theorem 1.1.

In the sections 6 and 7, we introduce and prove basic properties of Kato potentials and Brownian bridge measures, respectively.

Finally, section 8 is devoted to the applications of theorem 1.1.

We have included an appendix, in which certain heat kernel estimates for manifolds with bounded geometry have been collected, and in which readers who are not familiar with stochastic integrals may find a short introduction to this topic.

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2 Foundations of stochastic analysis on manifolds

2.1 Stochastic differential equations on manifolds

We are going to give a brief introduction to stochastic differential equations on smooth manifolds first. Since this concept naturally leads to solutions which may only be defined up to some "explosion time", it is natural to assume that the processes are only defined up to some stopping time which is not necessarily equal to ∞. We start with some definitions and remarks that are in the spirit of this observation.

Throughout this thesis, we assume that any manifold under consideration is paracompact and without boundary. Unless otherwise stated, measurability will always be understood with respect to the corresponding Borel-σ-algebra. Let M be a smooth connected manifold with $\hat{M} := M \cup \{\infty_M\}$ its Alexandroff compactification and $m := \dim M$. Let $(\Omega, \mathscr{F}, \mathscr{F}_*, \mathbb{P})$ be a filtered probability space. Whenever necessary, we will make the filtration $\mathscr{F}_*$ right-continuous and complete without changing the notation. For stopping times η and ζ we use the usual notation

$$[\eta, \zeta) \times \Omega := \Big\{ (t, \omega) \Big| \ \eta(\omega) \leq t < \zeta(\omega) \Big\},$$

and in general, a process X with values in M will be a map

$$X : [0, \zeta_X) \times \Omega \longrightarrow M, \tag{21}$$

where ζ_X is a $\mathbb{P}$-a.s. positive predictable stopping time, such that

$$X_t : \Big\{ t < \zeta_X \Big\} \longrightarrow M \tag{22}$$

is $\mathscr{F}$-measurable for any $t \geq 0$. We will then say that X is *defined up to* ζ_X. Similarly, X will be called

- *adapted*, if (22) is $\mathscr{F}_t$-measurable for any $t \geq 0$,
- *continuous*, if for $\mathbb{P}$-a.e. $\omega \in \Omega$ the map

  $$X_\bullet(\omega) : [0, \zeta_X(\omega)) \longrightarrow M$$

  is continuous, and
- *maximally defined*, if

$$\lim_{t \nearrow \zeta_X(\omega)} X_t(\omega) = \infty_M \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Big\{ \zeta_X < \infty \Big\}. \tag{23}$$

The concept of $\mathbb{R}^m$-valued semi-martingales can be carried over to manifolds as follows:

Definition 2.1 *A continuous adapted process X with values in M is called a continuous semi-martingale on M, if for any stopping time ζ with $\zeta < \zeta_X$ $\mathbb{P}$-a.s. and any real-valued $f \in \mathrm{C}^\infty(M)$, the stopped process*

$$f(X^\zeta) : [0, \infty) \times \Omega \to \mathbb{R} \tag{24}$$

is a usual continuous semi-martingale.

The notion of semi-martingales on manifolds has been introduced by L. Schwartz in [69]. It follows from the Itô formula that this definition coincides with the usual one for processes with values in $\mathbb{R}^m$.

Remark 2.2 1. If X is a continuous adapted process with values in M, then X is a continuous semi-martingale, if and only if for any real-valued $f \in \mathrm{C}^\infty(M)$ there is a sequence of stopping times (ζ_n) which announces ζ such that for any $n \in \mathbb{N}$ the process $f(X^{\zeta_n})$ is a usual continuous semi-martingale. A proof of this simple fact can be found in [69], p.104.

2. The notion of "continuous local martingales" on manifolds can be defined in a complete analogy to definition 2.1.

□

Having this definition, we can now give a precise definition of stochastic differential equations (and their solutions) on manifolds:

Definition 2.3 a) *A stochastic differential equation in M is a pair (A, Z), where $A : M \times \mathbb{R}^l \to \mathrm{T}M$ is a morphism of smooth vector bundles and Z is a continuous semi-martingale with values in $\mathbb{R}^l$ and $\zeta_Z = \infty$. The process Z is called the driving semi-martingale of (A, Z).*

b) *A continuous semi-martingale X with values in M is called a solution of the stochastic differential (A, Z), if for any stopping time ζ with $\zeta < \zeta_X$ $\mathbb{P}$-a.s. and any real-valued $f \in \mathrm{C}^\infty(M)$, the process X^ζ satisfies the following (Itô) formula,*

$$f(X_t^\zeta) = f(X_0) + \int_0^{t\wedge\zeta} \mathrm{d}f(X_s^\zeta)(A(X_s^\zeta))\underline{\mathrm{d}}Z_s \quad \mathbb{P}\text{-a.s. for any } t \geq 0. \tag{25}$$

Here, we have used the usual notation $a \wedge b := \min\{a, b\}$ and the symbol $\underline{\mathrm{d}}$ stands for the Stratonovic differential. We will write d for Itô differentials in the following. The analogue of remark 2.2 also holds for definition 2.3:

Remark 2.4 It follows directly from the continuity of the involved processes and the Stratonovic stopping rule that a continuous semi-martingale X with values in M is a solution of (A, Z), if and only if for any real-valued $f \in \mathrm{C}^{\infty}(M)$ there is a sequence of stopping times (ζ_n) that announces ζ, such that for any $n \in \mathbb{N}$ the process $f(X^{\zeta_n})$ satisfies (25) with ζ replaced by ζ_n.

$\square$

One usually uses the symbolic notation

$$\mathrm{d}X = A(X)\underline{\mathrm{d}}Z \tag{26}$$

in order to express that a process X is a solution of (A, Z). Note that if $M = \mathbb{R}^m$, then the usual Itô formula implies that definition 2.3 is equivalent to the usual definition of (strong) solutions of stochastic differential equations.

We fix the standard orthonormal basis $e_1, \ldots, e_l$ of $\mathbb{R}^l$. If $A : M \times \mathbb{R}^l \to \mathrm{T}M$ and $Z = (Z^1, \ldots, Z^l)^t$ are as in definition 2.3, then $A_j := A(\bullet)e_j \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M)$ and the formulae (25) and (26) can be written as

$$f(X_t^\zeta) = f(X_0) + \sum_{j=1}^{l} \int_0^{t\wedge\zeta} A_j(f)(X_s^\zeta)\underline{\mathrm{d}}Z_s^j \tag{27}$$

and

$$\mathrm{d}X = \sum_{j=1}^{l} A_j(X)\underline{\mathrm{d}}Z^j, \tag{28}$$

respectively. Conversely, if $A_1, \ldots, A_l \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M)$, then there is an unique morphism of smooth vector bundles $A : M \times \mathbb{R}^l \to \mathrm{T}M$ such that $A_j = A(\bullet)e_j$.

An application of the Itô formula implies that the coordinate maps from an embedding $M \hookrightarrow \mathbb{R}^l$ can serve as a set of test functions for (26). In detail, this means:

Proposition 2.5 *Let (A, Z) be a stochastic differential equation in M and let*

$$\Psi = (\Psi^1, \ldots, \Psi^l)^t : M \hookrightarrow \mathbb{R}^l$$

be a smooth embedding for some $l \in \mathbb{N}$ such that $\Psi(M)$ is a closed subset of $\mathbb{R}^l$. Then a continuous semi-martingale X with values in M is a solution of (A, Z), if and only if for any stopping time ζ with $\zeta < \zeta_X$ $\mathbb{P}$-a.s. and any $k = 1, \ldots, l$, the process $\Psi^k(X^\zeta)$ satisfies

$$\Psi^k(X_t^\zeta) = \Psi^k(X_0) + \sum_{j=1}^{l} \int_0^{t\wedge\zeta} A_j(\Psi^k)(X_s^\zeta)\underline{\mathrm{d}}Z_s^j \quad \mathbb{P}\text{-a.s. for any } t \geq 0.$$

Proof. Proposition 1.2.7 in [41].

■

The following existence and uniqueness theorem for stochastic differential equations on manifolds will be central in the following:

Theorem 2.6 *Let (A, Z) be a stochastic differential equation in M. Then for any $\mathscr{F}_0$-measurable map $x_0 : \Omega \to M$ there is a unique maximally defined solution X of*

$$\mathrm{d}X = A(X)\underline{\mathrm{d}}Z \tag{29}$$

with $X_0 = x_0$ $\mathbb{P}$-a.s.

Remark 2.7 1. Here, the uniqueness of X is understood in the following sense: If $\tilde{X}$ is a solution of (A, Z) with $\tilde{X} = x_0$ $\mathbb{P}$-a.s., then $\zeta_{\tilde{X}} \leq \zeta_X$ $\mathbb{P}$-a.s. and $\tilde{X}$ is indistinguishable from $X\,|_{[0,\zeta_{\tilde{X}})\times\Omega}$, that is,

$$\mathbb{P}\Big\{ \text{ for all } t \text{ with } 0 \leq t < \zeta_{\tilde{X}} \text{ one has } X_t = \tilde{X}_t \Big\} = 1.$$

2. Of course, one can also consider stochastic differential equations with an arbitrary nonnegative starting time. The analogue of theorem 2.6 remains true in this slightly more general context [27].

□

The conclusion of theorem 2.6 will be written symbolically as

$$\mathrm{d}X = A(X)\underline{\mathrm{d}}Z, \;\; X_0 = x_0, \tag{30}$$

and X will be called the *maximal solution* of (30). In order to prove theorem 2.6, one can use the corresponding result for $M = \mathbb{R}^m$ (which can be proved by a typical stopping time argument, if one cuts off the given stochastic differential equation on the elements of an exhaustion of $\mathbb{R}^m$ and then applies the usual existence and uniqueness theorem for stochastic differential equations with globally Lipschitz coefficients) and then apply this result to the general case by using the Whitney embedding theorem. This is the proof given in [27], and a text book version of this proof can be found in [37]. It is also possible to give a proof that use a localization and patching argument [43], however, this approach leads to several technical difficulties.

The uniqueness part of theorem 2.6 easily implies:

Corollary 2.8 *Under the assumptions of theorem 2.6, let N be a smooth submanifold of M which is closed as a subset, and let $A(x)v \in \mathrm{T}_x N$ for all $x \in N$, $v \in \mathbb{R}^l$. Any maximal solution X of (A, Z) with $X_0 \in N$ $\mathbb{P}$-a.s. stays in N up to ζ_X, that is,*

$$\mathbb{P}\Big\{\text{for all } t \text{ with } 0 \leq t < \zeta_X \text{ one has } X_t \in N\Big\} = 1.$$

Proof. This is the corollary on p. 371 of [37].

□

For the sake of completeness, we state the following theorem concerning the solution flows of stochastic differential equations. We refer the reader to [28] and [27] for a proof (see also [52] for $M = \mathbb{R}^m$):

Theorem 2.9 *For any stochastic differential equation (A, Z) in M there is a family*

$$\zeta : \Omega \times M \to [0, \infty], \;\; (\omega, x) \longmapsto \zeta(\omega, x)$$

of $\mathbb{P}$-a.s. positive predictable stopping times, and a map

$$X : [0, \infty) \times \Omega \times M \to \hat{M}, \;\; (t, \omega, x) \longmapsto X_t(\omega, x),$$

such that if one sets

$$M_t(\omega) := \Big\{x \Big| \, t < \zeta(\omega, x)\Big\},$$

then for $\mathbb{P}$-a.e. $\omega \in \Omega$ one has:

i) *$M_t(\omega)$ is an open subset of M for any $t \geq 0$, in particular,*

$$\zeta(\omega, \bullet) : M \longrightarrow [0, \infty]$$

is lower semi-continuous.

ii) *The map*

$$X_t(\omega, \bullet) : M_t(\omega) \longrightarrow M$$

is a smooth diffeomorphism onto some open subset of M for any $t \geq 0$.

iii) *The pair $(X_\bullet(\bullet, x), \zeta(\bullet, x))$ is the maximal solution of*

$$\mathrm{d}X = A(X)\underline{\mathrm{d}}Z, \;\; X_0 = x \tag{31}$$

for any $x \in M$.

iv) *The map*
$$[0,t] \longrightarrow \mathrm{C}^\infty(M_t(\omega), M), \;\; s \longmapsto X_s(\omega, \bullet)$$
is continuous for any $t \geq 0$*, if* $\mathrm{C}^\infty(M_t(\omega), M)$ *is equipped with its canonical* C^∞*-topology.*

v) *If* $\mathrm{supp}(A_j) \subset M$ *is compact for any* $j = 1, \ldots, l$*, then* $\zeta(\bullet, \bullet) = \infty$ *and*
$$X_t(\omega, \bullet) : M \longrightarrow M$$
is a smooth diffeomorphism onto M *for any* $t \geq 0$*.*

2.2 Horizontal lifts of semi-martingales

In this section, we are going to explain the concept of stochastic horizontal lifts of continuous semi-martingales to arbitrary smooth principal bundles. We will only consider processes with starting time 0, but it is clear that, with obvious adaptions, all results from section 2.2 and section 2.3 carry over to processes with an arbitrary nonnegative starting time. Let us first remark the following proposition, which characterizes the semi-martingale property of manifold valued processes:

Proposition 2.10 *Any solution of a stochastic differential equation in* M *is a continuous semi-martingale. Conversely, any continuous semi-martingale* X *in* M *with* $\zeta_X = \infty$ *solves a stochastic differential equation in the sense of definition 2.3.*

Proof. It is clear that the solutions of stochastic differential equations are continuous semi-martingales. The other direction can be seen as follows: Let $\Psi : M \hookrightarrow \mathbb{R}^l$ be a smooth embedding for some $l \in \mathbb{N}$ such that $\Psi(M)$ is a closed subset of $\mathbb{R}^l$, and let A be given as the orthogonal projection $A(x) : \mathbb{R}^l \to \mathrm{T}_x M$ for any $x \in M$. Then X is a solution of the stochastic differential equation $(A, \Psi(X))$. The technical details can be found in [41], lemma 2.3.3.

■

Actually, a somewhat stronger statement holds: One can generalize the notion of (solutions of) stochastic differential equations [41] by allowing the driving semi-martingale itself to be defined only up to some predictable stopping time, where then an analogue of theorem 2.6 holds. With this generalized definition, and keeping the construction of the driving semi-martingale in the proof of proposition 2.10 in mind, one finds that a process is a continuous semi-martingale, if and only if it is the solution of a stochastic differential equation.

In order to avoid technical difficulties (like stochastic localization procedures that would be necessary in order to define the involved stochastic integrals correctly), we will assume for the abstract results of this section that the given continuous semi-martingale on M is defined up to ∞. The assertions for the general case can then be derived with typical stopping time arguments. Furthermore, we remark that one can always change the filtration to achieve an infinite lifetime ([37], p.371). This approach is certainly satisfactory, if one is interested in abstract existence results as in proposition 2.10 or in theorem 2.15.

Let us now introduce the concept of stochastic line integrals, which will help us to extend the notion of "horizontal lifts" to continuous semi-martingales. In the following, any two processes which are defined up to ∞ will be identified, if they are indistinguishable.

Proposition and definition 2.11 *Let N be a smooth manifold and let Y be a continuous semi-martingale on N which is defined up to ∞. Furthermore, let F be a finite dimensional $\mathbb{K}$-linear* [4] *space. Then there is a unique morphism of $\mathbb{K}$-linear spaces*

$$\begin{aligned}\Omega^1(M,F) &\longrightarrow \Big\{F\text{-valued continuous semi-martingales defined up to } \infty\Big\},\\ \alpha &\longmapsto \int \alpha(\underline{\mathrm{d}}Y)\end{aligned}$$

such that for any $f \in \mathrm{C}^\infty(N,F)$,

$$\int f(Y)\alpha(\underline{\mathrm{d}}Y) := \int f(Y)\underline{\mathrm{d}}\int \alpha(\underline{\mathrm{d}}Y) = \int (f\alpha)(\underline{\mathrm{d}}Y) \tag{32}$$

$$\int (\mathrm{d}f)(\underline{\mathrm{d}}Y) = f(Y) - f(Y_0). \tag{33}$$

The process $\int \alpha(\underline{\mathrm{d}}Y)$ is called the Stratonovic stochastic line integral of α along Y.

Proof. We explain the construction: Let

$$h = (h^1,\ldots,h^q)^t : N \hookrightarrow \mathbb{R}^q$$

be such that $h(N)$ is a closed subset of $\mathbb{R}^q$, and let $\alpha_1,\ldots,\alpha_q \in \mathrm{C}^\infty(N,F)$ be such that

$$\alpha = \sum_{j=1}^q \alpha_j \mathrm{d}h^j.$$

[4] $\mathbb{K} = \mathbb{R}, \mathbb{C}$

Then one can define

$$\int_0^t \alpha(\underline{\mathrm{d}}Y_s) := \sum_{j=1}^q \int_0^t \alpha_j(Y_s)\underline{\mathrm{d}}h^j(Y_s). \tag{34}$$

The details of the proof (in particular, the well-definedness of (34)) can be found in [37] (Satz 7.62).

■

Remark 2.12 1. Proposition 2.11 extends the definition of usual line integrals along deterministic smooth curves: Let $\gamma : [0,\infty) \to N$ be a deterministic smooth curve on N. By the chain rule, γ is clearly a continuous semi-martingale in the sense of definition 2.1. One has

$$\int_0^t \alpha(\underline{\mathrm{d}}\gamma_s) = \int_\gamma \alpha \mid_{[0,t]} := \int_0^t \alpha_{\gamma(s)}(\dot\gamma(s))\mathrm{d}s.$$

This follows from the uniqueness part of proposition 2.11.

2. In principle, it is possible to define $\int \alpha(\underline{\mathrm{d}}Y)$ by using charts (this is carried out in [43]). However, the patching procedure is complicated. Furthermore, the use of Whitney's embedding theorem makes it easier to check the well-definedness.

3. It is possible to give a third equivalent definition of stochastic line integrals, using the so called *anti-development* (with respect to some initial value) of the given continuous semi-martingale [41].

□

There is a canonic way to calculate line integrals along solutions of stochastic differential equations:

Corollary 2.13 *In the setting of proposition 2.11, let Y be given as the maximal solution of*

$$\mathrm{d}Y = \sum_{j=1}^l A_j(Y)\underline{\mathrm{d}}Z^j, \;\; Y_0 = y_0 \tag{35}$$

for some stochastic differential equation (A, Z) on N and some $\mathscr{F}_0$-measurable $x_0 : \Omega \to N$, and let $\alpha \in \Omega^1(N, F)$. Then

$$\int \alpha(\underline{\mathrm{d}}Y) = \sum_{j=1}^l \int \alpha(A_j(Y))\underline{\mathrm{d}}Z^j.$$

Proof. This follows easily from the uniqueness part of proposition 2.11. ■

In particular, proposition 2.10 and corollary 2.13 lead to another equivalent definition of stochastic line integrals along continuous semi-martingales.

For the rest of this section, let X be a continuous semi-martingale with values in M which is defined up to ∞ and which starts from some $\mathscr{F}_0$-measurable $x_0 : \Omega \to M$. We also fix a smooth principal bundle $\pi : P \to M$ with structure group G and the associated Lie algebra $\mathfrak{g}$, and a connection 1-form $\alpha_0 \in \Omega^1(P, \mathfrak{g})$. Since we now have the probabilistic notion of line integrals, we can give the following definition:

Definition 2.14 *A continuous semi-martingale U on P which is defined up to ∞ is called a horizontal lift of X to P (with respect to the connection α_0), if $\pi(U) = X$ and*

$$\int \alpha_0(\underline{\mathrm{d}}U) = 0. \tag{36}$$

Clearly, this definition is motivated from remark 2.12 and the fact that if a $\gamma : [0, \infty) \to M$ is a deterministic smooth curve in M, then for any $u_0 \in P$ with $\pi(u_0) = \gamma_0$ there is a unique horizontal lift (in the usual sense) $u : [0, \infty) \to P$ from u_0. This lift clearly satisfies

$$\int_u \alpha_0 \mid_{[0,t]} = 0 \quad \text{for any } t \geq 0.$$

Being equipped with this notion, one can prove:

Theorem 2.15 *For any $\mathscr{F}_0$-measurable $u_0 : \Omega \to P$ with $\pi(u_0) = x_0$ $\mathbb{P}$-a.s. there is a unique horizontal lift U of X to P with $U_0 = u_0$ $\mathbb{P}$-a.s.*

In this generality, this result first appeared in [72] (where the construction of U is given, but an argument for $\zeta_U = \infty$ is missing). A full proof of theorem 2.15 can be found in [37], Satz 7.141. In the general case, that is, if the given semi-martingale on M is not necessarily defined up to ∞, then the corresponding horizontal lift lives on P as long as the first process lives on M. This assertion is included in [37], Satz 7.141.

By proposition 2.10, the process X satisfies a stochastic differential equation of the form (30). This can be used to derive a corresponding equation for the lift U, which turns out to be very useful in applications:

Proposition 2.16 *Let X be the maximal solution of*

$$\mathrm{d}X = \sum_{j=1}^{l} A_j(X)\underline{\mathrm{d}}Z^j, \;\; X_0 = x_0 \tag{37}$$

for some stochastic differential equation (A, Z) and some $\mathscr{F}_0$-measurable $x_0 : \Omega \to M$, and let $u_0 : \Omega \to P$ be $\mathscr{F}_0$-measurable with $\pi(u_0) = x_0$ $\mathbb{P}$-a.s. Then the horizontal lift U of X to P from u_0 is uniquely determined as the maximal solution $\tilde{U}$ of

$$\mathrm{d}\tilde{U} = \sum_{j=1}^{l} A_j^*(\tilde{U})\underline{\mathrm{d}}Z^j, \;\; \tilde{U}_0 = u_0, \tag{38}$$

where A_j^* is the horizontal lift of A_j to P (with respect to α_0) for any $j = 1, \ldots, l$.

Proof. First of all, $\zeta_{\tilde{U}} = \infty$ has been shown in the proof of theorem 13C, p.175, in [28]. The equality $\pi(\tilde{U}) = X$ then follows from checking that $\pi(\tilde{U})$ satisfies (37), which follows directly from the definition of A_j^* and the chain rule. The fact that

$$\int \alpha_0(\underline{\mathrm{d}}\tilde{U}) = 0 \tag{39}$$

follows from the (local) considerations of lemma 3.2 in [72]. However, being equipped with our embedding approach to stochastic line integrals, the proof of (39) becomes trivial: Corollary 2.13 implies

$$\int \alpha_0(\underline{\mathrm{d}}\tilde{U}) = \sum_{j=1}^{l} \int \alpha_0(A_j^*)(\tilde{U})\underline{\mathrm{d}}Z^j = 0,$$

where the last equality follows from the fact that the vector fields A_j^* are horizontal.

■

2.3 Stochastic parallel transport

Throughout section 2.3, X will again be a continuous semi-martingale with values in M which is defined up to ∞ and which starts from some $\mathscr{F}_0$-measurable $x_0 : \Omega \to M$. Let $E \to M$ be a smooth d-dimensional complex vector bundle with a fixed smooth Hermitian structure

$$(\bullet, \bullet)_x : E_x \times E_x \longrightarrow \mathbb{C}, \;\; x \in M,$$

and a fixed Hermitian covariant derivative

$$\nabla : \Gamma_{\mathrm{C}^\infty}(M, E) \longrightarrow \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}^*M \otimes E).$$

It is implicit in the notation that we have complexified T^*M and $\mathrm{T}M$. Let $\pi : \mathrm{P}(E) \to M$ be the $\mathrm{U}(d)$-principal bundle corresponding to $(E, (\bullet, \bullet)_x)$, that is,

$$\mathrm{P}(E) = \bigcup_{x \in M} \Big\{ u \Big| \, u : \mathbb{C}^d \xrightarrow{\simeq} E_x \text{ is an isometry} \Big\}.$$

That ∇ is Hermitian means that it is compatible with $(\bullet, \bullet)_x$ in the usual sense: For any $\Psi_1, \Psi_2 \in \Gamma_{\mathrm{C}^\infty}(M, E)$, $A \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M)$ one has

$$A\Big((\Psi_1, \Psi_2)\Big)(x) = (\nabla_A \Psi_1(x), \Psi_2(x))_x + (\Psi_1(x), \nabla_A \Psi_2(x))_x \quad \text{for any } x \in M,$$

where $(\Psi_1, \Psi_2) \in \mathrm{C}^\infty(M)$ is given by

$$x \longmapsto (\Psi_1(x), \Psi_2(x))_x.$$

We write $\mathscr{U}(d) := \mathrm{Lie}(\mathrm{U}(d))$ for the anti-Hermitian elements of

$$\mathrm{Mat}(\mathbb{C}^d) := \mathrm{Mat}_{\mathbb{C}}(d \times d).$$

Since ∇ is Hermitian, it follows from proposition 1.5 on p. 117 in [51] that ∇ induces a connection $\alpha_0 \in \Omega(\mathrm{P}(E), \mathscr{U}(d))$ on $\mathrm{P}(E)$, and the considerations of the previous section show that for any $\mathscr{F}_0$-measurable $u_0 : \Omega \to \mathrm{P}(E)$ with $\pi(u_0) = x_0$ $\mathbb{P}$-a.s. there is a unique horizontal lift of X to $\mathrm{P}(E)$ starting from u_0. The aim of this section is to introduce the parallel transport map associated to X and to derive a formula for the stochastic differential of it. Let $E \boxtimes E^* \to M \times M$ be the exterior tensor bundle corresponding to E, that is,

$$E \boxtimes E^* \mid_{(x,y)} = E_x \otimes E_y^* = \mathrm{Hom}(E_y, E_x). \tag{40}$$

The following assertion is certainly well-known. Nevertheless, we have not been able to find a proof in the literature, so we give one here:

Proposition and definition 2.17 *Let U be a lift of X to $\mathrm{P}(E)$. Then the continuous adapted process given by*

$$//^X := U U_0^{-1} : [0, \infty) \times \Omega \longrightarrow E \boxtimes E^*$$

does not depend on the particular choice of the lift U, and $//^X$ will be called the stochastic parallel transport in E along X.

Proof. Note that whatever lift has been taken to define $//^X$, one has

$$//_t^X \in \mathrm{Hom}(E_{x_0}, E_{X_t}) \quad \mathbb{P}\text{-a.s. for all } t \geq 0.$$

Let U be the horizontal lift starting from u_0 and let $\tilde{U}$ be the horizontal lift from starting from $\tilde{u_0}$. We define a $\mathscr{F}_0$-measurable map g_0 by setting

$$g_0 := u_0^{-1}\tilde{u_0} : \Omega \longrightarrow \mathrm{U}(d).$$

Clearly, Ug_0 is a lift of X starting from $\tilde{u}_0$. It is also horizontal: For example, one can use Shigekawa's pull back formula (lemma 3.4 in [72]) to deduce that

$$\int \alpha(\underline{\mathrm{d}}Ug_0) = \mathrm{Ad}(g_0^{-1}) \int \alpha(\underline{\mathrm{d}}U),$$

which is equal to zero by assumption. The uniqueness part of theorem 2.15 now implies $Ug_0 = \tilde{U}$.

■

Note that, as we have already done, the dependence of $//^X$ on ∇ and $(\bullet, \bullet)_x$ will usually be omitted in our notation, since these data will always be fixed. Furthermore, for all $t \geq 0$ the linear maps

$$//_t^X : E_{x_0} \xrightarrow{\simeq} E_{X_t}$$

are isometries $\mathbb{P}$-a.s.

Next, we remark the following purely geometric lemma:

Lemma 2.18 *For any $\Psi \in \Gamma_{\mathrm{C}^\infty}(M, E)$ let F_Ψ be the smooth function defined by*

$$F_\Psi : \mathrm{P}(E) \longrightarrow \mathbb{C}^d, \;\; F_\Psi(u) = u^{-1}\Big(\Psi \circ \pi(u)\Big).$$

Let $A \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M)$ and let $A^* \in \Gamma_{\mathrm{C}^\infty}(\mathrm{P}(E), \mathrm{TP}(E))$ be the lift of A to $\mathrm{P}(E)$ (with respect to ∇). Then one has

$$A^* F_\Psi = F_{\nabla_A \Psi} \quad \textit{for any } \Psi \in \Gamma_{\mathrm{C}^\infty}(M, E).$$

A *proof* can be found in [51], p. 115.

In case X is given as the solution of a stochastic differential equation (this is no restriction), one can proceed as follows:

Proposition 2.19 *Assume that X is the maximal solution of*

$$\mathrm{d}X = \sum_{j=1}^{l} A_j(X)\underline{\mathrm{d}}Z^j, \;\; X_0 = x_0 \tag{41}$$

for some stochastic differential equation (A, Z) and some $\mathscr{F}_0$-measurable $x_0 : \Omega \to M$, and that U is a horizontal lift of X to $\mathrm{P}(E)$. Then the following formulae hold for any $\Psi \in \Gamma_{\mathrm{C}^\infty}(M, E)$,

$$\mathrm{d}\Big(U^{-1}\Psi(X)\Big) = U^{-1}\sum_{j=1}^{l}(\nabla_{A_j}\Psi)(X)\underline{\mathrm{d}}Z^j \tag{42}$$

$$= U^{-1}\left(\sum_{j=1}^{l}(\nabla_{A_j}\Psi)(X)\mathrm{d}Z^j + \frac{1}{2}\sum_{i,j=1}^{l}(\nabla_{A_i}\nabla_{A_j}\Psi)(X)\mathrm{d}[Z^i, Z^j]\right). \tag{43}$$

Proof. If one applies the formula in proposition 2.16 to each component of F_Ψ, that is,

$$\mathrm{d}F_\Psi(U) = \sum_{j=1}^{l} A_j^*(F_\Psi)(U)\underline{\mathrm{d}}Z^j,$$

then formula (42) follows immediately from $A_j^* F_\Psi = F_{\nabla_{A_j}\Psi}$ (lemma 2.18). In order to derive formula (43), one now only has to apply proposition 2.16 to $F_{\nabla_{A_j}\Psi}$, in order to convert the Stratonovic differential to an Itô differential.

■

Corollary 2.20 *Under the assumptions of proposition 2.19, let X start from a deterministic point $x_0 \in M$. Then the following formulae hold,*

$$\mathrm{d}\Big(//^{X,-1}\Psi(X)\Big) = //^{X,-1}\sum_{j=1}^{l}(\nabla_{A_j}\Psi)(X)\underline{\mathrm{d}}Z^j \tag{44}$$

$$= //^{X,-1}\left(\sum_{j=1}^{l}(\nabla_{A_j}\Psi)(X)\mathrm{d}Z^j + \frac{1}{2}\sum_{i,j=1}^{l}(\nabla_{A_i}\nabla_{A_j}\Psi)(X)\mathrm{d}[Z^i, Z^j]\right). \tag{45}$$

Proof. This follows from multiplying the formulae from proposition 2.19 with some $u_0 \in \mathrm{P}(E)$ with $\pi(u_0) = x_0$.

■

We close this section with some specific remarks about stochastic parallel transport in trivial vector bundles: Firstly, we prove that the (inverse) stochastic parallel transport on trivial vector bundles can essentially be defined as the solution of a linear stochastic differential equation:

Proposition 2.21 *Assume that X starts from a deterministic point $x_0 \in M$. Let $E = M \times \mathbb{C}^d$ with its standard Hermitian structure, let $f_1, \dots, f_d \in \Gamma_{C^\infty}(M, E)$ be the standard global orthonormal frame*

$$f_j(x) := (x, e_j) \in E_x = \{x\} \times \mathbb{C}^d, \ \ j = 1, \dots, d,$$

and let $\alpha \in \Omega^1(M, \mathscr{U}(d))$ be the connection-1-form of ∇ (with respect to $f_1, \dots, f_d$). If the process $\mathscr{A}(X)$ is defined by

$$\begin{aligned} &\mathscr{A}(X) : [0, \infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d), \\ &\mathscr{A}(X)^k_l := \left(//^{X,-1} f_k(X), f_l(x_0) \right)_{x_0}, \end{aligned} \tag{46}$$

then $\mathscr{A}(X)$ is uniquely determined as the maximal solution [5] *of the following linear stochastic differential equation in* $\mathrm{Mat}(\mathbb{C}^d)$,

$$\mathrm{d}\mathscr{A}^\alpha(X) = -\alpha(\underline{\mathrm{d}}X)\mathscr{A}^\alpha(X), \ \ \mathscr{A}^\alpha_0(X) = \mathbf{1}. \tag{47}$$

Proof. Firstly, the linearity of (47) implies that $\mathscr{A}^\alpha(X)$ can clearly be defined up to ∞.
We can assume that X is given as the maximal solution of (37), and we set $//^{-1} := //^{X,-1}$ and $\mathscr{A} := \mathscr{A}(X)$. Then one has

$$\begin{aligned} \mathrm{d}\mathscr{A}^k_l &= \left(\mathrm{d}//^{-1} f_k(X), f_l(x_0) \right)_{x_0} \\ &= \sum_{j=1}^{l} \left(//^{-1} \nabla_{A_j} f_k(X) \underline{\mathrm{d}}Z^j, f_l(x_0) \right)_{x_0} \\ &= \sum_{j=1}^{l} \sum_{i=1}^{d} \left(//^{-1} f_i(X) \alpha(A_j)^i_k \underline{\mathrm{d}}Z^j, f_l(x_0) \right)_{x_0} \\ &= \sum_{i=1}^{d} \left(//^{-1} f_i(X) \alpha(\underline{\mathrm{d}}X)^i_k, f_l(x_0) \right)_{x_0} \\ &= -\Big(\alpha(\underline{\mathrm{d}}X)\mathscr{A} \Big)^k_l, \end{aligned}$$

where we have used corollary 2.20 for the second equality, corollary 2.13 for the fourth equality and $\alpha(\underline{\mathrm{d}}X)^i_k = -\overline{\alpha(\underline{\mathrm{d}}X)^k_i}$ for the last equality.

■

Proposition 2.21 should be read as follows:

[5] Remember the notation $\alpha(\underline{\mathrm{d}}X) = \underline{\mathrm{d}} \int \alpha(\underline{\mathrm{d}}X)$.

Remark 2.22 1. We fix the situation of proposition 2.21. If for any $t \geq 0$ we choose $f_1(X_t), \dots, f_d(X_t)$ as the orthonormal basis of E_{X_t}, then $//_t^X$ is represented by the unitary matrix $\mathscr{A}_t^{\alpha}(X)$ (see also proposition C.29), which is uniquely determined by

$$\mathrm{d}\mathscr{A}^{\alpha,-1}(X) = \mathscr{A}^{\alpha,-1}(X)\alpha(\underline{\mathrm{d}}X), \ \ \mathscr{A}_0^{\alpha,-1}(X) = \mathbf{1}. \tag{48}$$

Since furthermore the global orthonormal frame $f_1, \dots, f_d$ induces an isomorphism $\Gamma_{\mathscr{K}}(M, E) \cong \mathscr{K}(M, \mathbb{C}^d)$ of $\mathscr{K}(M)$-modules, where for example $\mathscr{K}$ may stand for "smooth" or "continuous" or "measurable", one can consider $\mathscr{A}^{\alpha}(X)$ itself as the stochastic parallel transport along X. Note that in view of remark 2.12.1, equation (48) certainly represents another analogy to the usual theory of deterministic smooth curves.

2. If $E = M \times \mathbb{C}$, then the first part of this remark reduces to

$$\mathscr{A}^{\alpha,-1}(X) = \mathrm{e}^{\int \alpha(\underline{\mathrm{d}}X)}. \tag{49}$$

In particular, if $M = \mathbb{R}^m$, if $\alpha = \mathrm{i}\beta$ for some

$$\beta = \sum_{j=1}^{m} \beta_j \mathrm{d}x^j \in \Omega^1_{\mathbb{R}}(\mathbb{R}^m),$$

and if X is equal to an Euclidean Brownian motion in $\mathbb{R}^m$, then the Itô formula and $[X^j, X^k]_t = \delta^{jk}\mathrm{d}t$ show

$$\begin{aligned}\mathscr{A}_t^{\mathrm{i}\beta,-1}(X) &= \exp\left(\mathrm{i}\sum_{j=1}^{m}\int_0^t \beta_j(X_s)\underline{\mathrm{d}}X_s^j\right)\\ &= \exp\left(\mathrm{i}\sum_{j=1}^{m}\int_0^t \beta_j(X_s)\mathrm{d}X_s^j + \frac{\mathrm{i}}{2}\int_0^t \mathrm{div}\beta(X_s)\mathrm{d}s\right).\end{aligned}$$

The latter expression is well-known from the classical Feynman-Kac-Itô formula in $\mathbb{R}^m$ [77], if one interprets β as a magnetic potential. In particular, the equality

$$\mathscr{A}_t^{\mathrm{i}\beta,-1}(X) = \exp\left(\mathrm{i}\sum_{j=1}^{m}\int_0^t \beta_j(X_s)\mathrm{d}X_s^j + \frac{\mathrm{i}}{2}\int_0^t \mathrm{div}\beta(X_s)\mathrm{d}s\right)$$

gives some geometric insight into the classical Feynman-Kac-Itô formula.

□

2.4 Brownian motions and stochastic completeness

For the rest of this thesis, we assume that M is equipped with a fixed Riemannian structure $g \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}^*M \otimes \mathrm{T}^*M)$. The Laplace-Beltrami operator corresponding to this structure will be denoted with

$$\Delta = -\mathrm{d}^*\mathrm{d} : \mathrm{C}^\infty(M) \longrightarrow \mathrm{C}^\infty(M).$$

Since g will remain fixed, we will usually omit the dependence on data depending on g in our notation, as we have already done for Δ.
Brownian motions can be defined as follows in arbitrary Riemannian manifolds:

Definition 2.23 *A continuous adapted maximally defined process*

$$B(x) : [0, \zeta_{B(x)}) \times \Omega \longrightarrow M$$

is called a Brownian motion on $M = (M, g)$ with starting point $x \in M$, if $B_0(x) = x$ $\mathbb{P}$-a.s. and if for any stopping time ζ with $\zeta < \zeta_{B(x)}$ $\mathbb{P}$-a.s. and any real-valued $f \in \mathrm{C}^\infty(M)$, the process

$$B_t^{f,\zeta}(x) := f(B_t^\zeta(x)) - \frac{1}{2}\int_0^{t\wedge\zeta} \Delta f(B_s^\zeta(x))\mathrm{d}s \tag{50}$$

is a continuous local martingale.

Clearly, definition 2.23 implies that Brownian motions are continuous semimartingales. Furthermore, by using the Lévy-characterization of Brownian motions and the Itô formula, one easily finds that this definition is consistent with the usual definition of Brownian motions in the Euclidean $\mathbb{R}^m$.

We leave the question of existence of Brownian motions on M aside for a moment. In the following, $\mathrm{vol}(\bullet)$ stands for the canonical volume measure (on the Borel-σ-algebra) associated to the given Riemannian structure on M. It is well-known that the minimal heat kernel $p_t(x, y)$ of M exists and is uniquely determined in the sense of the following theorem, which also connects Brownian motions to $p_t(x, y)$ by asserting that the transition density of Brownian motions is given by $p_t(x, y)$.

Theorem 2.24 *There exists a unique minimal positive fundamental solution*

$$p : (0, \infty) \times M \times M \longrightarrow (0, \infty), \ \ (t, x, y) \longmapsto p_t(x, y)$$

of the heat equation

$$\frac{\partial}{\partial t}h(t, x) = \frac{1}{2}\Delta_x h(t, x).$$

The map p will be called the minimal heat kernel of $M = (M, g)$, and it has the following additional properties:

i) *p is smooth,*

ii) $p_t(x,y) = p_t(y,x)$ *for all* $t > 0$, $x, y \in M$,

iii) *one has*

$$\int_M p_t(x,y)\mathrm{vol}(\mathrm{d}y) \leq 1 \quad \textit{for all } t > 0,\ x \in M, \tag{51}$$

iv) *the Chapman-Kolmogorov equations hold,*

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)\mathrm{vol}(\mathrm{d}z) \quad \textit{for all } t,s > 0,\ x,y \in M,$$

v) $p_t(x,\bullet) \in \mathrm{L}^1(M) \cap \mathrm{L}^2(M)$ *for all* $t > 0$, $x \in M$,

vi) *if* $B(x)$ *is a Brownian motion on* M *which starts in* x *and if* $N \subset M$ *is a measurable set, then*

$$\mathbb{P}\Big\{ t < \zeta_{B(x)} \text{ and } \ B_t(x) \in N \Big\} = \int_N p_t(x,y)\mathrm{vol}(\mathrm{d}y) \quad \textit{for any } t > 0. \tag{52}$$

Proof. As the theorem is certainly well-known, we only explain possible constructions of the minimal heat kernel: Let (M_n) be a smooth exhaustion of M and let $p^{(n)}_t(x,y)$ denote the Dirichlet heat kernel corresponding to M_n. Then it has been shown in [21] that

$$p_t(x,y) := \lim_{n\to\infty} p^{(n)}_t(x,y)$$

is independent of (M_n) and has the properties i)-iv). The property $p_t(x,\bullet) \in \mathrm{L}^2(M)$ follows obviously from ii) and iv), and vi) is included in [41], proposition 4.1.6.
Let us mention that it is also possible to define $p_t(x,y)$ somewhat more directly as the integral kernel corresponding to e^{-tH}, where H is the Friedrichs extension of

$$-\Delta/2 : \mathrm{C}^\infty_0(M) \longrightarrow \mathrm{L}^2(M).$$

The equivalence of this approach to the first one has also been shown in [21].

■

Next, let us review the question under which assumptions on $M = (M,g)$ the corresponding Brownian motions can be defined up to ∞. More precisely:

Definition 2.25 *$M = (M, g)$ is called stochastically complete, if*

$$\mathbb{P}\Big\{\zeta_{B(x)} = \infty\Big\} = 1$$

for any $x \in M$ and any Brownian motion $B(x)$ on M with starting point x.

A simple characterization of this property can be found in part a) of the following proposition. Part b) will be used implicitely throughout in the following:

Proposition 2.26 a) *The following statements are equivalent:*

i) *M is stochastically complete.*

ii) *One has*

$$\int_M p_t(x, y)\mathrm{vol}(\mathrm{d}y) = 1 \quad \textit{for any } t > 0 \textit{ and any } x \in M.$$

iii) *One has*

$$\int_M p_{t_0}(x_0, y)\mathrm{vol}(\mathrm{d}y) = 1 \quad \textit{for some } t_0 > 0 \textit{ and some } x_0 \in M.$$

b) *Let M be stochastically complete, let $B(x)$ be a Brownian motion in M starting from x and let $t > 0$. If $N \subset M$ is measurable with* $\mathrm{vol}(N) = 0$*, then one has*

$$\left|\left\{s \,\middle|\, 0 \leq s \leq t,\ B_s(x) \in N\right\}\right| = 0 \quad \mathbb{P}\text{-}a.s.,$$

where $|\bullet|$ stands for the Lebesgue measure in $\mathbb{R}^1$.

Proof. a) The equivalence i) $\Leftrightarrow$ ii) follows from property vi) of proposition 2.24. The equivalence iii) $\Leftrightarrow$ ii) follows easily from the Chapman-Kolmogorov equation, if one first considers the case $t < t_0$ and then the case $t \geq t_0$. See for example theorem 6.2 in [33].

b) One has

$$\begin{aligned} \mathbb{E}\left[\int_0^t 1_{\{s|0\leq s\leq t,\, B_s(x)\in N\}}(s)\mathrm{d}s\right] &= \int_0^t \mathbb{P}\{B_s(x) \in N\}\mathrm{d}s \\ = \int_0^t \int_N p_s(x, y)\mathrm{vol}(\mathrm{d}y)\mathrm{d}s &= 0, \end{aligned} \tag{53}$$

where we have used theorem 2.24 vi) for the second equality.

■

In general, geodesic completeness is not implied by stochastic completeness ($M = \mathbb{R}^2 \setminus \{0\}$ is a simple counter example) and there are geodesically complete manifolds which are not stochastically complete (see e.g. [29], (5.49), p.69). However, one has the following theorem:

Theorem 2.27 *If M is geodesically complete and if for some $x \in M$ and some $r_0 > 0$ one has*

$$\int_{r_0}^{\infty} \frac{r\mathrm{d}r}{\log(\mathrm{vol}(x,r))} = \infty, \tag{54}$$

then M is stochastically complete.

Here, $\mathrm{vol}(x,r)$ stands for the volume of the open geodesic ball with radius r around x. A detailed proof of theorem 2.27 can be found in the survey article of A. Grigor'yan [33] (theorem 9.1), who first proved this result in [34]. As an application, one has [41][37]:

Theorem 2.28 *Let M be geodesically complete. If the Ricci curvature of M is bounded from below in radial direction by some quadratic function of the Riemannian distance function (for some fixed reference point), then M is stochastically complete. In particular, if the Ricci curvature of M is bounded from below by a constant, then M is stochastically complete.*

Theorem 2.28 can be derived from theorem 2.27 by using the Bishop-Gromov inequality [41]. The fact that stochastic completeness is implied by geodesic completeness if the Ricci curvature is bounded from below is known as *Yau's (stochastic completeness) theorem* and has first been proved in [89] with analytic methods. A probabilistic approach to Yau's theorem can be found in [27].

By these considerations it is clear that the Euclidean $\mathbb{R}^m$, compact manifolds or, more generally, manifolds with a bounded geometry are examples of stochastically complete manifolds (see also inequality (213)).

Remark 2.29 1. Grigor'yan's stochastic completeness criterion has been generalized by K.-T. Sturm [81] to certain local Dirichlet forms on locally compact seperable Hausdorff spaces with a positive Radon measure with full support.

2. It is also possible to define the notion of stochastic completeness for metric graphs, but, interestingly, the analogue of theorem 2.27 does not hold in this situation. This follows from corollary 3.10 in [88]. [6]

□

We now turn to the question of how to construct Brownian motions. Definition 2.23 actually means that Brownian motions on M are diffusion processes

[6] This reference has been pointed out to the author by Matthias Keller.

corresponding to the generator $\Delta/2$. Thus, in principle, one can construct Brownian motions as follows: In order to include the possibility of explosion, one considers the generalized path space

$$\mathscr{W}(\hat{M}) := \left\{ \omega \middle| \ \omega \in \mathrm{C}([0,\infty),\hat{M}),\, \omega(t) = \infty_M \text{ for all } t \geq \zeta^M(\omega) \ \right\},$$

where

$$\zeta^M(\omega) = \inf \left\{ t \middle| \ \omega(t) = \infty_M \right\}$$

stands for the explosion time of the coordinate process $(t,\omega) \mapsto \omega(t)$ on $\mathscr{W}(\hat{M})$. Then one takes $\mathscr{F}^M$ and $\mathscr{F}^M_*$ to be the σ-algebra and, respectively, the filtration which is generated by the coordinate process. Using the Whitney embedding theorem, one can show ([41], proposition 3.2.1):

Theorem 2.30 *For any $x \in M$ there is a unique measure $\mathbb{P}^x$, the Wiener measure, on $(\mathscr{W}(\hat{M}), \mathscr{F}^M)$ such that the coordinate process on the filtered probability space $(\mathscr{W}(\hat{M}), \mathscr{F}^M, \mathscr{F}^M_*, \mathbb{P}^x)$ is a Brownian motion on M starting in x, which is defined up to ζ^M.*

This construction of Brownian motions on M is satisfactory, if one wants to derive path integral formulae for Schrödinger operators on functions. However, if one is interested in Schrödinger type operators on arbitrary vector bundles, where parallel transports are involved naturally, then it is convenient (see corollary 2.20) to start with an Euclidean Brownian motion and to define Brownian motions on M by solving a stochastic differential equation. In the following, we explain two constructions that are in the spirit of this remark.
Let $\nabla^{\mathrm{T}M}$ denote the Levi-Civita connection. An extrinsic construction principle of Brownian motions can be given in terms of the Nash embedding theorem. The construction of diffusion processes on manifolds by the use of embedding theorems goes back to L. Schwartz [71]. In the following theorem, it is understood that $\mathbb{R}^l$ is equipped with its Euclidean metric.

Theorem 2.31 *Let $M \hookrightarrow \mathbb{R}^l$ isometrically for some $l \in \mathbb{N}$ and let the morphism of smooth vector bundles $A : M \times \mathbb{R}^l \to \mathrm{T}M$ be given as the orthogonal projection $A(x) : \mathbb{R}^l \to \mathrm{T}_x M$ for any $x \in M$. Let W be a Brownian motion in $\mathbb{R}^l$. Then the maximal solution of*

$$\mathrm{d}X = \sum_{j=1}^{l} A_j(X) \underline{\mathrm{d}} W^j, \ \ X_0 = x \tag{55}$$

is a Brownian motion on M with starting point x.

Proof. Let $f \in \mathrm{C}^\infty(M)$ and let ζ be a stopping time with $\zeta < \zeta_{B(x)}$ $\mathbb{P}$-a.s. Then

$$f(X_t^\zeta) - f(x) = \sum_{j=1}^{l} \int_0^{t\wedge\zeta} A_j(f)(X_t^\zeta)\mathrm{d}W^j + \frac{1}{2}\sum_{i,j=1}^{l}\int_0^{t\wedge\zeta} A_iA_j(f)(X_t^\zeta)\delta^{ij}\mathrm{d}t,$$

thus it is sufficient to prove that

$$\sum_{j=1}^{l} A_j^2 f = \Delta f. \tag{56}$$

Fix an arbitrary $y \in M$. Let $w_1, \dots, w_m$ be an orthonormal basis of T_yM and let $v_{m+1}, \dots, v_l \in \mathbb{R}^l$ be such that $v_1, \dots, v_l$ is an orthonormal basis of $\mathbb{R}^l$, where $v_j := A(x)^* w_j$ for $j = 1, \dots, m$. Then

$$\begin{aligned}
\Delta f \mid_y &= \sum_{j=1}^{m} w_j^2 f \mid_y - \nabla^{\mathrm{T}M}_{\nabla^{\mathrm{T}M}_{w_j} w_j} f \mid_y \\
&= \sum_{j=1}^{l} (A(\bullet)v_j)^2 f \mid_y - \nabla^{\mathrm{T}M}_{\nabla^{\mathrm{T}M}_{A(\bullet)v_j} A(\bullet)v_j} f \mid_y \\
&= \sum_{j=1}^{l} (A(\bullet)e_j)^2 f \mid_y - \nabla^{\mathrm{T}M}_{\nabla^{\mathrm{T}M}_{A(\bullet)e_j} A(\bullet)e_j} f \mid_y .
\end{aligned}$$

Since

$$\sum_{j=1}^{l} \nabla^{\mathrm{T}M}_{A_j} A_j = 0, \tag{57}$$

which has been proved in [37], Satz 7.119, the theorem follows.

■

If $B(x)$ is constructed in this way, then one can apply corollary 2.20 in order to get the following formula for the stochastic differential of the corresponding stochastic parallel transport:

Proposition 2.32 *Let M be stochastically complete, let $B(x)$ be constructed as in theorem 2.31 and let E be a Hermitian vector bundle over M with a Hermitian covariant derivative. Furthermore, let $\Psi \in \Gamma_{\mathrm{C}^\infty}(M, E)$ and let $//^x$ be the stochastic parallel transport in E along $B(x)$. Then*

$$\mathrm{d}\Big(//^{x,-1}\Psi(B(x))\Big) = //^{x,-1}\sum_{j=1}^{l}(\nabla_{A_j}\Psi)(B(x))\underline{\mathrm{d}}W^j. \tag{58}$$

This approach to $B(x)$ will be discussed further in section 5.1.

We close this section with an intrinsic construction of Brownian motions on M using horizontal lifts to the frame bundle, which is due to Malliavin [56]. Since we won't follow this approach, we will only state the result without proof. Let $\pi : \mathrm{O}(M) \to M$ be the orthonormal frame bundle of $M = (M, g)$. We equip $\mathrm{O}(M)$ with the Levi-Civita connection

$$\vartheta : \pi^*\mathrm{T}M \xrightarrow{\simeq} \mathrm{T}_H\mathrm{O}(M) \hookrightarrow \mathrm{T}_H\mathrm{O}(M) \oplus \mathrm{T}_V\mathrm{O}(M) = \mathrm{T}\,\mathrm{O}(M).$$

The standard horizontal vector fields $L_1, \ldots, L_m$ on $\mathrm{O}(M)$ are defined by $L_j(u) := \vartheta(u)(ue_j)$ for $j = 1, \ldots, m$, and we also define a differential operator in Hörmander form by

$$\tilde{\Delta} := \sum_{j=1}^{m} L_j^2 : \mathrm{C}^\infty(\mathrm{O}(M)) \longrightarrow \mathrm{C}^\infty(\mathrm{O}(M)).$$

The key observation for the following theorem (which is discussed in detail in [41] and [27]) is that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}^\infty(M) & \xrightarrow{\pi^*} & \mathrm{C}^\infty(\mathrm{O}(M)) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\tilde{\Delta}} \\ \mathrm{C}^\infty(M) & \xrightarrow{\pi^*} & \mathrm{C}^\infty(\mathrm{O}(M)) \end{array}$$

In the spirit of this diagram, $\tilde{\Delta}$ is called the *horizontal lift of* Δ *to* $\mathrm{O}(M)$.

Theorem 2.33 *Let W be a Brownian motion in the Euclidean $\mathbb{R}^m$ and let $u \in \mathrm{O}(M)$ with $\pi(u) = x$. If the continuous semi-martingale U on $\mathrm{O}(M)$ is defined as the maximal solution of*

$$\mathrm{d}U = \sum_{j=1}^{m} L_j(U)\underline{\mathrm{d}}W^j, \;\; U_0 = u, \tag{59}$$

then $B(x) := \pi(U)$ with $\zeta_{B(x)} := \zeta_U$ is a Brownian motion on M starting from x, and U is the horizontal lift of $B(x)$ to $\mathrm{O}(M)$ from u.

As we have already remarked, this construction has an intrinsic character: The driving semi-martingale of (59), which in this case is equal to the *anti-development of* $B(x)$ (with respect to u), lives in $\mathbb{R}^m$, where $m = \dim M$.

3 Essential self-adjointness of Schrödinger type operators with locally square integrable potentials

The aim of this section is to find natural conditions that garantee the essential self-adjointness of operators of the form $\nabla^*\nabla/2 + V$ in a Hilbert space of square integrable sections, if this operator is initially defined on the smooth sections with compact support. Let us introduce:

Hypothesis 3.1 *$E \to M$ is a smooth d-dimensional complex vector bundle with a smooth Hermitian structure $(\bullet, \bullet)_x$ and a Hermitian covariant derivative ∇. The symbol $\|\bullet\|_x$ stands for the corresponding norm and operator norm of E_x. The seperable complex Hilbert space of square integrable sections in E, defined as the completion of $\Gamma_{\mathrm{C}_0^\infty}(M, E)$ with respect to the scalar product*

$$\langle f, g \rangle = \int_M (f(x), g(x))_x \mathrm{vol}(\mathrm{d}x),$$

with $\|f\|^2 := \langle f, f \rangle$, will be denoted with $\Gamma_{\mathrm{L}^2}(M, E)$.

Since ∇ and $(\bullet, \bullet)_x$ will always be fixed, we will omit the dependence on ∇ or $(\bullet, \bullet)_x$ of data corresponding to (E, ∇) or $(E, (\bullet, \bullet)_x)$ or $(E, \nabla, (\bullet, \bullet)_x)$ in our notation, as we have already done for example for $\mathrm{P}(E)$, for the parallel transport or for $\Gamma_{\mathrm{L}^2}(M, E)$.

For section 3 we fix a bundle E as in hypothesis 3.1 and we will use a similar notation for all vector bundles of this type that are constructed from E, like $E \otimes \mathrm{T}M$. There is a canonical second order elliptic differential operator associated to ∇ (and g):

Definition 3.2 *The Bochner Laplacian*

$$\nabla^*\nabla : \Gamma_{\mathrm{C}^\infty}(M, E) \longrightarrow \Gamma_{\mathrm{C}^\infty}(M, E)$$

is the second order elliptic differential operator given by the composition

$$\begin{aligned}\nabla^*\nabla : \Gamma_{\mathrm{C}^\infty}(M, E) &\xrightarrow{\nabla} \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M \otimes E) \\ &\xrightarrow{\nabla^{\mathrm{T}M} \otimes 1 + 1 \otimes \nabla} \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M \otimes \mathrm{T}M \otimes E) \xrightarrow{-\mathrm{tr}_g} \Gamma_{\mathrm{C}^\infty}(M, E).\end{aligned}$$

If $v_1, \ldots, v_m$ is a local orthonormal frame for $\mathrm{T}M$ over $N \subset M$ and if $\Psi \in \Gamma_{\mathrm{C}^\infty}(M, E)$, then

$$\nabla^*\nabla\Psi = -\sum_{j=1}^{m} \nabla^2_{v_j}\Psi + \nabla_{\nabla^{\mathrm{T}M}_{v_j} v_j}\Psi \quad \text{in } N. \tag{60}$$

We consider $\nabla^*\nabla/2$ as a linear operator in $\Gamma_{\mathrm{L}^2}(M,E)$. As the notation indicates, one has:

Lemma 3.3 *The operator* $\nabla^*\nabla/2$ *with domain of definition* $\mathrm{D}(\nabla^*\nabla/2) = \Gamma_{\mathrm{C}_0^\infty}(M,E)$ *is a symmetric nonnegative operator.*

Proof. The assertion is certainly well-known, although we have not been able to find a detailed proof in the literature. We know the following proof from L. Habermann's lecture notes: Let $\Psi_1, \Psi_2 \in \Gamma_{\mathrm{C}_0^\infty}(M,E)$. Then we can define a function $f_{\Psi_1,\Psi_2} \in \mathrm{C}_0^\infty(M)$ by setting

$$f_{\Psi_1,\Psi_2}(x) := \sum_{j=1}^{m} \big(\nabla_{v_j}\Psi_1(x), \nabla_{v_j}\Psi_2(x)\big)_x\,,$$

where $v_1,\dots,v_m$ is some local orthonormal frame for $\mathrm{T}M$ in a neighbourhood of x (this definition does not depend on the particular choice of the local orthonormal frame). We will prove in a moment that

$$(\nabla^*\nabla\Psi_1,\Psi_2) = -\mathrm{div}\Big(X_{\Psi_1,\Psi_2}\Big) + f_{\Psi_1,\Psi_2}, \tag{61}$$

where

$$X_{\Psi_1,\Psi_2} \in \Gamma_{\mathrm{C}_0^\infty}(M,\mathrm{T}M)$$

is defined by

$$g(X_{\Psi_1,\Psi_2},Y) = (\nabla_Y\Psi_1,\Psi_2) \quad \text{for all } Y \in \Gamma_{\mathrm{C}^\infty}(M,\mathrm{T}M).$$

As a consequence, the identity

$$\langle\nabla^*\nabla\Psi_1,\Psi_2\rangle = \int_M f_{\Psi_1,\Psi_2}(x)\mathrm{vol}(\mathrm{d}x) = \overline{\langle\nabla^*\nabla\Psi_2,\Psi_1\rangle}$$

follows from the divergence theorem and the definition of f_{Ψ_1,Ψ_2}, and noting that $f_{\Psi_1,\Psi_1} \geq 0$, this proves the lemma.
It remains to prove (61): Let $x \in M$ be arbitrary and let $v_1,\dots,v_n$ be a local orthonormal frame for $\mathrm{T}M$ in a neighbourhood of x with

$$\nabla^{\mathrm{T}M}_{v_j} v_k \mid_x = 0 \quad \text{for } j,k = 1,\dots,m. \tag{62}$$

Using (62) and that $\nabla^{\mathrm{T}M}$ is compatible with g we see that

$$\mathrm{div}\Big(X_{\Psi_1,\Psi_2}\Big) \mid_x = \sum_{j=1}^{m} v_j\Big((\nabla_{v_j}\Psi_1,\Psi_2)\Big) \mid_x\,. \tag{63}$$

On the other hand, (60), (62) and the assumption that ∇ is compatible with $(\bullet,\bullet)_x$ imply that

$$\begin{aligned}(\nabla^*\nabla\Psi_1,\Psi_2)\mid_x &= -\sum_{j=1}^{m}(\nabla_{v_j}\nabla_{v_j}\Psi_1,\Psi_2)\mid_x\\ &= -\sum_{j=1}^{m} v_j\Big((\nabla_{v_j}\nabla_{v_j}\Psi_1,\Psi_2)\Big)\mid_x+\sum_{j=1}^{m}(\nabla_{v_j}\Psi_1,\nabla_{v_j}\Psi_2)\mid_x,\end{aligned}$$

which together with (63) proves (61).

■

The following definition will be convenient for us:

Definition 3.4 *Any*[7] $V\in\Gamma_{\mathrm{L}^0}(M,\mathrm{End}(E))$ *will be called a potential, if the morphism of complex Hilbert spaces* $V(x):E_x\to E_x$ *is Hermitian for a.e.* $x\in M$.

Whenever V is a potential such that the operator $\nabla^*\nabla/2+V$ with

$$\mathrm{D}(\nabla^*\nabla/2+V)=\Gamma_{\mathrm{C}_0^\infty}(M,E)$$

makes sense and is essentially self-adjoint, we will denote the corresponding closure with $H(V)$. Firstly, in this context it is certainly natural to assume that V is locally square integrable, because then $V\Psi\in\Gamma_{\mathrm{L}^2}(M,E)$ for any $\Psi\in\Gamma_{\mathrm{C}_0^\infty}(M,E)$. Secondly, since we want derive a formula for $\mathrm{e}^{-tH(V)}$, $t\geq 0$, it is natural to assume that V is bounded from below. The aim of this section is to prove the following theorem, which asserts that these two assumptions on the potential imply essential self-adjointness, if M is geodesically complete. For any, say locally square integrable, potential V we define the corresponding maximal domain of definition

$$\mathrm{D}(H(V)_{\max}):=\left\{f\,\Big|\,f,\ (\nabla^*\nabla/2+V)f\in\Gamma_{\mathrm{L}^2}(M,E)\right\},$$

where $(\nabla^*\nabla/2+V)f$ has to be understood in the sense of distributions.

Theorem 3.5 *Let* M *be geodesically complete and let* V *be a potential with*

$$C_V\mathbf{1}\leq V\in\Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M,\mathrm{End}(E))\quad\textit{for some } C_V\in\mathbb{R}.$$

The operator $\nabla^*\nabla/2+V$ *with*

$$\mathrm{D}(\nabla^*\nabla/2+V)=\Gamma_{\mathrm{C}_0^\infty}(M,E)$$

is essentially self-adjoint, the closure $H(V)$ *is semi-bounded from below and one has* $\mathrm{D}(H(V))=\mathrm{D}(H(V)_{\max})$.

[7] L^0 stands for “measurable”.

Here, $V \geq C_V \mathbf{1}$ is understood pointwise, that is, $V(x) \geq C_V \mathbf{1}_x$ for a.e. $x \in M$ in the sense of Hermitian morphisms in E_x. The factor $1/2$ in $\nabla^*\nabla/2$ is motivated by the definition of Stratonovic integrals and its necessity will become clear in the proof of theorem 5.3.
A proof of theorem 3.5 has been given in [11]. Since we believe that the methods from [11] are not standard, we are going to explain their proof below. In addition, we will present a new rather elementary proof for the case that M has a bounded geometry.

Theorem 3.5 in particular implies the following well-known result:

Theorem 3.6 *If M is geodesically complete, then the operator $\nabla^*\nabla/2$ with*

$$\mathrm{D}(\nabla^*\nabla/2) = \Gamma_{\mathrm{C}_0^\infty}(M, E)$$

is essentially self-adjoint and the self-adjoint extension $H(0)$ is nonnegative and given by $\mathrm{D}(H(0)) = \mathrm{D}(H(0)_{\max})$.

Before we give a proof of theorem 3.5 for the general case, let us first explain that in case M has a bounded geometry, theorem 3.5 can be proved in complete analogy to the case where M is the Euclidean $\mathbb{R}^m$, $E = \mathbb{R}^m \times \mathbb{C}$ and ∇ is the usual exterior derivative. The latter case has been carried out by T. Kato in [49] (see also theorem X.28 in [64] for a textbook version of this proof). To this end, we first remark:

Proposition 3.7 *Let M have a bounded geometry, let b be a positive real number and let $f \in \mathrm{L}^2(M)$ be real-valued with*

$$(b - \Delta) f \geq 0 \quad \textit{in the sense of distributions.} \tag{64}$$

Then $f \geq 0$ a.e. in M.

Proof. Proposition B.3 in [11].

■

Remark 3.8 There is an elementary proof of proposition 3.7 for the Laplace operator in the Euclidean $\mathbb{R}^m$: By using the Fourier transform one finds that $b - \Delta$ maps [8]

$$b - \Delta : \mathscr{S}(\mathbb{R}^m) \xrightarrow{\simeq} \mathscr{S}(\mathbb{R}^m) \quad \text{as complex linear spaces.} \tag{65}$$

[8] $\mathscr{S}(\mathbb{R}^m)$ stands for the Schwartz functions and $\mathscr{S}'(\mathbb{R}^m)$ for the tempered distributions.

Furthermore, since $(b-\Delta)^{-1}$ is an integral operator with a positive integral kernel [64], one has

$$(b-\Delta)^{-1}: \mathscr{S}_{+}(\mathbb{R}^m) \longrightarrow \mathscr{S}_{+}(\mathbb{R}^m), \tag{66}$$

where $\mathscr{S}_{+}(\mathbb{R}^m)$ stands for the positive elements of $\mathscr{S}(\mathbb{R}^m)$. Now (64), (65) and (66) easily imply

$$\int_{\mathbb{R}^m} f(x)\Psi(x)\mathrm{d}x \geq 0 \quad \text{for all } \Psi \in \mathscr{S}_{+}(\mathbb{R}^m),$$

so $f \geq 0$ a.e. in $\mathbb{R}^m$.

□

We use the notation $|f|(x) := \|f(x)\|_x$ for any section f in E. One has the following abstract Kato inequality:

Theorem 3.9 *Let $f \in \Gamma_{\mathrm{L}^1_{\mathrm{loc}}}(M,E)$ be such that $\nabla^*\nabla f \in \Gamma_{\mathrm{L}^1_{\mathrm{loc}}}(M,E)$ in the sense of distributions. Then*

$$-\Delta|f| \leq \mathrm{Re}(\nabla^*\nabla f, \mathrm{sign} f) \quad \textit{in the sense of distributions},$$

where

$$\mathrm{sign} f(x) := \begin{cases} \frac{f(x)}{\|f(x)\|_x}, & f(x) \neq 0 \\ \\ 0, & f(x) = 0. \end{cases}$$

Proof. Theorem 5.7 in [11].

■

Proof of theorem 3.5 for the case when M has a bounded geometry. We use ideas from the proof of lemma 3.9 in [58] and remark that by considering $H(V - C_V)$ instead of $H(V)$, one can assume $V \geq 0$. By theorem X.26 in [64] it is sufficient to prove that

$$\mathrm{Ker}((\nabla^*\nabla + V + 1)^*) = \{0\}.$$

Let

$$f \in \mathrm{D}((\nabla^*\nabla + V + 1)^*) \subset \Gamma_{\mathrm{L}^2}(M,E)$$

with

$$(\nabla^*\nabla + V + 1)^* f = 0 \quad \text{a.e. in } M.$$

Then one has

$$(\nabla^*\nabla + V + 1) f = 0 \quad \text{in the sense of distributions},$$

so that $\nabla^*\nabla f \in \Gamma_{\mathrm{L}^1_{\mathrm{loc}}}(M,E)$ by the Cauchy-Schwartz inequality. As a consequence, theorem 3.9 implies

$$\begin{aligned} -\Delta |f| &\leq \operatorname{Re}(\nabla^*\nabla f, \operatorname{sign} f) = (-(V+1)f, \operatorname{sign} f) \\ &\leq -|f|, \end{aligned} \tag{67}$$

where we have used that $V \geq 0$. Thus

$$(1-\Delta)(-|f|) \geq 0,$$

and finally $f = 0$ a.e. in M follows from proposition 3.7.

■

Since it is not known whether proposition 3.7 holds on arbitrary geodesically complete Riemannian manifolds, one has to simulate this result in some sense. This can be done with the following lemma together with the fact, that one has "good" cut-off functions on geodesically complete manifolds:

Lemma 3.10 *Under the assumptions of theorem 3.5, let $V \geq 0$, let $f \in \mathrm{D}(H(V)_{\max})$ and let $\varphi : M \to \mathbb{R}$ be a Lipschitz (thus a.e. differentiable) function with compact support. Then one has (see also theorem 3.13)*

$$\begin{aligned} &\|\nabla(\varphi f)\|^2 + \int_M (V(x)\varphi(x)f(x), \varphi(x)f(x))_x \operatorname{vol}(\mathrm{d}x) \\ &= \operatorname{Re}\langle \varphi(\nabla^*\nabla + V)f, \varphi f\rangle + \|\mathrm{d}\varphi \otimes f\|^2 . \end{aligned} \tag{68}$$

Proof. This is an adaption of lemma 8.10 in [11] to our situation.

■

Lemma 3.11 *Let M be geodesically complete. There is a sequence (φ_n) of Lipschitz functions on M with compact support such that*

i) $0 \leq \varphi_n(x) \leq 1$, $\varphi_n(x) \to 1$ *as* $n \to \infty$ *for any* $x \in M$, *and*

ii) $\|\mathrm{d}\varphi_n\|_\infty \to 0$ *as* $n \to \infty$.

Proof. This is lemma 8.9 in [75]. Since the proof is simple, we include it for the convenience of the reader: Fix $x_0 \in M$ and take a function $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood of 0 and

$$0 \leq \chi(a) \leq 1, \ |\chi'(a)| \leq 1 \quad \text{for all } a \in \mathbb{R}.$$

If $\mathrm{d}(\bullet,\bullet)$ denotes the metric which is induced by g, then $\mathrm{d}(\bullet, x_0)$ is a Lipschitz function with Lipschitz constant ≤ 1, and

$$\varphi_n(x) := \chi\left(\frac{\mathrm{d}(x,x_0)}{n}\right)$$

has the desired properties, where the compactness of $\mathrm{supp}(\varphi_n)$ follows from the completeness of M.

■

Proof of theorem 3.5 for the general case. We can assume $V \geq 0$, and it is sufficient to prove that the assumptions

$$f \in \Gamma_{\mathrm{L}^2}(M, E), \;\; (\nabla^*\nabla + V + 1)f = 0 \;\; \text{in the sense of distributions}$$

imply $f = 0$ a.e. in M. Clearly, one has $(\nabla^*\nabla + V)f = -f \in \Gamma_{\mathrm{L}^2}(M, E)$, so $f \in \mathrm{D}(H(V)_{\max})$. Thus, we can use $V \geq 0$ together with lemma 3.10 applied to $\varphi := \varphi_n$, where φ_n is as in lemma 3.11, to conclude

$$-\left\|\varphi_n f\right\|^2 + \left\|\mathrm{d}\varphi_n \otimes f\right\|^2 \geq 0,$$

so that

$$\begin{aligned} \left\|f\right\|^2 = \lim_{n\to\infty} \left\|\varphi_n f\right\|^2 &\leq \lim_{n\to\infty} \left\|\mathrm{d}\varphi_n \otimes f\right\|^2 \\ &\leq \left\|f\right\|^2 \lim_{n\to\infty} \left\|\mathrm{d}\varphi_n\right\|_\infty^2 = 0. \end{aligned} \tag{69}$$

This proves the theorem.

■

For the sake of completeness, we state an "accessible" generalization of theorem 3.5 to potentials that are not necessarily bounded from below:

Theorem 3.12 *Let M be geodesically complete and let V be a potential which satisfies the following two assumptions:*

i) $V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E))$

ii) $V = V_1 + V_2$, *where V_1 and V_2 are potentials with $V_1 \geq 0$ and $V_2 \leq 0$ and $V_2 \in \Gamma_{\mathrm{L}^p_{\mathrm{loc}}}(M, \mathrm{End}(E))$ with $p \geq m/2$ if $m \geq 5$, $p > 2$ if $m = 4$, and $p = 2$ if $m \leq 3$.*

If the operator $\nabla^*\nabla/2 + V$ with

$$\mathrm{D}(\nabla^*\nabla/2 + V) = \Gamma_{\mathrm{C}_0^\infty}(M, E)$$

is semi-bounded from below, then this operator is essentially self-adjoint, its closure $H(V)$ is semi-bounded from below with $\mathrm{D}(H(V)) = \mathrm{D}(H(V)_{\max})$.

Proof. This follows from combining theorem 2.13, remark 2.2 and theorem 2.3 in [11].

$\blacksquare$

We refer the reader to [11] for more general results on the essential self-adjointness for operators of the form $D^*D + V$. Here, E and $\tilde{E}$ are vector bundles as in hypothesis 3.1,

$$D : \Gamma_{\mathrm{C}^\infty}(M, E) \longrightarrow \Gamma_{\mathrm{C}^\infty}(M, \tilde{E})$$

is a first order differential operator, and V is a locally square integrable potential which is not necessarily bounded from below. The results from [11] (partly) extend an earlier result by M. Lesch [55] for operators of this type with V locally bounded.

We would also like to mention [58] and [59], where O. Milatovic determines conditions on M, E, ∇ under which one can still explicitely (= by giving the domain of definition) define a self-adjoint operator corresponding to $\nabla^*\nabla + V$ with V only in $\Gamma_{\mathrm{L}^1_{\mathrm{loc}}}(M, \mathrm{End}(E))$, which extends classical results for usual scalar Schrödinger operators in the Euclidean $\mathbb{R}^m$ by H.L. Cycon [18] and T. Kato [48].

We close this section with some remarks concerning the regularity of sections in $\mathrm{D}(H(V)_{\max})$. For any $k \in \mathbb{N}$ the local Sobolev space of order k with respect to M and E will be denoted with $\Gamma_{\mathrm{H}^k_{\mathrm{loc}}}(M, E)$. For example, these complex linear spaces can be defined with a standard localization procedure as follows: Some $f \in \Gamma_{\mathrm{L}^0}(M, E)$ is an element of $\Gamma_{\mathrm{H}^k_{\mathrm{loc}}}(M, E)$, if and only if for any chart

$$h : \mathbb{R}^m \supset N \longrightarrow h(N) \subset M$$

in which E is trivial, one has

$$h^*(f\mid_{h(N)}) \in \mathrm{H}^k_{\mathrm{loc}}(N, \mathbb{C}^d), \tag{70}$$

in the sense that the local frame for E in N is used to identify $f\mid_N$ with a map $N \to \mathbb{C}^d$. Of course, this is the same as saying that $f \in \Gamma_{\mathrm{L}^0}(M, E)$ is in $\Gamma_{\mathrm{H}^k_{\mathrm{loc}}}(M, E)$, if and only if $f \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, E)$ and for any h as above one has

$$\partial^\alpha h^*(f\mid_{h(N)}) \in \mathrm{L}^2_{\mathrm{loc}}(N, \mathbb{C}^d) \quad \text{for any multiindex } \alpha \text{ with } |\alpha| \leq k \tag{71}$$

in the sense of distributions. We refer the reader to [74] and [25] for details about local Sobolev spaces, in particular for the spaces $\Gamma_{\mathrm{H}^s_{\mathrm{loc}}}(M, E)$ with arbitrary $s \in \mathbb{R}$ (which can be defined either with a localization as above or, somewhat more intrinsically, using pseudodifferential operator techniques) and for the construction of a canonic locally convex topology on $\Gamma_{\mathrm{H}^s_{\mathrm{loc}}}(M, E)$. Note that these spaces don't depend on the particular choice of the Riemannian structure on M and the Hermitian structure on E.

One of the main results in [11] implies the following:

Theorem 3.13 *Let V be a potential in $\Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E))$.*
a) *If V satisfies assumption ii) of theorem 3.12 then*

$$\mathrm{D}(H(V)_{\max}) \subset \Gamma_{\mathrm{H}^1_{\mathrm{loc}}}(M, E). \tag{72}$$

In particular, if V is bounded from below, then one has (72).
b) *If $m \leq 3$ or if $V \in \Gamma_{\mathrm{L}^p_{\mathrm{loc}}}(M, E)$ with $p > m/2$ for $m \geq 4$, then*

$$\mathrm{D}(H(V)_{\max}) \subset \Gamma_{\mathrm{H}^2_{\mathrm{loc}}}(M, E).$$

Proof. a) This follows from combining theorem 2.3 (ii) with remark 2.2 in [11].
b) This follows from theorem 2.3 (i) in [11].

■

4 Some general assumptions and notations

We will work under the following assumptions and with the following notations for the rest of this thesis:

1. Throughout, E will denote a vector bundle over M satisfying hypothesis 3.1.

2. We assume that the underlying filtered probability space is equal to some Euclidean Wiener space equipped with the Wiener measure. To be more specific, with the notation of example C.8, we set

$$(\Omega, \mathscr{F}, \mathscr{F}_*, \mathbb{P}) := (\mathrm{C}([0,\infty), \mathbb{R}^l), \mathscr{F}^l, \mathscr{F}^l_*, \mathbb{P}^0),$$

and we take W to be the coordinate process on $\mathrm{C}([0,\infty), \mathbb{R}^l)$, so that W is an Euclidean Brownian motion in $\mathbb{R}^l$. Here, l is a fixed number, such that there is an isometric embedding of M into the Euclidean $\mathbb{R}^l$. The reason for this assumption is that it will allow us to use the Kolmogorov consistency theorem in the form of Satz 1.25' in [37] in order to construct the Brownian bridge measure in section 6.

3. Whenever M is stochastically complete, we will construct the Brownian motion $B(x)$ with $\zeta_{B(x)} = \infty$ for any $x \in M$ in the spirit of Nash's embedding theorem and assumption 2, that is, $B(x)$ is defined by theorem 2.31 as the maximal solution of

$$\mathrm{d}B(x) = \sum_{j=1}^{l} A_j(B(x)) \underline{\mathrm{d}} W^j, \ \ B_0(x) = x, \tag{73}$$

where $M \hookrightarrow \mathbb{R}^l$ isometrically [9], where $A : M \times \mathbb{R}^l \to \mathrm{T}M$ is given as the orthogonal projection $A(x) : \mathbb{R}^l \to \mathrm{T}_x M$ and where W is a Brownian motion in $\mathbb{R}^l$ which starts in 0. Throughout, $//^x$ will denote the stochastic parallel transport in E along $B(x)$. Then using

$$\sum_{j=1}^{l} \nabla^{\mathrm{T}M}_{A_j} A_j = 0$$

as in the proof of theorem 2.31 implies

$$\nabla^* \nabla \Psi = -\sum_{j=1}^{l} \nabla_{A_j} \nabla_{A_j} \Psi \quad \text{for all } \Psi \in \Gamma_{\mathrm{C}^\infty}(M, E), \tag{74}$$

so that by corollary 2.20 (see also proposition 2.32) the following formula holds:

$$\begin{aligned} &\mathrm{d}\Big(//^{x,-1} \Psi(B(x)) \Big) \\ &= //^{x,-1} \left(\sum_{j=1}^{l} (\nabla_{A_j} \Psi)(B(x)) \mathrm{d}W^j - H_0 \Psi(B(x)) \mathrm{d}t \right). \end{aligned} \tag{75}$$

5 Probabilistic representations of Schrödinger semigroups

Throughout section 5, we will assume that M is geodesically and stochastically complete.

5.1 The Feynman-Kac formula for bounded potentials

The aim of this section is to prove the Feynman-Kac formula for essentially bounded potentials. This formula will then imply the general Feynman-Kac formula for locally square potentials with an approximation argument and dominated convergence.

We fix a potential[10]

$$V \in \Gamma_{\mathrm{L}^\infty}(M, \mathrm{End}(E))$$

for the rest of section 5.1. The following lemma will be needed to deduce the Feynman-Kac formula for potentials in L^∞ from the one for continuous bounded potentials:

[9] If M is the Euclidean $\mathbb{R}^m$, we will of course take $l = m$ and $B(x) := W + x$.

[10] L^∞ stands for "measurable, essentially bounded".

Lemma 5.1 *There is a sequence* (V_n) *of potentials in* [11] $\Gamma_{\mathrm{C_b}}(M, \mathrm{End}(E))$ *and a* $C > 0$ *such that for all* $n \in \mathbb{N}$ *and a.e.* $x \in M$ *one has*

$$\|V_n(x)\|_x \leq C, \quad \|V(x)\|_x \leq C, \quad \lim_{n\to\infty} \|V_n(x) - V(x)\|_x = 0 \tag{76}$$

and

$$\lim_{n\to\infty} \|V_n f - V f\| = 0 \quad \textit{for any } f \in \Gamma_{\mathrm{C}_0^\infty}(M, E). \tag{77}$$

Proof. Assume first that V is supported in a relatively compact coordinate neighbourhood N (which will be identified with a bounded open subset of $\mathbb{R}^m$) in which E is trivial. In this case, the assertion follows from a Friedrichs mollifier argument: Let V be given as

$$V^i_j \in \mathrm{L}^\infty(N) \quad \text{for } i, j = 1, \ldots d.$$

We fix some $p \geq 1$ and remark that since $|N| < \infty$, we have $V^i_j \in \mathrm{L}^p(N)$. Let $(\varphi_n)_n$ be a nonnegative sequence in $\mathrm{C}_0^\infty(\mathbb{R}^m)$ with [12]

$$\int_{\mathbb{R}^m} \varphi_n(x)\mathrm{d}x = 1, \ \ \mathrm{supp}(\varphi_n) \subset \mathrm{K}_{1/n}(0) \quad \text{for any } n.$$

We set

$$N_n := \left\{ x \,\middle|\, \mathrm{K}_{\frac{1}{n}}(x) \subset N \right\}, \ \ D_n := N_n \cap \mathrm{K}_n(0).$$

Then by standard arguments (see for example [3], p.111) one finds that the functions

$$x \longmapsto \tilde{V}^i_{j,n}(x) := \int_{D_n} \varphi_n(x-y) V^i_j(y)\mathrm{d}y$$

are in $\mathrm{C}_0^\infty(N)$ for any n and that $\tilde{V}^i_{j,n} \to V^i_j$ as $n \to \infty$ in the norm of $\mathrm{L}^p(N)$. As a consequence, for some subsequence $(V^i_{j,n})_n$ of $(\tilde{V}^i_{j,n})_n$ and some $K_d > 0$ we have

$$\left|V^i_{j,n}(x)\right| \leq K_d \|V\|_\infty, \ \ V^i_{j,n}(x) \to V^i_j(x) \quad \text{as } n \to \infty, \text{ for a.e. } x \in N.$$

Thus if $V_n \in \mathrm{C}_0^\infty(N, \mathrm{Mat}(\mathbb{C}^d))$ is the self-adjoint matrix given by $V^i_{j,n}$, then one has a $L_d > 0$ such that

$$\|V_n(x)\|_{\mathrm{Mat}(\mathbb{C}^d)} \leq L_d \|V\|_\infty, \ \ V_n(x) \to V(x) \quad \text{as } n \to \infty, \text{ for a.e. } x \in N.$$

[11] $\mathrm{C_b}$ stands for "continuous, bounded".

[12] In a metric space, we will write $\mathrm{K}_r(x)$ for the open ball with radius r around x.

The existence of a sequence satisfying (76) now follows from a standard partition of unity argument.
This sequence also satisfies (77): For a.e. $x \in M$ one has

$$\lim_{n\to\infty} \|(V_n(x) - V(x))f(x)\|_x^2 = 0$$

and

$$\|(V_n(x) - V(x))f(x)\|_x^2 \leq 4C^2 \|f(x)\|_x^2 ,$$

so that (77) follows from dominated convergence.

$\blacksquare$

We fix an arbitrary $x \in M$. Since

$$t \longmapsto V_t^{(x)}(\omega) \ \text{ is in } \ \mathrm{L}^1_{\mathrm{loc}}\Big([0,\infty), \mathrm{End}(E)_x\Big) \ \text{ for any fixed } \omega \in \Omega,$$

with the process $V^{(x)}$ given by

$$V^{(x)} : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x, \ V_t^{(x)} := -//_t^{x,-1} V(B_t(x))//_t^x,$$

it follows in the usual way from the Banach fixed point theorem that one can define a process

$$\mathscr{V}^x : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

as the unique weak (= locally absolutely continuous) pathwise solution of the ordinary initial value problem

$$\mathrm{d}\mathscr{V}_t^x = \mathscr{V}_t^x V_t^{(x)} \mathrm{d}t, \ \mathscr{V}_0^x = \mathbf{1}.$$

$\mathscr{V}^x$ will sometimes also be called the *path ordered exponential* corresponding to V, a definition that is motivated from formula (78) below. Let us note the following elementary estimate, which will be used throughout in the following: Since

$$\mathscr{V}_t^x = \mathbf{1} + \sum_{n=1}^{\infty} \int_{t\Delta_n} V_{t_1}^{(x)} \cdots V_{t_n}^{(x)} \mathrm{d}t_1 \cdots \mathrm{d}t_n, \tag{78}$$

one finds that $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\|\mathscr{V}_t^x\|_x \\
&\leq 1 + \sum_{n=1}^{\infty} \int_{t\Delta_n} \left\|//_{t_1}^{x,-1} V(B_{t_1}(x))//_{t_1}^x\right\|_x \cdots \left\|//_{t_n}^{x,-1} V(B_{t_n}(x))//_{t_n}^x\right\|_x \mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\leq \mathrm{e}^{t\|V\|_\infty}. \qquad (79)
\end{aligned}$$

Next, let us prove that the convergence from lemma 5.1 implies the convergence of the corresponding path ordered exponentials:

Lemma 5.2 *Let (V_n) be a sequence as in lemma 5.1 and for any $n \in \mathbb{N}$ let the process*

$$\mathscr{V}_n^x : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

be given as the solution of the ordinary initial value problem

$$\mathrm{d}\mathscr{V}_{n,t}^x = -\mathscr{V}_{n,t}^x //_t^{x,-1} V_n(B_t(x)) //_t^x \mathrm{d}t, \;\; \mathscr{V}_{n,0}^x = \mathbf{1}.$$

Then for any fixed $t > 0$,

$$\lim_{n\to\infty} \left\| \mathscr{V}_{n,t}^x - \mathscr{V}_t^x \right\|_x = 0 \;\; \mathbb{P}\text{-}a.s.$$

Proof. The following equalities and inequalities are all valid $\mathbb{P}$-a.s. It follows that

$$\left\| //_s^{x,-1} V_n(B_s(x)) //_s^x - //_s^{x,-1} V(B_s(x)) //_s^x \right\|_x \to 0 \quad \text{as } n \to \infty$$

for a.e. $s \in [0,t]$. Also, there is a $\tilde{C} > 0$ such that for a.e. $s \in [0,t]$ and all n one has

$$\begin{aligned}
&\left\| //_s^{x,-1} V_n(B_s(x)) //_s^x - //_s^{x,-1} V(B_s(x)) //_s^x \right\|_x \leq \tilde{C}, \\
&\left\| //_s^{x,-1} V_n(B_s(x)) //_s^x \right\|_x \leq \tilde{C}, \\
&\left\| //_s^{x,-1} V(B_s(x)) //_s^x \right\|_x \leq \tilde{C}
\end{aligned} \tag{80}$$

(one can take $\tilde{C} := 2C$, where C is chosen as in lemma 5.1), thus proposition A.2 and dominated convergence imply

$$\begin{aligned}
&\left\| \mathscr{V}_{n,t}^x - \mathscr{V}_t^x \right\|_x \\
&\leq \mathrm{e}^{3\tilde{C}t} \int_0^t \left\| //_s^{x,-1} V_n(B_s(x)) //_s^x - //_s^{x,-1} V(B_s(x)) //_s^x \right\|_x \mathrm{d}s \to 0
\end{aligned}$$

as $n \to \infty$, and the lemma is proved. ■

By theorem 3.6, $\nabla^*\nabla/2 + V$ is essentially self-adjoint on the domain $\Gamma_{\mathrm{C}_0^\infty}(M,E)$, the closure $H(V)$ is given by $\mathrm{D}(H(V)) = \mathrm{D}(H(0))$ and one clearly has $H(V) \geq -\|V\|_\infty$.

Now we can prove:

Theorem 5.3 *Let V be a potential in $\Gamma_{\mathrm{L}^\infty}(M, \mathrm{End}(E))$. For any $x \in M$ let*

$$\mathscr{V}^x : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

be the process which is given as the weak pathwise solution of the ordinary inital value problem

$$\mathrm{d}\mathscr{V}_t^x = -\mathscr{V}_t^x //_t^{x,-1} V(B_t(x)) //_t^x \mathrm{d}t, \;\; \mathscr{V}_0^x = \mathbf{1}. \tag{81}$$

If $f \in \Gamma_{\mathrm{L}^2}(M, E)$, $t \geq 0$, then the following identity holds

$$\mathrm{e}^{-tH(V)} f(x) = \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right] \quad \textit{for a.e. } x \in M. \tag{82}$$

Proof. The theorem will be proved in two steps:

1. We first assume that $V \in \Gamma_{\mathrm{C_b}}(M, \mathrm{End}(E))$. Let

$$\begin{aligned} &Q_t : \Gamma_{\mathrm{L}^2}(M, E) \longrightarrow \Gamma_{\mathrm{L}^0}(M, E), \\ &Q_t h(x) := \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} h(B_t(x))\right] \end{aligned}$$

and remember that $p_t(x, y)$ denotes the minimal heat kernel of M. Since $p_t(x, \bullet)$ is nonnegative and in $\mathrm{L}^1(M)$, we can use the Hölder inequality to estimate as follows for all $t > 0$,

$$\begin{aligned} \|Q_t h\|^2 &\leq \mathrm{e}^{2t\|V\|_\infty} \int_M \mathbb{E}\left[\|h(B_t(x))\|_{B_t(x)}\right]^2 \mathrm{vol}(\mathrm{d}x) \\ &= \mathrm{e}^{2t\|V\|_\infty} \int_M \left(\int_M \|h(y)\|_y \, p_t(x, y) \mathrm{vol}(\mathrm{d}y)\right)^2 \mathrm{vol}(\mathrm{d}x) \\ &\leq \mathrm{e}^{2t\|V\|_\infty} \int_M \int_M p_t(x, y) \|h(y)\|_y^2 \, \mathrm{vol}(\mathrm{d}y) \int_M p_t(x, z) \mathrm{vol}(\mathrm{d}z) \mathrm{vol}(\mathrm{d}x) \\ &= \mathrm{e}^{2t\|V\|_\infty} \|h\|^2. \end{aligned}$$

Thus we have[13]

$$Q_t \in \mathscr{L}(\Gamma_{\mathrm{L}^2}(M, E)) \text{ for all } t \geq 0.$$

If $\Psi \in \Gamma_{\mathrm{C}_0^\infty}(M, E)$, then the Itô product rule and (75) imply

$$\begin{aligned} \mathrm{d}\Big(\mathscr{V}_t^x //_t^{x,-1} \Psi(B_t(x))\Big) &= -\mathscr{V}_t^x //_t^{x,-1} V(B_t(x)) //_t^x //_t^{x,-1} \Psi(B_t(x)) \mathrm{d}t \\ &\quad + \mathscr{V}_t^x //_t^{x,-1} \left(\sum_{j=1}^r (\nabla_{A_j} \Psi)(B_t(x)) \mathrm{d}W_t^j - H_0 \Psi(B_t(x)) \mathrm{d}t\right) \end{aligned} \tag{83}$$

for the $E_x \cong \mathbb{C}^d$ valued continuous semi-martingale $\mathscr{V}^x //^{x,-1} \Psi(B(x))$. Since

$$\mathrm{supp}(\nabla_{A_j} \Psi) \subset \mathrm{supp}(\Psi), \;\; C_{\Psi,j} := \max_{y \in \mathrm{supp}(\Psi)} \left\|\nabla_{A_j} \Psi(y)\right\|_y < \infty,$$

[13] $\mathscr{L}(X, Y)$ stands for the Banach space of linear bounded operators $X \to Y$ between Banach spaces X, Y.

one has

$$\int_0^t \mathbb{E}\left[\left\|\mathscr{V}_s^x //_s^{x,-1}(\nabla_{A_j}\Psi)(B_s(x))\right\|_x^2\right] \mathrm{d}s \le C_{\Psi,j}^2 \int_0^t \mathrm{e}^{2s\|V\|_\infty}\mathrm{d}s < \infty,$$

thus

$$Z := \int \sum_{j=1}^r \mathscr{V}^x //^{x,-1}(\nabla_{A_j}\Psi)(B(x))\mathrm{d}W^j$$

is a (continuous) martingale with values in E_x, and taking $\mathbb{E}[\bullet]$ on both sides of equation (83) shows that

$$Q_t\Psi(x) = \Psi(x) - \int_0^t Q_s H(V)\Psi(x)\mathrm{d}s \quad \text{in } E_x.$$

This implies the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q_t\Psi, h\rangle = -\langle Q_t H(V)\Psi, h\rangle \quad \text{for all } h \in \Gamma_{\mathrm{L}^2}(M,E). \tag{84}$$

Since $\Gamma_{\mathrm{L}^2}(M,E)$ is seperable, this shows that $t \mapsto Q_t\Psi$ is strongly differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_t\Psi = -Q_t H(V)\Psi,\ \ Q_0\Psi = \Psi,$$

so $Q_t\Psi = \mathrm{e}^{-tH}\Psi$ for all $\Psi \in \Gamma_{\mathrm{C}_0^\infty}(M,E)$, and finally $Q_t = \mathrm{e}^{-tH}$ follows from $Q_t \in \mathscr{L}(\Gamma_{\mathrm{L}^2}(M,E))$.

2. Now let $V \in \Gamma_{\mathrm{L}^\infty}(M, \mathrm{End}(E))$. We can assume $t > 0$. By lemma 5.1, we can find a sequence of potentials (V_n) in $\Gamma_{\mathrm{C_b}}(M, \mathrm{End}(E))$ such that

$$\|H(V_n)\Psi - H(V)\Psi\| \to 0 \quad \text{as } n \to \infty,$$

so since $\Gamma_{\mathrm{C}_0^\infty}(M,E)$ is a common core for $H(V)$ and $H(V_n)$, theorem VIII 25, theorem VIII 20 in [63] and the Feynman-Kac formula applied to $H(V_n)$ show that we can assume

$$\mathbb{E}\left[\mathscr{V}_{n,t}^x //_t^{x,-1} f(B_t(x))\right] \to \mathrm{e}^{-tH(V)}f(x) \quad \text{as } n \to \infty\ ,$$

with the notations from lemma 5.2. Finally, the theorem follows from dominated convergence, since by lemma 5.2,

$$\mathscr{V}_{n,t}^x //_t^{x,-1} f(B_t(x)) \to \mathscr{V}_t^x //_t^{x,-1} f(B_t(x)) \quad \text{as } n \to \infty,\ \mathbb{P}\text{-a.s.}$$

and

$$\left\|\mathscr{V}_{n,t}^x //_t^{x,-1} f(B_t(x))\right\|_x \le \mathrm{e}^{tC}\left\|f(B_t(x))\right\|_{B_t(x)},\ \ \mathbb{E}\left[\left\|f(B_t(x))\right\|_{B_t(x)}\right] < \infty,$$

where we have used lemma 5.1, (79) and

$$\mathbb{E}\left[\|f(B_t(x))\|_{B_t(x)}\right] = \mathrm{e}^{\frac{t}{2}\Delta}\|f(x)\|_x < \infty,$$

since $x \mapsto \|f(x)\|_x$ is in $\mathrm{L}^2(M)$.

■

Let us note the following corollary to the above proof:

Corollary 5.4 *Let $v : M \to \mathbb{R}$ be measurable and essentially bounded and let $H_0(v)$ be the self-adjoint realization of $-\Delta/2 + v$ in $\mathrm{L}^2(M)$ in the sense of theorem 3.5. Then one has the following formula for any $t \geq 0$, $f \in \mathrm{L}^2(M)$ and a.e. $x \in M$,*

$$\mathrm{e}^{-tH_0(v)}f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x))\right]. \tag{85}$$

Proof. If v is continuous and bounded and $\Psi \in \mathrm{C}_0^\infty(M)$, then (56) and (73) imply the following identity for any $t \geq 0$, and $x \in M$,

$$\begin{aligned} \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s}\Psi(B_t(x)) &= \Big(\text{ a martingale which starts in } 0 \Big) + \Psi(x) \\ &- \int_0^t \mathrm{e}^{-\int_0^s v(B_r(x))\mathrm{d}r} H_0(v)\Psi(B_s(x))\mathrm{d}s \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{86}$$

Now one can continue in the same way as has been done in the first part of the proof theorem 5.3 to conclude that (85) holds for continuous and bounded potentials. If the potential v is not necessarily continuous, then one can approximate v by continuous bounded potentials and deduce (85) with the same arguments that have been used in the second part of the proof theorem 5.3.

■

Of course, using standard identifications in trivial line bundles, it is straightforward to derive corollary 5.4 somewhat directly from theorem 5.3. We have given the direct argument (formula (86)) here, just because of its simplicity.

5.2 The Feynman-Kac formula for locally square integrable potentials

As we have already remarked, we want to use theorem 5.3 now in order to extend the Feynman-Kac formula for Schrödinger type operators that have been considered in theorem 3.5:

Theorem 5.5 *Let V be a potential with*

$$C_V\mathbf{1} \le V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}$$

and let $H(V)$ be given as in theorem 3.5. Then for a.e. $x \in M$ there is a unique process

$$\mathscr{V}^x : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

which satisfies

$$\mathrm{d}\mathscr{V}^x_t = -\mathscr{V}^x_t //_t^{x,-1} V(B_t(x)) //_t^x \mathrm{d}t, \;\; \mathscr{V}^x_0 = \mathbf{1} \tag{87}$$

pathwise in the weak sense, and for any $f \in \Gamma_{\mathrm{L}^2}(M,E)$, $t \ge 0$ the following identity holds

$$\mathrm{e}^{-tH(V)} f(x) = \mathbb{E}\left[\mathscr{V}^x_t //_t^{x,-1} f(B_t(x))\right] \quad \textit{for a.e. } x \in M. \tag{88}$$

For the rest of section 5.2, we work under the assumptions of theorem 5.5, so V will always stand for a square integrable potential which is bounded from below by some $C_V \in \mathbb{R}$.

We first consider the scalar case. Assume we are given a nonnegative real-valued function $v \in \mathrm{L}^2_{\mathrm{loc}}(M)$ and let $H_0(v)$ be the corresponding self-adjoint realization in $\mathrm{L}^2(M)$ of $-\Delta/2 + v$ in the sense of theorem 3.5. The following proposition, a Feynman-Kac formula for functions, will be useful for the proof of theorem 5.5. The consideration of operators of the form $H_0(v)$ is natural for the study of the spectrum of $H(V)$: In some sense, the energy of $H(V)$ can be bounded from below by any operator $H_0(v)$ with $V \ge v\mathbf{1}$ (see section 8).

Proposition 5.6 *The map $t \mapsto v(B_t(x))$ is in $\mathrm{L}^1_{\mathrm{loc}}[0,\infty)$ $\mathbb{P}$-a.s. for a.e. $x \in M$ and the following formula holds for any $f \in \mathrm{L}^2(M)$, $t \ge 0$ and a.e. $x \in M$,*

$$\mathrm{e}^{-tH_0(v)} f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x))\right]. \tag{89}$$

Proof. We can assume $t > 0$. The asserted integrability follows from

$$\begin{aligned} &\int_M \mathbb{E}\left[\int_0^t |\tilde{v}(B_s(x))|\,\mathrm{d}s\right] \mathrm{vol}(\mathrm{d}x) \\ &= \int_0^t \int_M \int_M p_s(x,y)\mathrm{vol}(\mathrm{d}x)\,|\tilde{v}(y)|\,\mathrm{vol}(\mathrm{d}y) \\ &\le t\,\|\tilde{v}\|_{\mathrm{L}^1(M)} \end{aligned} \tag{90}$$

for any $\tilde{v} \in \mathrm{L}^1(M)$, and a typical localization argument (see for example he proof of lemma 6.5 below).

For the proof of (89), let (v_n) be a sequence of measurable, essentially bounded, nonnegative functions on M such that $v_n(x) \nearrow v(x)$ as $n \to \infty$ for a.e. $x \in M$ (e.g., one can take $v_n := \min\{n, v\}$). If $\tilde{f} \in \mathrm{C}_0^\infty(M)$, then since $|v_n - v|^2 \leq 4|v|^2$ and since $|v|^2$ is in $\mathrm{L}^1_{\mathrm{loc}}(M)$, one has

$$\begin{aligned}
&\int_M |v_n(x)\tilde{f}(x) - v(x)\tilde{f}(x)|^2 \mathrm{vol}(\mathrm{d}x) \\
&\leq C_{\tilde{f}} \int_{\mathrm{supp}(\tilde{f})} |v_n(x) - v(x)|^2 \mathrm{vol}(\mathrm{d}x) \to 0 \quad \text{as } n \to \infty
\end{aligned} \tag{91}$$

by dominated convergence. The following (in-)equalities are all valid for a.e. $x \in M$. By using the Feynman-Kac formula for bounded potentials, one finds that the heat semigroup corresponding to $H(v_n)$ is given by

$$\mathrm{e}^{-tH(v_n)} f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v_n(B_s(x))\mathrm{d}s} f(B_t(x))\right].$$

So (91) and the same arguments as in the second part of the proof theorem 5.3 show

$$\mathbb{E}\left[\mathrm{e}^{-\int_0^t v_n(B_s(x))\mathrm{d}s} f(B_t(x))\right] \to \mathrm{e}^{-tH_0(v)} f(x) \text{ as } n \to \infty. \tag{92}$$

Using that $\mathbb{P}$-a.s.

$$\begin{aligned}
&0 \leq v_n(B_s(x)) \leq v_{n+1}(B_s(x)) \leq v(B_s(x)), \\
&v_n(B_s(x)) \to v(B_s(x)) \quad \text{as } n \to \infty \text{ for a.e. } s \in [0,t],
\end{aligned}$$

one has

$$\int_0^t v_n(B_s(x))\mathrm{d}s \to \int_0^t v(B_s(x))\mathrm{d}s, \tag{93}$$

and as a consequence

$$\mathrm{e}^{-\int_0^t v_n(B_s(x))\mathrm{d}s} \to \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} \quad \text{as } n \to \infty, \mathbb{P}\text{-a.s.}, \tag{94}$$

by monotone convergence. Writing $f = f_1 - f_2 + \mathrm{i}f_3 - \mathrm{i}f_4$ with nonnegative functions f_j, we can assume that f is nonnegative in the following. Since

$$\begin{aligned}
&\mathrm{e}^{-\int_0^t v_1(B_s(x))\mathrm{d}s} f(B_t(x)) \geq \mathrm{e}^{-\int_0^t v_n(B_s(x))\mathrm{d}s} f(B_t(x)) \\
&\geq \mathrm{e}^{-\int_0^t v_{n+1}(B_s(x))\mathrm{d}s} f(B_t(x)) \geq \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x)), \\
&\mathrm{e}^{-\int_0^t v_n(B_s(x))\mathrm{d}s} f(B_t(x)) \to \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} f(B_t(x)) \quad \text{as } n \to \infty, \mathbb{P}\text{-a.s.}
\end{aligned} \tag{95}$$

one can use a generalized monotone convergence theorem (theorem 9.17 in [90]), where the necessary finiteness is assured by (92), to conclude that the desired formula holds.

■

The proof of theorem 5.5 will be prepared further with three lemmata. In order to be able to use the Feynman-Kac formula for potentials in L^∞, we first remark:

Lemma 5.7 *Let V be a potential with*

$$0 \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)).$$

Then there is a sequence of nonnegative potentials (V_n) in $\Gamma_{\mathrm{L}^\infty}(M, \mathrm{End}(E))$ such that

i) *for all $n \in \mathbb{N}$, a.e. $x \in M$,*

$$\|V_n(x)\|_x \leq \|V_{n+1}(x)\|_x \leq \|V(x)\|_x\,,$$

ii) *for a.e. $x \in M$ and all* [14] $f \in \Gamma_{\mathrm{L}^\infty_0}(M, E)$,

$$\lim_{n\to\infty} \|V_n(x) - V(x)\|_x = 0, \quad \lim_{n\to\infty} \|V_n f - V f\| = 0.$$

Proof. Since we are only interested in the measurable structure of E, we can use a partition of unity argument to trivialize E globally in a measurable way, so let $V \in \mathrm{L}^2_{\mathrm{loc}}(M, \mathrm{Mat}(\mathbb{C}^d))$. We can also assume that V is diagonal,

$$V = \mathrm{diag}(v_1, \ldots, v_d) \text{ with } 0 \leq v_j \in \mathrm{L}^2_{\mathrm{loc}}(M) \text{ real-valued.}$$

Then setting $v_{n,j} := \min\{n, v_j\}$, it is clear that with

$$V_n := \mathrm{diag}(v_{n,1}, \ldots, v_{n,d}) \in \mathrm{L}^\infty(M, \mathrm{Mat}(\mathbb{C}^d)),$$

one has

$$\begin{aligned} \|V_n(x)\|_{\mathrm{Mat}(\mathbb{C}^d)} \leq \|V_{n+1}(x)\|_{\mathrm{Mat}(\mathbb{C}^d)} \leq \|V(x)\|_{\mathrm{Mat}(\mathbb{C}^d)}\,, \\ \|V_n(x) - V(x)\|_{\mathrm{Mat}(\mathbb{C}^d)} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{96}$$

The convergence

$$\|V_n f - V f\|_{\mathrm{L}^2(M,\mathbb{C}^d)} \to 0 \quad \text{as } n \to \infty \text{ for any } f \in \mathrm{L}^\infty_0(M, \mathbb{C}^d)$$

follows as in (91).

■

[14] L^∞_0 stands for “measurable, essentially bounded with compact support”.

Lemma 5.8 *Let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then the map

$$[0,\infty) \longrightarrow [0,\infty), \;\; t \longmapsto \left\| V\Big(B_t(x)\Big) \right\|_{B_t(x)} \tag{97}$$

is in $\mathrm{L}^1_{\mathrm{loc}}[0,\infty)$ $\mathbb{P}$-a.s. for a.e. $x \in M$. In particular, for a.e. $x \in M$ there is a unique process

$$\mathscr{V}^x : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

which solves the ordinary initial value problem

$$\mathrm{d}\mathscr{V}^x_t = -\mathscr{V}^x_t //^{x,-1}_t V(B_t(x)) //^x_t \mathrm{d}t, \;\; \mathscr{V}^x_0 = \mathbf{1} \tag{98}$$

pathwise in the weak sense. Furthermore, for all $x \in M$ such that $\mathscr{V}^x$ is defined, one has

$$\|\mathscr{V}^x_t\|_x \leq \mathrm{e}^{-tC_V} \quad \mathbb{P}\textit{-a.s. for any } t \geq 0. \tag{99}$$

Proof. Clearly, the second assertion follows from the first one and the Banach fixed point theorem since

$$\left\| //^{x,-1}_t V(B_t(x)) //^x_t \right\|_x \leq \|V(B_t(x))\|_{B_t(x)}.$$

The fact that (97) is locally integrable in the above sense is implied by proposition 5.6 by setting $v(x) := \|V(x)\|_x$, and (99) follows from applying proposition A.1 b) pathwise to

$$F(t) := -\Big(//^{x,-1}_t V(B_t(x)) //^x_t \Big)$$

and $c(t) := C_V$.

■

A statement analogous to lemma 5.2 holds for unbounded potentials:

Lemma 5.9 *Let V be a potential with*

$$0 \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)),$$

let (V_n) be a sequence as in lemma 5.7, and for any $n \in \mathbb{N}$ and any $x \in M$ let the process

$$\mathscr{V}^x_n : [0,\infty) \times \Omega \longrightarrow \mathrm{End}(E)_x$$

be given as the weak pathwise solution of the ordinary initial value problem

$$\mathrm{d}\mathscr{V}^x_{n,t} = -\mathscr{V}^x_{n,t}//^{x,-1}_t V_n(B_t(x))//^x_t \mathrm{d}t, \;\; \mathscr{V}^x_{n,0} = \mathbf{1}.$$

Then for any fixed $x \in M$ such that $s \mapsto \|V(B_s(x))\|_{B_s(x)}$ is locally integrable $\mathbb{P}$-a.s. and any $t > 0$ one has

$$\lim_{n\to\infty} \left\|\mathscr{V}^x_{n,t} - \mathscr{V}^x_t\right\|_x = 0 \quad \mathbb{P}\text{-}a.s.$$

Proof. The proof is similar to the proof of lemma 5.2. By proposition A.2, lemma 5.7 and lemma 5.8 one has the following inequality $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left\|\mathscr{V}^x_{n,t} - \mathscr{V}^x_t\right\|_x \\
&\leq \mathrm{e}^{3\int_0^t \|V(B_s(x))\|_{B_s(x)}\mathrm{d}s} \\
&\quad\times \int_0^t \left\|//^{x,-1}_s V_n(B_s(x))//^x_s - //^{x,-1}_s V(B_s(x))//^x_s\right\|_x \mathrm{d}s.
\end{aligned}$$

Using lemma 5.7 and lemma 5.8 again, the assertion follows from dominated convergence.

■

Proof of theorem 5.5. Since

$$\mathrm{e}^{-t(H(V)-C_V)} = \mathrm{e}^{tC_V}\mathrm{e}^{-tH(V)},$$

and since if $t \mapsto \mathscr{V}^x_t$ is the solution of

$$\mathrm{d}\mathscr{V}^x_t = -\mathscr{V}^x_t//^{x,-1}_t V(B_t(x))//^x_t \mathrm{d}t, \;\; \mathscr{V}^x_0 = \mathbf{1}, \tag{100}$$

then $t \mapsto \mathrm{e}^{tC_V}\mathscr{V}^x_t$ is the solution of (100) with V replaced with $V - C_V$, we can assume that $V \geq 0$. Furthermore, we can assume $t > 0$. Let (V_n) be as in lemma 5.7. By the same arguments as in the second part of the proof of theorem 5.3, one has

$$\mathbb{E}\left[\mathscr{V}^x_{n,t}//^{x,-1}_t f(B_t(x))\right] \to \mathrm{e}^{-tH(V)}f(x) \quad \text{as } n \to \infty,$$

with the notations from lemma 5.9. Since by lemma 5.9,

$$\mathscr{V}^x_{n,t}//^{x,-1}_t f(B_t(x)) \to \mathscr{V}^x_t//^{x,-1}_t f(B_t(x)) \quad \text{as } n \to \infty, \mathbb{P}\text{-a.s.}$$

and since by applying (99) to V_n,

$$\left\|\mathscr{V}^x_{n,t}//^{x,-1}_t f(B_t(x))\right\|_x \leq \|f(B_t(x))\|_{B_t(x)}, \tag{101}$$

the theorem follows from dominated convergence, in view of

$$\mathbb{E}\left[\|f(B_t(x))\|_{B_t(x)}\right] = \mathrm{e}^{\frac{t}{2}\Delta}\,\|f(x)\|_x < \infty. \tag{102}$$

■

Being motivated from [76], we will call the family of bounded self-adjoint operators

$$(\mathrm{e}^{-tH(V)})_{t\geq 0} \subset \mathscr{L}(\Gamma_{\mathrm{L}^2}(M,E))$$

the *Schrödinger semigroup corresponding to* $H(V)$. We directly get the following corollary:

Corollary 5.10 *Under the assumptions of theorem 5.5, one has*

$$\left\|\mathrm{e}^{-tH(V)}\right\| \leq \mathrm{e}^{-tC_V} \quad \textit{for any } t \geq 0.$$

In particular, $V \geq 0$ *implies that the Schrödinger semigroup corresponding to* $H(V)$ *is contractive and that* $H(V) \geq 0$.

Proof. The inequalities

$$\begin{aligned}\left\|\mathrm{e}^{-tH(V)}f\right\|^2 &\leq \mathrm{e}^{-2tC_V}\int_M \mathbb{E}\left[\|f(B_t(x))\|_{B_t(x)}\right]^2 \mathrm{vol}(\mathrm{d}x)\\ &\leq \mathrm{e}^{-2tC_V}\,\|f\|^2 \end{aligned} \tag{103}$$

can be seen by using (99) and the same arguments as in the first part of the proof of theorem 5.3.

■

6 The Kato class

Now we introduce the (local) Kato class corresponding to vector bundles. The importance of local Kato potentials for us is, that it is a very large class (see theorem 6.4), under which one can still expect pointwise results for $H(V)$. In particular, expressions like $\mathrm{e}^{-tH(V)}f(x)$ given by the Feynman-Kac formula will turn out to make sense *for all* $x \in M$, if V is in the local Kato class (see lemma 6.5 and also lemma 7.6). Of course, this is of importance, if one is interested in pointwise results like continuity.

Definition 6.1 *Let M be stochastically complete. A measurable section V in* $\mathrm{End}(E)$ *is said to be in the Kato class, if*

$$\lim_{t\searrow 0}\sup_{x\in M}\mathbb{E}\left[\int_0^t \left\|V(B_s(x))\right\|_{B_s(x)}\,\mathrm{d}s\right]=0,$$

and V is said to be in the local Kato class, if $1_K V$ is in the Kato class for any compact $K\subset M$. We write $\Gamma_{\mathcal{K}}(M,\mathrm{End}(E))$ and $\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E))$ for the Kato and the local Kato class, respectively.

Remark 6.2 It is clear that in general $\Gamma_{\mathcal{K}}(M,\mathrm{End}(E))$ depends on the Riemannian structure of M and the Hermitian structure of E, but since we have fixed these structures, they don't appear in our notation. On the other hand, $\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E))$ does not depend on the Hermitian structure of E, but still depends on the Riemannian structure of M, so the term "local" should be understood rather in the sense of Markov processes with values in a metric space, than in the usual sense of manifolds.

□

If $E=M\times\mathbb{C}$ (with its standard Euclidean structure), then we set

$$\mathcal{K}(M):=\Gamma_{\mathcal{K}}(M,\mathrm{End}(E)),\ \ \mathcal{K}_{\mathrm{loc}}(M):=\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E)),$$

where the elements of these sets will be identified with functions $M\to\mathbb{C}$ in the usual way. We collect some facts about $\Gamma_{\mathcal{K}}(M,\mathrm{End}(E))$: If V is in the Kato class, then obviously

$$\sup_{x\in M}\mathbb{E}\left[\int_0^{t_0} \left\|V(B_s(x))\right\|_{B_s(x)}\,\mathrm{d}s\right]<\infty\ \text{ for some (small) } t_0>0. \tag{104}$$

Using the Chapman-Kolomogorov equation, we can prove:

Lemma 6.3 *Let M be stochastically complete.*
a) *For a measurable section V in* $\mathrm{End}(E)$ *one has*

$$\sup_{x\in M}\mathbb{E}\left[\int_0^{t_0} \left\|V(B_s(x))\right\|_{B_s(x)}\,\mathrm{d}s\right]<\infty\ \ \textit{for some } t_0>0, \tag{105}$$

if and only if

$$\sup_{x\in M}\mathbb{E}\left[\int_0^{t} \left\|V(B_s(x))\right\|_{B_s(x)}\,\mathrm{d}s\right]<\infty\ \ \textit{for all } t>0. \tag{106}$$

b) *One has*

$$\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E))\subset\Gamma_{\mathrm{L}^1_{\mathrm{loc}}}(M,\mathrm{End}(E)).$$

Proof. a) The proof is probably standard in the theory of PCAF's (positive continuous additive functionals; see for example [53]). Take some $n \in \mathbb{N}$ with $t \leq nt_0$. Then we can estimate as follows:

$$
\begin{aligned}
&\sup_{x\in M} \mathbb{E}\left[\int_0^t \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right] \\
&= \sup_{x\in M} \int_M \int_0^t p_s(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&\leq \sup_{x\in M} \int_M \int_0^{nt_0} p_s(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&\leq \sum_{k=2}^{n} \sup_{x\in M} \int_M \int_0^{t_0} p_{(k-1)t_0+s}(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&\qquad + \sup_{x\in M} \int_M \int_0^{t_0} p_s(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&= \sum_{k=2}^{n} \sup_{x\in M} \int_0^{t_0} \int_M p_{(k-1)t_0}(x,z) \int_M p_s(z,y)\,\|V(y)\|_y\,\mathrm{vol}(\mathrm{d}y)\mathrm{vol}(\mathrm{d}z)\mathrm{d}s \\
&\qquad + \sup_{x\in M} \int_M \int_0^{t_0} p_s(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&\leq \left(\sum_{k=2}^{n} \sup_{x\in M} \int_M p_{(k-1)t_0}(x,z)\mathrm{vol}(\mathrm{d}z)\right) \sup_{z\in M} \int_0^{t_0} \int_M p_s(z,y)\,\|V(y)\|_y\,\mathrm{vol}(\mathrm{d}y)\mathrm{d}s \\
&\qquad + \sup_{x\in M} \int_M \int_0^{t_0} p_s(x,y)\,\|V(y)\|_y\,\mathrm{d}s\ \mathrm{vol}(\mathrm{d}y) \\
&\leq n \sup_{z\in M} \int_0^{t_0} \int_M p_s(z,y)\,\|V(y)\|_y\,\mathrm{vol}(\mathrm{d}y)\mathrm{d}s < \infty, \qquad (107)
\end{aligned}
$$

where we have used the Chapman-Kolmogorov identity for the fourth step and (51) for the sixth step.

b) Let V be in the local Kato class, let $K \subset M$ be compact and let $0 < \varepsilon_1 < \varepsilon_2$. Take a $C > 0$ such that for all $s \in [\varepsilon_1, \varepsilon_2]$ and all $x, y \in K$ one has $p_s(x,y) > C$. Thus,

$$
C(\varepsilon_2 - \varepsilon_1) \int_K \|V(y)\|_y\,\mathrm{vol}(\mathrm{d}y) \leq \sup_{x\in M} \int_K \int_0^{\varepsilon_2} p_s(x,y)\mathrm{d}s\,\|V(y)\|_y\,\mathrm{vol}(\mathrm{d}y),
$$

which is finite by part a).

■

The set of measurable sections V in $\mathrm{End}(E)$ satisfying (105) or equivalently (106) can be called the *Dynkin class*. This definition is motivated by the scalar case [54].

For any $p \geq 1$ let

$$\Gamma_{\mathrm{L}^p_{\mathrm{u,loc}}}(M, \mathrm{End}(E)) := \left\{ V \,\Big|\, \sup_{x \in M} \int_{\mathrm{K}_1(x)} \|V(y)\|_y^p \,\mathrm{vol}(\mathrm{d}y) < \infty \right\}$$

denote the set of uniformly locally p-integrable measurable sections in $\mathrm{End}(E)$. Clearly, one has

$$\Gamma_{\mathrm{L}^p_{\mathrm{u,loc}}}(M, \mathrm{End}(E)) \subset \Gamma_{\mathrm{L}^p_{\mathrm{loc}}}(M, \mathrm{End}(E)).$$

The following theorem shows that under relatively mild assumptions on the Riemannian structure of M, being locally Kato is not a restrictive assumption on the class of locally square integrable sections, if $1 \leq m \leq 3$:

Theorem 6.4 *Let M be geodesically complete with Ricci curvature bounded from below* [15] *and positive injectivity radius.*
a) *A measurable section V in $\mathrm{End}(E)$ is in $\Gamma_{\mathcal{K}}(M, \mathrm{End}(E))$, if and only if*

$$V \in \Gamma_{\mathrm{L}^1_{\mathrm{u,loc}}}(M, \mathrm{End}(E)), \quad \textit{if } m = 1$$

and

$$\lim_{r \searrow 0} \sup_{x \in M} \int_{\mathrm{K}_r(x)} \|V(y)\|_y \, G_m(\mathrm{d}(x,y)) \mathrm{vol}(\mathrm{d}y) = 0, \quad \textit{if } m \geq 2.$$

Here, $G_m : (0, \infty) \to (0, \infty)$ is the function such that $G_m(\|v - w\|_{\mathbb{R}^m})$ is the Green's function of the Laplace operator in the Euclidean $\mathbb{R}^m$, that is,

$$G_m(r) := \begin{cases} r^{2-m}, & \textit{if } m > 2 \\ \log(r^{-1}), & \textit{if } m = 2. \end{cases} \tag{108}$$

b) *For any p such that $p \geq 1$ if $m = 1$, and $p > m/2$ if $m \geq 2$, one has*

$$\Gamma_{\mathrm{L}^p_{\mathrm{u,loc}}}(M, \mathrm{End}(E)) \subset \Gamma_{\mathcal{K}}(M, \mathrm{End}(E)), \tag{109}$$

$$\Gamma_{\mathrm{L}^p_{\mathrm{loc}}}(M, \mathrm{End}(E)) \subset \Gamma_{\mathcal{K}_{\mathrm{loc}}}(M, \mathrm{End}(E)). \tag{110}$$

Proof. a) This follows from the scalar results of [54]. The main point for the proof clearly is a two sided Gaussian bound for $p_t(x,y)$, which is valid under these assumptions on M (see lemma 7.1).
b) The inclusion (109) has also been proved in [54] and the inclusion (110) follows easily from (109).

■

The following lemma shows that the local Kato assumption on the potential is sufficient to ensure that $\mathrm{e}^{-tH(V)} f(x)$ can be defined *for all $x \in M$, $t > 0$*:

[15] Then M is automatically stochastically complete.

Lemma 6.5 *Let M be stochastically complete and let $V \in \Gamma_{\mathcal{K}_{\mathrm{loc}}}(M, \mathrm{End}(E))$. Then for any $x \in M$ and $t > 0$ one has*

$$\int_0^t \|V(B_s(x))\|_{B_s(x)}\, \mathrm{d}s < \infty \quad \mathbb{P}\text{-}a.s. \tag{111}$$

Proof. The arguments of [13], p.45, can be adapted to our situation. We use the fact that the Brownian motion is continuous and (under our assumptions on M) nonexplosive. Let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion of M with relatively compact open subsets, and for any n let $\tau_n(x)$ be the first exit time of $B(x)$ from K_n. Then one has

$$\begin{aligned}
&\mathbb{P}\left\{\int_0^t \|V(B_s(x))\|_{B_s(x)}\, \mathrm{d}s = \infty\right\} \\
=\ &\mathbb{P}\left\{\int_0^t \left\|\Big(1_{K_n}(B_s(x)) + 1_{M\setminus K_n}(B_s(x))\Big)V(B_s(x))\right\|_{B_s(x)} \mathrm{d}s = \infty\right\} \\
\leq\ &\mathbb{P}\left\{\int_0^t \|(1_{K_n}V)(B_s(x))\|_{B_s(x)}\, \mathrm{d}s = \infty\right\} \\
&+ \mathbb{P}\left\{t \geq \tau_n(x)\right\}.
\end{aligned} \tag{112}$$

The last term tends to 0 as $n \to \infty$, so the lemma is proved if (111) holds for any $V \in \Gamma_{\mathcal{K}}(M, \mathrm{End}(E))$. But this follows directly from (106).

■

7 Brownian bridges

Next, we introduce the Brownian bridge measures $\mathbb{E}_t^{x,y}[\bullet]$, which are a rigorous version of the conditional expectation values $\mathbb{E}[\bullet \mid B_t(x) = y]$, and as such, they will provide us with a probabilistic formula for the integral kernel of the Schrödinger semigroup. Although the construction of the Brownian bridge measures as measures on the path space $\mathscr{W}(M)$ with M closed is certainly well-known [41], we believe that the construction below, which corresponds to our embedding approach to Brownian motion, might not be known as well. Our arguments are motivated by [10], where $\mathbb{E}_t^{x,y}[\bullet]$ has been defined for closed manifolds and for $B(x)$ defined intrinsically by means of theorem 2.33.

For the following considerations, it will be convenient to note the following lemma:

Lemma 7.1 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. Then for any $t > 0$ there are $A_t, B_t, C_t, D_t > 0$ such that for all $0 < s \leq t$ and all $x, y \in M$ one has*

$$\frac{A_t \mathrm{e}^{-B_t \frac{\mathrm{d}(x,y)^2}{s}}}{s^{m/2}} \leq p_s(x,y) \leq \frac{C_t \mathrm{e}^{-D_t \frac{\mathrm{d}(x,y)^2}{s}}}{s^{m/2}}. \tag{113}$$

In particular, one has

$$\sup_{x,y \in M} p_t(x,y) < \infty \quad \textit{for all } t > 0. \tag{114}$$

Proof. The estimates in (113) follow easily from the considerations of p.110 in [54] and a simple rescaling argument for the Riemannian structure g.

■

Remark 7.2 For example, one may assume that M has a bounded geometry in order to be in the setting of lemma 7.1. □

Let $\mathscr{F}_*$ be the filtration which is generated by the coordinate process W on Ω and let $\tilde{\mathscr{F}}_* \supset \mathscr{F}_*$ be the corresponding augmented right-continuous filtration. By definition, the process $B(x)$ is adapted to $\tilde{\mathscr{F}}_*$. However, since $B(x)$ is the solution of a stochastic differential equation which is driven by W, it follows that $B(x)$ is also adapted to $\mathscr{F}_*$ (see for example theorem 2.1 on p.375 in [66]). We first note the following elementary lemma:

Lemma 7.3 *Let M be stochastically complete with*

$$\sup_{x,y \in M} p_t(x,y) < \infty \quad \textit{for all } t > 0. \tag{115}$$

Then the process

$$p_{t-\bullet}(B(x), y) : [0,t) \times \Omega \longrightarrow (0, \infty)$$

is a $(\mathscr{F}_s)_{0 \leq s < t}$-martingale.

Proof. Since $B(x)$ is adapted to $\mathscr{F}_* \subset \tilde{\mathscr{F}}_*$, it is sufficient to prove that $p_{t-\bullet}(B(x), y)$ is a $(\tilde{\mathscr{F}}_s)_{0 \leq s < t}$ martingale.
The integrability of $p_{t-\bullet}(B(x), y)$ follows from the Chapman-Kolmogorov equations. Let $0 \leq s_1 < s_2 < t$. Then $\mathbb{P}$-a.s. one has

$$\begin{aligned}
\mathbb{E}\left[p_{t-s_2}(B_{s_2}(x), y) \mid \tilde{\mathscr{F}}_{s_1}\right](\bullet) &= \int_\Omega p_{t-s_2}\Big(B_{s_2-s_1}\Big(B_{s_1}(x)(\bullet)\Big)(\omega), y\Big)\, \mathbb{P}(\mathrm{d}\omega) \\
&= \int_M p_{s_2-s_1}(B_{s_1}(x)(\bullet), z) p_{t-s_2}(z,y) \mathrm{vol}(\mathrm{d}z) \\
&= p_{t-s_1}(B_{s_1}(x)(\bullet), y),
\end{aligned} \tag{116}$$

where we have used the Markov property of $B(x)$ (Bemerkung 7.250 in [37] together with (115)) for the first step, theorem 2.24 vi) for the second step and the Chapman-Kolmogorov identity for the third step.

$\blacksquare$

Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. We fix arbitrary $x, y \in M$ and $t > 0$ for the rest of this section. Lemma 7.3 shows that we can define the conditional Brownian bridge measure $\mathbb{P}^{x,y}_t$ as the unique probability measure on $(\Omega, \mathscr{F}_{t-})$, where

$$\mathscr{F}_{t-} := \sigma\left(\bigcup_{0\leq s<t} \mathscr{F}_s\right),$$

such that

$$\left.\frac{\mathrm{d}\mathbb{P}^{x,y}_t}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_s} = \frac{p_{t-s}(B_s(x), y)}{p_t(x,y)} \quad \text{for any } s < t. \tag{117}$$

The existence of this measure follows from a small adaption of Satz 1.25' and the remarks on p.249 in [37], and the uniqueness follows from the monotone class theorem (see [61], p.143, for an abstract argument). Note also that $\mathscr{F}_{t-} = \mathscr{F}_t$, which is implied by the (left-)continuity of W (see for example p.89 in [46]).

Due to the strict positivity of the Radon-Nikodym derivative, the measures $\mathbb{P}$ and $\mathbb{P}^{x,y}_t$ are actually equivalent on $\mathscr{F}_s$, if $s < t$, so that the process

$$\left(\frac{1}{p_{t-\bullet}(B(x), y)}, \mathbb{P}^{x,y}_t\right) : [0,t) \times \Omega \longrightarrow (0, \infty)$$

is a nonnegative continuous martingale, which, by martingale convergence, necessarily has a limit as $s \nearrow t$ (see for example [36], theorem 1.3). Using the upper bound (113) again, these considerations imply

$$\lim_{s\nearrow t} B_s(x) = y \quad \mathbb{P}^{x,y}_t\text{-a.s.} \tag{118}$$

Furthermore, it follows from (117) that the process

$$(B(x), \mathbb{P}^{x,y}_t) : [0,t] \times \Omega \longrightarrow M \tag{119}$$

is a Brownian bridge from x to y at the terminal time t in the sense of the defining finite dimensional distributions given in [41]. As such, it is well-known that one has the following time reversal property: The law of

$$(B_{t-\bullet}(x), \mathbb{P}^{x,y}_t) : [0,t] \times \Omega \longrightarrow M$$

is equal to the law of

$$(B(y), \mathbb{P}_t^{y,x}) : [0,t] \times \Omega \longrightarrow M.$$

Let $\mathbb{E}_t^{x,y}[\bullet]$ denote the expectation value with respect to $\mathbb{P}_t^{x,y}$. Our initial remark that $\mathbb{E}_t^{x,y}[\bullet]$ is a rigorous implementation of $\mathbb{E}[\bullet \mid B_t(x) = y]$ is finally justified by the following disintegration:

Lemma 7.4 *For any $A \in \mathscr{F}_{t-}$ $(= \mathscr{F}_t)$,*

$$\mathbb{P}(A) = \int_M \mathbb{P}_t^{x,y}(A) p_t(x,y) \mathrm{vol}(\mathrm{d}y). \tag{120}$$

Proof. By the monotone class theorem, it is sufficient to consider the case $A \in \mathscr{F}_s$ for some $0 \leq s < t$. Then we have

$$\mathbb{P}_t^{x,y}(A) = \mathbb{E}\left[1_A \frac{p_{t-s}(B_s(x), y)}{p_t(x,y)}\right],$$

so that with Fubini,

$$\begin{aligned}
\int_M \mathbb{P}_t^{x,y}(A) p_t(x,y) \mathrm{vol}(\mathrm{d}y) &= \int_M \mathbb{E}\left[1_A p_{t-s}(B_s(x), y)\right] \mathrm{vol}(\mathrm{d}y) \\
&= \int_A \int_M p_{t-s}(B_s(x)(\omega), y) \mathrm{vol}(\mathrm{d}y) \mathbb{P}(\mathrm{d}\omega) \\
&= \mathbb{P}(A),
\end{aligned}$$

where we have used (117) and the stochastic completeness of M.

■

Corollary 7.5 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, and let the potential $v \in \mathrm{L}^2_{\mathrm{loc}}(M)$ be bounded from below. If $H_0(v)$ denotes the self-adjoint realization of $-\Delta/2 + v$ in $\mathrm{L}^2(M)$, then for any $t > 0$, the function*

$$\begin{aligned}
&\mathrm{e}^{-tH_0(v)}(\bullet, \bullet) : M \times M \longrightarrow (0, \infty) \\
&\mathrm{e}^{-tH_0(v)}(x,y) := p_t(x,y) \mathbb{E}_t^{x,y}\left[\mathrm{e}^{-\int_0^t v(B_s(x)) \mathrm{d}s}\right]
\end{aligned}$$

is well-defined for a.e. $(x,y) \in M \times M$ and it defines an essentially bounded integral kernel for the operator $\mathrm{e}^{-tH_0(v)}$, in the sense that for any $f \in \mathrm{L}^2(M)$, a.e. $x \in M$ one has

$$\mathrm{e}^{-tH_0(v)} f(x) = \int_M \mathrm{e}^{-tH_0(v)}(x,y) f(x) \mathrm{vol}(\mathrm{d}y).$$

Proof. The equality (120) gives

$$\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} \leq \mathrm{e}^{-tC_v} \quad \mathbb{P}_t^{x,y}\text{-a.s. for a.e. } (x,y) \in M \times M,$$

where C_v is some lower bound of v. It follows that

$$M \times M \ni (x,y) \longmapsto \mathrm{e}^{-tH_0(v)}(x,y) \in (0,\infty)$$

is a well-defined measurable function, which is easily seen to have the desired properties, if one uses the Feynman-Kac formula, (120), (118).

■

We finally note the following proposition, which shows that a local Kato assumption is compatible with the Brownian bridge:

Proposition 7.6 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. If $V \in \Gamma_{\mathcal{K}_{\mathrm{loc}}}(M, \mathrm{End}(E))$, then one has*

$$\int_0^t \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s < \infty \quad \mathbb{P}_t^{x,y}\text{-}a.s. \tag{121}$$

Proof. By going through the same steps as in the proof of lemma 6.5, it is sufficient to prove that (121) holds for any $V \in \Gamma_{\mathcal{K}}(M, \mathrm{End}(E))$. By the time-reversal property of the Brownian bridge, one can write

$$\begin{aligned} &\mathbb{E}_t^{x,y}\left[\int_0^t \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right] \\ &= \mathbb{E}_t^{x,y}\left[\int_0^{\frac{t}{2}} \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right] + \mathbb{E}_t^{y,x}\left[\int_0^{\frac{t}{2}} \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right]. \end{aligned} \tag{122}$$

Both of the last two terms can be treated similarly, say,

$$\begin{aligned} &\mathbb{E}_t^{x,y}\left[\int_0^{\frac{t}{2}} \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^{\frac{t}{2}} \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s \frac{p_{\frac{t}{2}}(B_{\frac{t}{2}}(x),y)}{p_t(x,y)}\right] \\ &\leq \frac{\tilde{C}_t}{p_t(x,y)}\mathbb{E}\left[\int_0^{\frac{t}{2}} \|V(B_s(x))\|_{B_s(x)}\,\mathrm{d}s\right] < \infty, \end{aligned} \tag{123}$$

where we have used (117), the upper bound in (113) and (106).

■

8 Applications of the Feynman-Kac formula

In this section, we explain how the Feynman-Kac formula can be used to obtain regularity results and bounds for some spectral data of Schrödinger type operators on vector bundles over noncompact manifolds.

We continue to work under the assumptions and with the notations from section 4. If V is a potential and $x \in M$, $\omega \in \Omega$ are such that

$$[0, \infty) \longrightarrow [0, \infty), \ t \longmapsto \|V(B_t(x))\|_{B_t(x)}(\omega)$$

is locally integrable, then

$$\mathscr{V}^x_\bullet(\omega) : [0, \infty) \longrightarrow \mathrm{End}(E)_x$$

stands for the weak solution of

$$\begin{aligned} \mathrm{d}\mathscr{V}^x_t(\omega) &= -\mathscr{V}^x_t(\omega)\Big(//^{x,-1}_t V(B_t(x))//^x_t\Big)(\omega)\mathrm{d}t, \\ \mathscr{V}^x_0(\omega) &= \mathbf{1}. \end{aligned} \tag{124}$$

It is easily seen that $\mathscr{V}^x_\bullet(\omega)$ is invertible with

$$\begin{aligned} \mathrm{d}\mathscr{V}^{x,-1}_t(\omega) &= \Big(//^{x,-1}_t V(B_t(x))//^x_t\Big)(\omega)\mathscr{V}^{x,-1}_t(\omega)\mathrm{d}t, \\ \mathscr{V}^{x,-1}_0(\omega) &= \mathbf{1}, \end{aligned} \tag{125}$$

and furthermore one has

$$\begin{aligned} \mathrm{d}\mathscr{V}^{x,*}_t(\omega) &= -\Big(//^{x,-1}_t V(B_t(x))//^x_t\Big)^*(\omega)\mathscr{V}^{x,*}_t(\omega)\mathrm{d}t, \\ \mathscr{V}^{x,*}_0(\omega) &= \mathbf{1}. \end{aligned} \tag{126}$$

The self-adjoint realizations of $\nabla^*\nabla/2 + V$ in $\Gamma_{\mathrm{L}^2}(M, E)$ and of $-\Delta/2 + v$ in $\mathrm{L}^2(M)$ in the sense of theorem 3.5 will be denoted with $H(V)$ and with $H_0(v)$, respectively.

8.1 Bottom of the spectrum

In the following, we write q_H for the (minimal) closed, lower semi-bounded quadratic form corresponding to a self-adjoint, lower semi-bounded operator H in some Hilbert space $\mathscr{H}$ with norm $\|\bullet\|_{\mathscr{H}}$. If H is bounded from below by a constant C_1 and if $C_2 \leq C_1$, then q_H is given as follows:

$$\mathrm{D}(q_H) = \mathrm{D}\left((H - C_2)^{\frac{1}{2}}\right), \ q_H(f) = \left\|(H - C_2)^{\frac{1}{2}} f\right\|^2_{\mathscr{H}} + C_2 \|f\|^2_{\mathscr{H}}, \tag{127}$$

and (127) does not depend on C_2 ([47], p.332). The spectral theorem justifies the extension $q_H(f) := \infty$, if $f \in \mathscr{H} \setminus \mathrm{D}(q_H)$ (at least, if $H \geq 0$). See [47], chapter VI, for details on quadratic forms.

We remind the reader of the notation $|f|(x) := \|f(x)\|_x$ for any section f in E. A key observation is the following semigroup domination. We refer the reader to [6] and [38] for an abstract formulation of semigroup domination and applications.

Theorem 8.1 *Let M be stochastically and geodesically complete, let V be a potential with*

$$V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E))$$

and let v be a scalar potential with [16]

$$C_v \mathbf{1} \leq v\mathbf{1} \leq V \quad \text{for some } C_v \in \mathbb{R}.$$

Then the following inequality holds for any $f \in \Gamma_{\mathrm{L}^2}(M, E)$, $t \geq 0$ and a.e. $x \in M$,

$$\left\| \mathrm{e}^{-tH(V)} f(x) \right\|_x \leq \mathrm{e}^{-tH_0(v)} |f|(x). \tag{128}$$

In particular,

i) *for any $f \in \Gamma_{\mathrm{L}^2}(M, E)$, any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > C_v$ and a.e. $x \in M$,*

$$\left\| (H(V) + \lambda)^{-1} f(x) \right\|_x \leq (H_0(v) + \lambda)^{-1} |f|(x), \tag{129}$$

ii) *for any $f \in \Gamma_{\mathrm{L}^2}(M, E)$, $t \geq 0$,*

$$\left\langle \mathrm{e}^{-tH(V)} f, f \right\rangle \leq \left\langle \mathrm{e}^{-tH_0(v)} |f|, |f| \right\rangle_{\mathrm{L}^2(M)}, \tag{130}$$

iii) $|f| \in \mathrm{D}(q_{H_0(v)})$ *for any* $f \in \mathrm{D}(q_{H(V)})$,

iv) $\inf \sigma(H(V)) \geq \inf \sigma(H_0(v))$.

Proof. Let $f \in \Gamma_{\mathrm{L}^2}(M, E)$. The Feynman-Kac formulae for $H(V)$, $H_0(v)$ and the inequality

$$\|\mathscr{V}_t^x\|_x \leq \mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} \quad \mathbb{P}\text{-a.s.}, \tag{131}$$

[16]Note that v is automatically in $\mathrm{L}^2_{\mathrm{loc}}(M)$ under these assumptions.

which comes from proposition A.1 b) by defining

$$F(t) := -\left(//_t^{x,-1} V(B_t(x)) //_t^x\right)$$

and $c(t) := -v(B_t(x))$ pathwise, directly imply (128).
Now (129) follows from the Laplace transforms

$$(H(V)+\lambda)^{-1} f(x) = \int_0^\infty \mathrm{e}^{-t\lambda} \mathrm{e}^{-tH(V)} f(x) \mathrm{d}t$$

and

$$(H_0(v)+\lambda)^{-1} |f|(x) = \int_0^\infty \mathrm{e}^{-t\lambda} \mathrm{e}^{-tH_0(v)} |f|(x) \mathrm{d}t.$$

Multiplying (128) with $\|f(x)\|_x$ and integrating with respect to $\int_M (\bullet) \mathrm{vol}(\mathrm{d}x)$ proves (130) in view of the Cauchy-Schwarz inequality for $(\bullet, \bullet)_x$ and

$$\left\langle \mathrm{e}^{-tH(V)} f, f \right\rangle \geq 0.$$

For the other assertions, note that by (127) we may assume that $H(V)$ and $H_0(v)$ are nonnegative (otherwise, consider $H(V-C)$ and $H(v-C)$ with some $C \in \mathbb{R}$ small enough such that $H(V)-C$ and $H_0(v)-C$ are nonnegative). If $f \in \mathrm{D}(H(V)^{\frac{1}{2}})$, then by (130) and the spectral calculus,

$$\infty > \left\langle H(V)^{\frac{1}{2}} f, H(V)^{\frac{1}{2}} f \right\rangle = -\lim_{t \searrow 0} \left\langle \frac{\mathrm{e}^{-tH(V)} f - f}{t}, f \right\rangle \tag{132}$$

$$\geq -\lim_{t \searrow 0} \left\langle \frac{\mathrm{e}^{-tH_0(v)} |f| - |f|}{t}, |f| \right\rangle_{\mathrm{L}^2(M)} = \left\langle H_0(v)^{\frac{1}{2}} |f|, H_0(v)^{\frac{1}{2}} |f| \right\rangle_{\mathrm{L}^2(M)}, \tag{133}$$

so $|f|$ is in the form domain of $H_0(v)$ and

$$\left\langle H(V)^{\frac{1}{2}} f, H(V)^{\frac{1}{2}} f \right\rangle \geq \left\langle H_0(v)^{\frac{1}{2}} |f|, H_0(v)^{\frac{1}{2}} |f| \right\rangle_{\mathrm{L}^2(M)}.$$

Finally, by the variational principle (see for example [87], Satz 8.27),

$$\begin{aligned}
\inf \sigma(H(V)) &= \inf \left\{ \left\langle H(V)^{\frac{1}{2}} f, H(V)^{\frac{1}{2}} f \right\rangle \Big| \, f \in \mathrm{D}(H(V)^{\frac{1}{2}}), \|f\| = 1 \right\} \\
&\geq \inf \left\{ \left\langle H_0(v)^{\frac{1}{2}} |f|, H_0(v)^{\frac{1}{2}} |f| \right\rangle_{\mathrm{L}^2(M)} \Big| \, f \in \mathrm{D}(H(V)^{\frac{1}{2}}), \|f\| = 1 \right\} \\
&\geq \inf \left\{ \left\langle H_0(v)^{\frac{1}{2}} h, H_0(v)^{\frac{1}{2}} h \right\rangle_{\mathrm{L}^2(M)} \Big| \, h \in \mathrm{D}(H_0(v)^{\frac{1}{2}}), \|h\|_{\mathrm{L}^2(M)} = 1 \right\} \\
&= \inf \sigma(H_0(v)),
\end{aligned}$$

and the theorem is proved.

■

Remark 8.2 If V is bounded from below, then a possible choice for the scalar potential v in proposition 8.2 is $v(x) := \min \sigma(V(x))$.

$\square$

For the next proposition we set (whenever it makes sense)

$$\mathrm{e}^{-tH(V)} f(x) := \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right] \tag{134}$$

$$\mathrm{e}^{-tH_0(v)} h(x) := \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} h(B_t(x))\right] \tag{135}$$

for sections f in E and functions h on M. For any $p, q \in [1, \infty]$ let $\|\bullet\|_p$ denote the norm in $\Gamma_{\mathrm{L}^p}(M, E)$ and let $\|\bullet\|_{p,q}$ denote the norm corresponding to

$$\mathscr{L}\Big(\Gamma_{\mathrm{L}^p}(M, \bullet), \Gamma_{\mathrm{L}^q}(M, \bullet)\Big),$$

with the convention $\|\bullet\| = \|\bullet\|_{2,2}$ (and the same notation for operators on functions on M). We will prove the following proposition in a moment:

Proposition 8.3 a) *Let M be stochastically and geodesically complete, let V be a potential with*

$$V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E))$$

and let v be a scalar potential with

$$C_v \mathbf{1} \leq v\mathbf{1} \leq V \quad \textit{for some } C_v \in \mathbb{R}.$$

Furthermore, let $p, q \in [1, \infty]$ and $t > 0$. Then the assumption

$$\mathrm{e}^{-tH_0(v)} \in \mathscr{L}(\mathrm{L}^p(M), \mathrm{L}^q(M))$$

implies

$$\mathrm{e}^{-tH(V)} \in \mathscr{L}\Big(\Gamma_{\mathrm{L}^p}(M, \bullet), \Gamma_{\mathrm{L}^q}(M, \bullet)\Big)$$

and

$$\left\|\mathrm{e}^{-tH(V)}\right\|_{p,q} \leq \left\|\mathrm{e}^{-tH_0(v)}\right\|_{p,q}. \tag{136}$$

b) *Let M have a bounded geometry and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then for any $1 \leq p \leq q \leq \infty$, $t > 0$, one has

$$\mathrm{e}^{-tH(V)} \in \mathscr{L}\Big(\Gamma_{\mathrm{L}^p}(M, \bullet), \Gamma_{\mathrm{L}^q}(M, \bullet)\Big)$$

and there is a $C > 0$, which only depends on the Riemannian structure of M, such that

$$\left\| \mathrm{e}^{-tH(V)} \right\|_{p,q} \leq C^{\frac{1}{p}-\frac{1}{q}} \min\left\{ t^{\frac{m}{2}}, 1 \right\}^{-\frac{1}{p}+\frac{1}{q}} \mathrm{e}^{-tC_V}. \tag{137}$$

In particular, one has

$$\left\| \mathrm{e}^{-tH(V)} \right\|_{p,p} \leq \mathrm{e}^{-tC_V} \quad \text{for any } 1 \leq p \leq \infty,\ t > 0.$$

We will need the following assertion for the proof of part b) of proposition 8.3:

Proposition 8.4 *Let $t > 0$ and assume that*

$$C(t) := \sup_{x,y \in M} p_t(x,y) < \infty.$$

Then the assignment

$$\mathrm{e}^{\frac{t}{2}\Delta} h(x) := \int_M p_t(x,y) h(y) \mathrm{vol}(\mathrm{d}y)$$

defines an element of $\mathscr{L}(\mathrm{L}^p(M), \mathrm{L}^q(M))$ for all $1 \leq p \leq q \leq \infty$ and one has

$$\left\| \mathrm{e}^{\frac{t}{2}\Delta} \right\|_{p,q} \leq C(t)^{\frac{1}{p}-\frac{1}{q}}.$$

Proof. It should be possible to deduce this proposition with an abstract L^p-interpolation theorem like Riesz-Thorin. Nevertheless, here is an elementary proof which only uses the Hölder inequality. Let us first note the following simple fact: For any $r \geq 1$ and $x \in M$ one has

$$\| p_t(x, \bullet) \|_r \leq C(t,r) := C(t)^{1-\frac{1}{r}}. \tag{138}$$

Throughout, let $f \in \mathrm{L}^\mathrm{p}(M)$.

Case $1 < p < q < \infty$: Let r be given as $1 - 1/r = 1/p - 1/q$. Applying Hölder's inequality with the exponents

$$q_1 = q,\ \ q_2 = \frac{r}{1-\frac{r}{q}},\ \ q_3 = \frac{p}{1-\frac{p}{q}}$$

gives

$$\begin{aligned}
&\left\|\mathrm{e}^{\frac{t}{2}\Delta}f\right\|_q^q \\
&\leq \int_M\left(\int_M \left(p_t(x,y)^r|f(y)|^p\right)^{\frac{1}{q}} p_t(x,y)^{1-\frac{r}{q}}|f(y)|^{1-\frac{p}{q}}\mathrm{vol}(\mathrm{d}y)\right)^q \mathrm{vol}(\mathrm{d}x) \\
&\leq \int_M\left(\int_M p_t(x,y)^r|f(y)|^p\mathrm{vol}(\mathrm{d}y)\right)\left(\int_M p_t(x,y)^r\mathrm{vol}(\mathrm{d}y)\right)^{\frac{q}{r}\left(1-\frac{r}{q}\right)} \\
&\quad\times\left(\int_M |f(y)|^p\mathrm{vol}(\mathrm{d}y)\right)^{\frac{q}{p}\left(1-\frac{p}{q}\right)}\mathrm{vol}(\mathrm{d}x),
\end{aligned}$$

so that using Fubini's theorem and (138),

$$\begin{aligned}
&\left\|\mathrm{e}^{\frac{t}{2}\Delta}f\right\|_q^q \leq C(t,r)^{q\left(1-\frac{r}{q}\right)}\|f\|_p^{q\left(1-\frac{p}{q}\right)}\int_M|f(y)|^p\int_M p_t(x,y)^r\mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y) \\
&\leq C(t,r)^q\|f\|_p^q = C(t)^{q\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p^q. \qquad (139)
\end{aligned}$$

Case $1<p=q<\infty$: One has

$$\begin{aligned}
&\left\|\mathrm{e}^{\frac{t}{2}\Delta}f\right\|_p^p \leq \int_M\left(\int_M p_t(x,y)|f(y)|\mathrm{vol}(\mathrm{d}y)\right)^p\mathrm{vol}(\mathrm{d}x) \\
&\leq \int_M\int_M |f(y)|^p p_t(x,y)\mathrm{vol}(\mathrm{d}y)\mathrm{vol}(\mathrm{d}x) \qquad (140)\\
&= \int_M\int_M p_t(x,y)\mathrm{vol}(\mathrm{d}x)|f(y)|^p\mathrm{vol}(\mathrm{d}y) \\
&\leq \|f\|_p^p, \qquad (141)
\end{aligned}$$

where we have applied the Hölder inequality to the finite measure $\mu(\mathrm{d}y) = p_t(x,y)\mathrm{vol}(\mathrm{d}y)$ for the second inequality.

Case $1<p<q=\infty$: This works with the same argument that has been used for the inequality (140).

Case $1=p<q<\infty$: One has

$$\left\|\mathrm{e}^{\frac{t}{2}\Delta}f\right\|_q^q \leq \int_M\left(\int_M \left(p_t(x,y)^q|f(y)|\right)^{\frac{1}{q}}|f(y)|^{1-\frac{1}{q}}\mathrm{vol}(\mathrm{d}y)\right)^q\mathrm{vol}(\mathrm{d}x). \qquad (142)$$

Applying the Hölder inequality with the exponents

$$q_1 = q,\ \ q_2 = \frac{1}{1-\frac{1}{q}}$$

gives

$$\left\| \mathrm{e}^{\frac{t}{2}\Delta} f \right\|_q^q \leq \|f\|_1^{q-1} \int_M \int_M p_t(x,y)^q |f(y)| \mathrm{vol}(\mathrm{d}y) \mathrm{vol}(\mathrm{d}x), \tag{143}$$

so that the Fubini theorem and (138) imply

$$\left\| \mathrm{e}^{\frac{t}{2}\Delta} f \right\|_q^q \leq C(t)^{q\left(1-\frac{1}{q}\right)} \|f\|_1^q .$$

The cases $p = q = \infty$ and $p = q = 1$ and $p = 1, q = \infty$ are trivial.

■

Proof of proposition 8.3. a) Let $f \in \Gamma_{\mathrm{L}^p}(M, E)$ and remember (131). If $q < \infty$, then

$$\begin{aligned} \left\| \mathrm{e}^{-tH(V)} f \right\|_q &\leq \left(\int_M \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} \|f(B_t(x))\|_{B_t(x)} \right]^q \mathrm{vol}(\mathrm{d}x) \right)^{\frac{1}{q}} \\ &= \left\| \mathrm{e}^{-tH_0(v)} |f| \right\|_p \leq \left\| \mathrm{e}^{-tH_0(v)} \right\|_{p,q} \|f\|_p . \end{aligned}$$

If $q = \infty$,

$$\begin{aligned} \left\| \mathrm{e}^{-tH(V)} f \right\|_\infty &= \operatorname*{ess\,sup}_{x \in M} \left\| \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x)) \right] \right\|_x \\ &\leq \operatorname*{ess\,sup}_{x \in M} \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s} \|f(B_t(x))\|_{B_t(x)} \right] \\ &\leq \left\| \mathrm{e}^{-tH_0(v)} \right\|_{p,\infty} \|f\|_p . \end{aligned} \tag{144}$$

b) We have

$$\left\| \mathrm{e}^{-tH_0(C_V)} \right\|_{p,q} = \mathrm{e}^{-tC_V} \left\| \mathrm{e}^{\frac{t}{2}\Delta} \right\|_{p,q},$$

so that applying part a) to $v := C_V$ and using proposition 8.4 shows

$$\left\| \mathrm{e}^{-tH(V)} \right\|_{p,q} \leq \mathrm{e}^{-tC_V} \left\| \mathrm{e}^{\frac{t}{2}\Delta} \right\|_{p,q} \leq \mathrm{e}^{-tC_V} C(t)^{\frac{1}{p}-\frac{1}{q}}.$$

Since

$$C(t) = \sup_{x,y \in M} p_t(x,y) = \sup_{z \in M} p_t(z,z),$$

where the last equality follows easily from the properties of $p_t(x,y)$ (see for example [19], p.67 for details), the assertion follows from the heat kernel bound (214), noting that

$$\mathrm{e}^{-t \inf \sigma(-\Delta)} \leq 1.$$

■

Proposition 8.3 should be understood as follows:

Remark 8.5 Fix the assumptions of proposition 8.3 b). It follows from the arguments of the proof of proposition 8.14 that the operators $(Q_t^V)_{t>0}$ given formally by

$$Q_t^V f(x) := \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right]$$

form a semigroup of bounded operators in $\Gamma_{\mathrm{L}^p}(M, E)$. Under additional assumptions on V (probably local Kato will do), this semigroup will be strongly continuous. One could then define $H(V)_p$ to be the generator of $(Q_t^V)_{t>0}$ in $\Gamma_{\mathrm{L}^p}(M, E)$ and examine questions like the p-independence of $\sigma(H(V)_p)$. The reader may find assertions of this type for Schrödinger operators with magnetic fields in the Euclidean $\mathbb{R}^m$ in [42], and for certain uniformly elliptic operators on Riemannian manifolds in [80]. As we mainly have applications in quantum physics in mind, which corresponds to the Hilbert space case $p = 2$, we did not work further into this direction.

□

8.2 Integral kernels and trace estimates

Our next aim is to show that with some control on g and V, $\mathrm{e}^{-tH(V)}$ is an integral operator for any $t > 0$, and to derive a probabilistic formula for the corresponding integral kernel.

To this end, let $E \boxtimes E^* \to M \times M$ denote the tensor bundle corresponding to E, that is,

$$E \boxtimes E^* \mid_{(x,y)} = E_x \otimes E_y^* = \mathrm{Hom}(E_y, E_x). \tag{145}$$

We denote with $\|\bullet\|_{x,y}$ the operator norm in $\mathrm{Hom}(E_x, E_y)$ and equip the fibers of $E \boxtimes E^*$ with this norm in a continuous way, which means that for any [17] $\Psi \in \Gamma_{\mathrm{C}}(M \times M, E \boxtimes E^*)$ the function

$$M \times M \ni (x, y) \longmapsto \|\Psi(x, y)\|_{x,y} \in [0, \infty)$$

is continuous. Now we can prove:

Theorem 8.6 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius, and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then the following assertions hold for all $t > 0$:

[17] C stands for "continuous".

a) *The section*

$$\begin{aligned} M \times M \ni (x,y) &\longmapsto \mathrm{e}^{-tH(V)}(x,y) \in \mathrm{Hom}(E_y, E_x) \\ \mathrm{e}^{-tH(V)}(x,y) &:= p_t(x,y)\mathbb{E}_t^{x,y}\left[\mathscr{V}_t^x //_t^{x,-1}\right] \end{aligned}$$

in $E \boxtimes E^*$ *is well-defined for a.e.* $(x,y) \in M \times M$ *and it defines an integral kernel for the operator*

$$\mathrm{e}^{-tH(V)} : \Gamma_{\mathrm{L}^2}(M,E) \longrightarrow \Gamma_{\mathrm{L}^2}(M,E),$$

in the sense that for any $f \in \Gamma_{\mathrm{L}^2}(M,E)$*, a.e.* $x \in M$*,*

$$\mathrm{e}^{-tH(V)}f(x) = \int_M \mathrm{e}^{-tH(V)}(x,y)f(y)\mathrm{vol}(\mathrm{d}y). \tag{146}$$

b) *The integral kernel* $\mathrm{e}^{-tH(V)}(\bullet,\bullet)$ *is essentially bounded. More precisely, one has*

$$\left\|\mathrm{e}^{-tH(V)}(x,y)\right\|_{y,x} \leq C_t \mathrm{e}^{-tC_V} \quad \text{for a.e. } (x,y) \in M \times M,$$

where C_t *is an upper bound for* $p_t(\bullet,\bullet)$*.*

Proof. a),b) The proof is similar to the one of corollary 7.5.
Let us first remark the following fact: Since $//^{x,-1}$ is (essentially) defined as the solution of a stochastic differential equation that is driven by W, it follows that $//^{x,-1}$ is adapted to $\mathscr{F}_*$. By expanding $\mathscr{V}^x$ as a path ordered exponential as in (78) (even if V is not bounded, this is possible by theorem 4.3 in [22] and the remarks on p.55 there), one sees that $\mathscr{V}^x$ is also adapted to this filtration. Next, we note that in view of (120), lemma 5.8 implies

$$\|\mathscr{V}_t^x\|_x \leq \mathrm{e}^{-tC_V} \quad \mathbb{P}_t^{x,y}\text{-a.s. for a.e. } (x,y) \in M \times M,$$

and by (118), we also have that the parallel transport maps

$$//_t^{x,-1} : E_x \longrightarrow E_y$$

and that this map is an isometry, both under $\mathbb{P}_t^{x,y}$ for a.e. $(x,y) \in M \times M$. Altogether, it follows that $\mathbb{E}_t^{x,y}\left[\mathscr{V}_t^x //_t^{x,-1}\right]$ is well-defined for a.e. (x,y). Now (146) follows easily from the Feynman-Kac formula, (120), (118).

∎

In the following, we will consider the trace $\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right)$ as an element of $[0,\infty]$ and our next goal is to derive a probabilistic representation of this number. If $\mathrm{e}^{-tH(V)}(x,y)$ depended continuously on (x,y) (we believe that it is not even possible to *define* $\mathrm{e}^{-tH(V)}(x,y)$ *for all* $x,y\in M$ under our general assumptions on M and V ; see remark 8.7 for conditions under which this is possible), then it would be straightforward to derive a formula of the form

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right)=\int_M \mathrm{tr}_{E_x}\left(\mathrm{e}^{-tH(V)}(x,x)\right)\mathrm{vol}(\mathrm{d}x). \tag{147}$$

In order to avoid the necessity of this continuity, we proceed as follows: We will use the semigroup property of $(\mathrm{e}^{-tH(V)})_{t>0}$ to prove that

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right)=\int_M\int_M \mathrm{tr}_{E_y}\left(\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)^*\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right)\mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y), \tag{148}$$

which will be used in the following as a substitute for the literal interpretation of (147). In view of the first inequality of part b) of theorem 8.8, formula (148) will turn out to work equally well for our purpose, which is finding estimates for $\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right)$.

Remark 8.7 If one assumes that M has a bounded geometry, then one can use the Girsanov theorem and the same arguments as in [1] to prove that the bound (216) on the gradient of $p_t(x,y)$ implies the semi-martingale property of $B(x)\mid_{[0,t]\times\Omega}$ under $\mathbb{P}^{x,y}_t$. Assuming furthermore that

$$C_V\mathbf{1}\leq V\in\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E))\cap\Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M,\mathrm{End}(E))\quad\text{for some } C_V\in\mathbb{R},$$

one can use proposition 7.6 to see that $\mathrm{e}^{-tH(V)}(x,y)$ can be defined *for all* $x,y\in M$. We have not been able to prove or disprove the continuous dependence of $\mathrm{e}^{-tH(V)}(x,y)$ on (x,y) under these assumptions, which could be expected from the results of [13]. The main difficulty in proving this conjecture comes from the fact that the (local) Kato class seems to depend very sensitively on the Riemannian structure of M. This makes approximation arguments that could be motivated by proposition 2.3 in [13] not accessible for arbitrary manifolds with bounded geometry (the latter proposition asserts that in the Euclidean $\mathbb{R}^m$, one can locally approximate Kato decomposable potentials by smooth potentials with compact support in the so called Kato norm). We have only been able to derive a sketch of proof of the above conjecture under very restrictive additional assumptions on the Riemannian structure of M. $\square$

Now we prove:

Theorem 8.8 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. The following assertions hold for any $t > 0$:*

a) *Assume that V is a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then one has

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) = \int_M \int_M \mathrm{tr}_{E_y}\left(\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)^* \mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right) \mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y), \tag{149}$$

in the sense that either both sides are equal to a nonnegative real number or both sides are infinite.

b) *Assume that V is a potential with*

$$V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E))$$

and that v is a scalar potential with

$$C_v \mathbf{1} \leq v\mathbf{1} \leq V \quad \textit{for some } C_v \in \mathbb{R}.$$

Then one has [18]

$$\begin{aligned} \left\|\mathrm{e}^{-tH(V)}(x,y)\right\|_{y,x} &\leq \mathrm{e}^{-tH_0(v)}(x,y) \quad \textit{for a.e. } (x,y) \in M \times M, \textit{ and} \\ \mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) &\leq d \ \mathrm{tr}\left(\mathrm{e}^{-tH_0(v)}\right). \end{aligned} \tag{150}$$

Proof. a) Let $|||\bullet|||_{(x,y)}$ denote the Hilbert-Schmidt norm in $\mathrm{Hom}(E_x, E_y)$ and let $|||\bullet|||$ be the Hilbert-Schmidt norm in $\mathscr{L}(\Gamma_{\mathrm{L}^2}(M,E))$. The equality

$$\mathrm{e}^{-tH(V)} = \mathrm{e}^{-\frac{t}{2}H(V)}\mathrm{e}^{-\frac{t}{2}H(V)} \tag{151}$$

and the self-adjointness of $\mathrm{e}^{-\frac{t}{2}H(V)}$ imply that $\mathrm{e}^{-tH(V)}$ is trace class, if and only if $\mathrm{e}^{-\frac{t}{2}H(V)}$ is Hilbert-Schmidt and

$$\begin{aligned} \mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) &= \mathrm{tr}\left(\left(\mathrm{e}^{-\frac{t}{2}H(V)}\right)^* \mathrm{e}^{-\frac{t}{2}H(V)}\right) = \left|\left|\left|\mathrm{e}^{-\frac{t}{2}H(V)}\right|\right|\right|^2 \\ &= \int_M \int_M \left|\left|\left|\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right|\right|\right|^2_{(y,x)} \mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y). \end{aligned}$$

This proves formula (149).

[18] Remember that the number d stands for the dimension of the fibers of E.

b) The first inequality in (150) follows directly from the formulae for the integral kernels $\mathrm{e}^{-tH(V)}(\bullet,\bullet)$, $\mathrm{e}^{-tH_0(v)}(\bullet,\bullet)$ and proposition A.1 together with (120), so that in order to prove the second inequality, one can estimate as follows,

$$\begin{aligned}
\operatorname{tr}\left(\mathrm{e}^{-tH(V)}\right) &\leq \int_M\int_M \operatorname{tr}_{E_y}\left(\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)^*\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right)\mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y)\\
&\leq d\int_M\int_M \left\|\mathrm{e}^{-\frac{t}{2}H(V)}(x,y)\right\|^2_{(y,x)}\mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y)\\
&\leq d\ \operatorname{tr}\left(\mathrm{e}^{-tH_0(v)}\right).
\end{aligned}$$

This proves the theorem.

■

Remark 8.9 Let the assumptions of theorem 8.8 on M be satisfied. Local elliptic regularity [24] implies that there is a smooth integral kernel

$$(0,\infty)\times M\times M\ni(t,x,y)\longmapsto \mathrm{e}^{-t\widetilde{H}(0)}(x,y)\in\mathrm{Hom}(E_y,E_x)$$

such that for any $t>0$ one has

$$\mathrm{e}^{-t\widetilde{H}(0)}(x,y)=\mathrm{e}^{-tH(0)}(x,y)\quad\text{for a.e. }(x,y)\in M\times M.$$

The continuity of $\left\|\mathrm{e}^{-t\widetilde{H}(0)}(x,y)\right\|_{y,x}$ in (x,y) and theorem 8.8 b) imply

$$\left\|\mathrm{e}^{-t\widetilde{H}(0)}(x,y)\right\|_{y,x}\leq\ p_t(x,y),$$

for all $t>0$ and *all* $x,y\in M$. □

In the scalar Euclidean setting it is well-known that the quantum mechanical partition function is bounded by the classical partition function, that is, for a very large class of operators of the form $H_0(v)$ in $\mathrm{L}^2(\mathbb{R}^m)$ it holds that

$$\operatorname{tr}\left(\mathrm{e}^{-tH_0(v)}\right)\leq\frac{1}{(2\pi t)^{\frac{m}{2}}}\int_{\mathbb{R}^m}\mathrm{e}^{-tv(y)}\mathrm{d}y \tag{152}$$

$$=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\mathrm{e}^{-t\left(\frac{1}{2}\|x\|^2_{\mathbb{R}^m}+v(y)\right)}\mathrm{d}x\mathrm{d}y. \tag{153}$$

These inequalities are valid, for example, if v is Kato decomposable: $v_+\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^m)$ and $v_-\in\mathcal{K}(\mathbb{R}^m)$. The bound (152) is known as the *Golden-Thompson-Symanzik* inequality. We will prove direct extensions of these

bounds with classical (= probabilistic) methods from [77] to Yang-Mills Hamiltonians in the Euclidean $\mathbb{R}^m$ in section 8.4.
The following theorem asserts a Golden-Thompson-Symanzik type inequality for manifolds with bounded geometry, which is in the spirit of (152). Furthermore, this inequality easily implies a phase space bound for small times, which is an extension of (153) to this general setting. The classical proof of (152) uses the invariance of the Lebesgue measure in $\mathbb{R}^m$ under translations (see also the proof of (209)) and thus is not accessible in the setting of manifolds. However, one can use operator-theoretic methods to prove:

Theorem 8.10 *Let M have a bounded geometry and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

There is a constant $C > 0$, which only depends on the Riemannian structure of M, such that the following assertions hold:
a) *For any $t > 0$ one has*

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq \frac{Cd}{\min\{t^{\frac{m}{2}}, 1\}} \int_M \mathrm{e}^{-t\underline{V}(y)} \mathrm{vol}(\mathrm{d}y), \tag{154}$$

where $\underline{V}(y) := \min \sigma(V(y))$. In particular, if there is a $t > 0$ such that

$$\int_M \mathrm{e}^{-t\underline{V}(y)} \mathrm{vol}(\mathrm{d}y) < \infty, \tag{155}$$

then

$$\mathrm{e}^{-tH(V)} \in \mathscr{J}_1\left(\Gamma_{\mathrm{L}^2}(M,E)\right),$$

the trace class of $\Gamma_{\mathrm{L}^2}(M,E)$.
b) *One has the following phase space type bound for all $0 < t \leq 1$,*

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq Cd \int_M \int_{\mathrm{T}^*_y M} \mathrm{e}^{-t\left(\frac{1}{2}\|x\|^2_{\mathrm{T}^*_x M} + \underline{V}(y)\right)} \mathrm{vol}_{\mathrm{T}^*_y M}(\mathrm{d}x) \mathrm{vol}(\mathrm{d}y).$$

Remark 8.11 Here, the measure $\mathrm{vol}_{\mathrm{T}^*_y M}(\bullet)$ stands for the pushforward measure of the Lebesgue measure in $\mathbb{R}^m$ under some orthonormal frame

$$A : \mathbb{R}^m \longrightarrow \mathrm{T}^*_y M.$$

This definition does not depend on the particular choice of A. $\square$

Proof of theorem 8.10. a) In the following, we consider V as a maximally defined multiplication operator in $\Gamma_{\mathrm{L}^2}(M,E)$. So we have the domains of definition

$$\begin{aligned} \mathrm{D}(V) &= \Big\{ f \Big| \; f, Vf \in \Gamma_{\mathrm{L}^2}(M,E) \Big\}, \\ \mathrm{D}(H(0)) &= \Big\{ f \Big| \; f,\, \nabla^*\nabla f \text{ (as a distrib.) } \in \Gamma_{\mathrm{L}^2}(M,E) \Big\}. \end{aligned} \tag{156}$$

By the essential self-adjointness of $\nabla^*\nabla/2 + V$ on $\Gamma_{\mathrm{C}_0^\infty}(M,E)$ and

$$\Gamma_{\mathrm{C}_0^\infty}(M,E) \subset \mathrm{D}(H(0)) \cap \mathrm{D}(V),$$

it is clear that $H(0)+V$ is essentially self-adjoint on the domain $\mathrm{D}(H(0)) \cap \mathrm{D}(V)$. As a consequence, we can use the operator version of the Golden-Thompson inequality (theorem 4 in [67]) to obtain

$$\mathrm{tr}\left(\mathrm{e}^{-tH(V)}\right) \leq \mathrm{tr}\left(\mathrm{e}^{-\frac{t}{2}H(0)}\mathrm{e}^{-tV}\mathrm{e}^{-\frac{t}{2}H(0)}\right).$$

Since $\mathrm{e}^{-\frac{t}{2}V}\mathrm{e}^{-\frac{t}{2}H(0)}$ is an integral operator with an integral kernel given by

$$M \times M \ni (x,y) \longmapsto \mathrm{e}^{-\frac{t}{2}V(x)}\mathrm{e}^{-\frac{t}{2}H(0)}(x,y) \in \mathrm{Hom}(E_y, E_x),$$

one has

$$\begin{aligned} &\mathrm{tr}\left(\mathrm{e}^{-\frac{t}{2}H(0)}\mathrm{e}^{-tV}\mathrm{e}^{-\frac{t}{2}H(0)}\right) \\ &= \mathrm{tr}\left(\left(\mathrm{e}^{-\frac{t}{2}V}\mathrm{e}^{-\frac{t}{2}H(0)}\right)^* \mathrm{e}^{-\frac{t}{2}V}\mathrm{e}^{-\frac{t}{2}H(0)}\right) \\ &= \left|\left|\left| \mathrm{e}^{-\frac{t}{2}V}\mathrm{e}^{-\frac{t}{2}H(0)} \right|\right|\right|^2 \\ &= \int_M \int_M \mathrm{tr}_{E_y}\left(\left(\mathrm{e}^{-\frac{t}{2}V(x)}\mathrm{e}^{-\frac{t}{2}H(0)}(x,y)\right)^* \mathrm{e}^{-\frac{t}{2}V(x)}\mathrm{e}^{-\frac{t}{2}H(0)}(x,y)\right) \mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y), \end{aligned}$$

so that we can estimate as follows:

$$\begin{aligned} &\mathrm{tr}\left(\mathrm{e}^{-\frac{t}{2}H(0)}\mathrm{e}^{-tV}\mathrm{e}^{-\frac{t}{2}H(0)}\right) \\ &\leq d \int_M \int_M \left\| \mathrm{e}^{-tV(x)} \right\|_x \left\| \mathrm{e}^{-\frac{t}{2}H(0)}(x,y) \right\|_{y,x}^2 \mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y) \\ &\leq d \int_M \int_M \mathrm{e}^{\frac{t}{2}\Delta}(x,y)\mathrm{e}^{\frac{t}{2}\Delta}(y,x)\mathrm{vol}(\mathrm{d}y) \left\| \mathrm{e}^{-tV(x)} \right\|_x \mathrm{vol}(\mathrm{d}x) \\ &= d \int_M p_t(x,x) \left\| \mathrm{e}^{-tV(x)} \right\|_x \mathrm{vol}(\mathrm{d}x) \\ &\leq \frac{\tilde{C}d}{\min\{t^{\frac{m}{2}}, 1\}} \mathrm{e}^{-t \inf \sigma(-\Delta)} \int_M \mathrm{e}^{-t\underline{V}(x)} \mathrm{vol}(\mathrm{d}x). \end{aligned}$$

Here, we have used theorem 8.8 b) for the second step, the Chapman-Kolmogorov equation for $p_t(x,y)$ for the third step, and the bound (214) for the last step. Since

$$\mathrm{e}^{-t\inf\sigma(-\Delta)}\leq 1,$$

we are done.

b) This follows from part a) and

$$\frac{1}{(2\pi t)^{\frac{m}{2}}}=\int_{\mathbb{R}^m}\mathrm{e}^{-\frac{1}{2}t\|x\|^2_{\mathbb{R}^m}}\mathrm{d}x,$$

which is valid for all $t>0$.

■

8.3 Spacial continuity of the Schrödinger semigroup

As a next goal, we want to prove one of our main results: If one has the same control on the geometry of M that has been necessary to define the Brownian bridge measures in a satisfacorty way, and if V is in the local Kato class, then the operator $\mathrm{e}^{-tH(V)}$ maps

$$\mathrm{e}^{-tH(V)}:\Gamma_{\mathrm{L}^2}(M,E)\longrightarrow\Gamma_{\mathrm{C}_b}(M,E)\cap\Gamma_{\mathrm{L}^2}(M,E)$$

in the obvious sense. In detail, this is:

Theorem 8.12 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. Assume furthermore that V is a potential with*

$$C_V\mathbf{1}\leq V\in\Gamma_{\mathcal{K}_{\mathrm{loc}}}(M,\mathrm{End}(E))\cap\Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M,\mathrm{End}(E))\quad\textit{for some } C_V\in\mathbb{R}.$$

Then for any $t>0$, $f\in\Gamma_{\mathrm{L}^2}(M,E)$, the section

$$M\longrightarrow E,\;\; x\longmapsto\mathbb{E}\left[\mathscr{V}^x_t//^{x,-1}_t f(B_t(x))\right]\in E_x \tag{157}$$

is continuous and bounded. In particular, each eigensection of $H(V)$ can be chosen continuous and bounded.

Remark 8.13 Let us explain our approach for proving theorem 8.12: In the situation of theorem 8.12, let

$$Q^V_t f(x):=\mathbb{E}\left[\mathscr{V}^x_t//^{x,-1}_t f(B_t(x))\right]. \tag{158}$$

Note that under our assumptions on M and V, the right-hand side of (158) is indeed well-defined for *all* $x \in M$ (see lemma 6.5). We will use semigroup domination and a bound on $p_t(x,y)$ to prove that $Q_t^V f$ is bounded. Furthermore, one can prove that $Q_\bullet^V f(x)$ satisfies a semigroup property for all $x \in M$ (a priori, this is only clear for *a.e.* $x \in M$, and the proof that it remains true for *all* $x \in M$ is actually quite technical). From these considerations, it is clear that we may assume

$$f \in \Gamma_{\mathrm{L}^\infty}(M,E) \cap \Gamma_{\mathrm{L}^2}(M,E).$$

Next, note that one can expect from elliptic regularity that $Q_t^0 \tilde{f}$ is continuous (in fact, smooth) for any $t > 0$ and any essentially bounded square integrable $\tilde{f}$, so that the continuity of $Q_t^V f$ will follow, if we can locally uniformly approximate $Q_t^V f$ as $s \searrow 0$ by $Q_s^0 Q_{t-s}^V f$. This will in fact follow from the perturbation formula (161) below and the convergence (185). The latter of which strongly relies on the assumption that the potential is in the local Kato class. These techniques extend the corresponding ones from [17] (see also [13]) for usual scalar operators to our setting, where we remark that the proofs of assertions like proposition 8.14, proposition 8.15 or proposition 8.18 are almost trivial in the setting of [17]. □

The following four propositions will help us to turn the considerations of remark 8.13 into a full proof.
We first prove the asserted semigroup property and the perturbation formula:

Proposition 8.14 *Let M be geodesically and stochastically complete and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathcal{K}_{\mathrm{loc}}}(M, \mathrm{End}(E)) \cap \Gamma_{\mathrm{L}^2_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

We set

$$Q_t^V f(x) := \mathbb{E}\left[\mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right] \quad \textit{for any } t \geq 0,\ x \in M. \tag{159}$$

a) *$Q_\bullet^V f$ satisfies a pointwise semigroup identity,*

$$Q_{s+t}^V f(x) = Q_s^V Q_t^V f(x) \quad \textit{for any } s,t \geq 0,\ x \in M. \tag{160}$$

b) *One has the following perturbation formula for any $t \geq s \geq 0$, $x \in M$,*

$$Q_s^0 Q_{t-s}^V f(x) = \mathbb{E}\left[\mathscr{V}_s^{x,-1} \mathscr{V}_t^x //_t^{x,-1} f(B_t(x))\right]. \tag{161}$$

Proof. Firstly, note that lemma 6.5 implies that all expressions that are involved are well-defined for all $x \in M$. Furthermore, for any starting time $a \geq 0$ and any appropiate $\mathscr{F}_a$-measurable $h : \Omega \to M$, we define the processes $B^{a,h}$, $//^{a,h}$ and $\mathscr{V}^{a,h}$ as follows:

$$B^{a,h} : [a, \infty) \times \Omega \longrightarrow M$$

is defined as the maximal solution of

$$\mathrm{d}B^{a,h} = \sum_{j=1}^{l} A_j(B^{a,h})\underline{\mathrm{d}}W^j, \;\; B_a^{a,h} = h,$$

$//^{a,h}$ is defined as the stochastic parallel transport corresponding to $B^{a,h}$, so that

$$//_b^{a,h} : E_h \longrightarrow E_{B_b^{a,h}} \quad \text{for any } b \geq a,$$

and, finally, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the map

$$\mathscr{V}_\bullet^{a,h}(\omega) : [0, \infty) \longrightarrow \mathrm{End}(E)_{h(\omega)}$$

is defined as the weak solution of

$$\begin{aligned} \mathrm{d}\mathscr{V}_t^{a,h}(\omega) &= -\mathscr{V}_t^{a,h}(\omega)\Big(//_{a+t}^{a,h,-1} V(B_{a+t}^{a,h}) //_{a+t}^{a,h}\Big)(\omega)\mathrm{d}t, \\ \mathscr{V}_0^{a,h}(\omega) &= \mathbf{1}. \end{aligned}$$

Note that our usual notation implies

$$(B^{0,x}, //^{0,x}, \mathscr{V}^{0,x}) = (B(x), //^x, \mathscr{V}^x).$$

a) Let U^x be a lift of $B(x)$ and let $U^{s,B_s(x)}$ be the lift of $B^{s,B_s(x)}$ from U_s^x. Then proposition 2.17 implies $//^x = U^x U_0^{x,-1}$ and $//^{s,B_s(x)} = U^{s,B_s(x)} U_s^{x,-1}$, so that theorem 2.16 and the flow property [60] of the solutions of

$$\mathrm{d}U = \sum_{j=1}^{l} A_j^*(U)\underline{\mathrm{d}}W^j \tag{162}$$

show

$$//_{s+t}^x = //_{s+t}^{s,B_s(x)} //_s^x \quad \mathbb{P}\text{-a.s.} \tag{163}$$

Using (163) and the flow property $B_{s+t}(x) = B_{s+t}^{s,B_s(x)}$ of the solutions of

$$\mathrm{d}B = \sum_{j=1}^{l} A_j(B)\underline{\mathrm{d}}W^j, \tag{164}$$

one easily checks that for fixed s, the processes

$$\mathscr{V}^{x}_{s+\bullet} \quad \text{and} \quad \mathscr{V}^{x}_{s}//^{x,-1}_{s}\mathscr{V}^{s,B_s(x)}_{\bullet}//^{x}_{s}$$

both solve the same $\mathrm{End}(E)_x$-valued initial value problem, so that by uniqueness and (163) we get the multiplicative property

$$\mathscr{V}^{x}_{s+t}//^{x,-1}_{s+t} = \mathscr{V}^{x}_{s}//^{x,-1}_{s}\mathscr{V}^{s,B_s(x)}_{t}//^{s,B_s(x),-1}_{s+t} \quad \mathbb{P}\text{-a.s.} \tag{165}$$

With $\mathbb{E}^{\mathscr{F}_s}[\bullet] := \mathbb{E}[\bullet|\mathscr{F}_s]$ the last identity implies

$$\begin{aligned} Q^{V}_{s+t}f(x) &= \mathbb{E}\left[\mathscr{V}^{x}_{s}//^{x,-1}_{s}\mathscr{V}^{s,B_s(x)}_{t}//^{s,B_s(x),-1}_{s+t}f(B_{s+t}(x))\right] \\ &= \mathbb{E}\left[\mathscr{V}^{x}_{s}//^{x,-1}_{s}\mathbb{E}^{\mathscr{F}_s}\left[\mathscr{V}^{s,B_s(x)}_{t}//^{s,B_s(x),-1}_{s+t}f\left(B^{s,B_s(x)}_{s+t}\right)\right]\right]. \end{aligned} \tag{166}$$

Now lemma 6.31 in [37] gives

$$\begin{aligned} &\mathbb{E}\left[\mathscr{V}^{x}_{s}//^{x,-1}_{s}\mathbb{E}^{\mathscr{F}_s}\left[\mathscr{V}^{s,B_s(x)}_{t}//^{s,B_s(x),-1}_{s+t}f\left(B^{s,B_s(x)}_{s+t}\right)\right]\right] \\ &= \int_\Omega \mathscr{V}^{x}_{s}(\omega)//^{x,-1}_{s}(\omega)\int_\Omega Z^{s,B_s(x)(\omega)}_{t}(\tilde\omega)\mathbb{P}(\mathrm{d}\tilde\omega)\ \mathbb{P}(\mathrm{d}\omega), \end{aligned} \tag{167}$$

where we have set

$$Z^{a,y}_{t} := \mathscr{V}^{a,y}_{t}//^{a,y}_{a+t}f\left(B^{a,y}_{a+t}\right) \ \text{ for any } a \geq 0,\ y \in M.$$

(To be exact, lemma 6.31 in [37] is only directly applicable, if f is bounded. The general case can easily be deduced with a dominated convergence argument, if one applies this result to $f_n := 1_{\mathrm{K}_n(x)}f$ and lets $n \to \infty$.) It remains to prove that

$$\mathbb{E}\left[Z^{s,y}_{t}\right] = \mathbb{E}\left[Z^{0,y}_{t}\right] \quad \text{for any } y \in M. \tag{168}$$

To this end, we first remark that the processes $B^{s,y}_{s+\bullet}$ and $B(y)$ have the same law: Indeed, the smoothness of the vector fields A_j implies the uniqueness in law for (164) (this follows from theorem 1.1.10 in [41] and the Whitney embedding theorem). Now one can use the same arguments as in the proof of corollary 1 to Satz 6.40 in [37] to deduce that $B^{s,y}_{s+\bullet}$ and $B(y)$ are equal in law.
This equality in law shows that we can use the same arguments as in the proof of theorem 5.5 (see in particular lemma 5.8) to restrict ourselves to the case

$$V \in \Gamma_{\mathrm{L}^\infty}(M, \mathrm{End}(E)).$$

Let $\pi : \mathrm{P}(E) \to M$ denote the principal bundle projection, let U^y be a lift of $B(y)$ and let $U^{s,y}$ be the lift of $B^{s,y}$ from U_0^y. Proposition 2.17 implies $//^{s,y} = U^{s,y} U_0^{y,-1}$ and clearly we have $B^{s,y} = \pi(U^{s,y})$, $//^y = U^y U_0^{y,-1}$, $B(y) = \pi(U^y)$. For any $y \in M$ and $n \in \mathbb{N}$ we define a function $\mathscr{A}_n^{t,y}$ by setting

$$\begin{aligned}
&\mathscr{A}_n^{t,y} : \mathrm{C}([0,\infty), \mathrm{P}(E)) \longrightarrow E_y, \\
&\mathscr{A}_n^{t,y}(\gamma) := \left\{ \overrightarrow{\prod_{1\leq j\leq n}} \left(\mathbf{1} + \frac{t}{n} U_0^y \gamma[(tj)/n]^{-1} V\Big(\pi\left(\gamma[(tj)/n]\right)\Big)\gamma[(tj)/n] U_0^{y,-1} \right) \right\} \\
&\qquad \times U_0^y \gamma[t]^{-1} f\Big(\pi\left(\gamma[t]\right)\Big).
\end{aligned} \tag{169}$$

Then we have the following inequalities,

$$\left\| \mathscr{A}_n^{t,y}(U_{s+\bullet}^{s,y}) \right\|_y \leq \mathrm{e}^{t\|V\|_\infty} \left\| f(B_{s+t}^{s,y}) \right\|_{B_{s+t}^{s,y}} \quad \mathbb{P}\text{-a.s.} \tag{170}$$

and

$$\left\| \mathscr{A}_n^{t,y}(U^y) \right\|_y \leq \mathrm{e}^{t\|V\|_\infty} \left\| f(B_t(y)) \right\|_{B_t(y)} \quad \mathbb{P}\text{-a.s.} \tag{171}$$

Since $\mathscr{V}^{s,y}$ and $\mathscr{V}^y$ can be represented as product integrals (this follows from applying theorem 7.1 in [22] with $z \mapsto 1 + z$ together with the corresponding remarks on page 56), one has

$$\lim_{n\to\infty} \mathscr{A}_n^{t,y}(U_{s+\bullet}^{s,y}) = Z_t^{s,y} \quad \text{and} \quad \lim_{n\to\infty} \mathscr{A}_n^{t,y}(U^y) = Z_t^{0,y} \quad \mathbb{P}\text{-a.s.}$$

With the same arguments as above for $B^{\bullet,\bullet}$, one finds that $U_{s+\bullet}^{s,y}$ and U^y also have the same law, so that we can use dominated convergence (in view of (170) and (171)) to deduce

$$\mathbb{E}\left[Z_t^{s,y}\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathscr{A}_n^{t,y}(U_{s+\bullet}^{s,y})\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathscr{A}_n^{t,y}(U^y)\right] = \mathbb{E}\left[Z_t^{0,y}\right].$$

b) One has

$$\begin{aligned}
Q_s^0 Q_{t-s}^V f(x) &= \int_\Omega //_s^{x,-1}(\omega) \int_\Omega \mathscr{V}_{t-s}^{B_s(x)(\omega)}(\tilde{\omega}) //_{t-s}^{B_s(x)(\omega)}(\tilde{\omega}) \\
&\qquad\qquad \times f\left(B_{t-s}\Big(B_s(x)(\omega)\Big)(\tilde{\omega})\right) \mathbb{P}(\mathrm{d}\tilde{\omega})\, \mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega //_s^{x,-1}(\omega) \int_\Omega \mathscr{V}_{t-s}^{s,B_s(x)(\omega)}(\tilde{\omega}) //_t^{s,B_s(x)(\omega)}(\tilde{\omega}) \\
&\qquad\qquad \times f\left(B_t^{s,B_s(x)(\omega)}(\tilde{\omega})\right) \mathbb{P}(\mathrm{d}\tilde{\omega})\, \mathbb{P}(\mathrm{d}\omega) \\
&= \mathbb{E}\left[//_s^{x,-1} \mathbb{E}^{\mathscr{F}_s}\left[//_s^x \mathscr{V}_s^{x,-1} \mathscr{V}_t^x //_t^{x,-1} f\left(B_t^{s,B_s(x)}\right)\right]\right] \\
&= \mathbb{E}\left[\mathscr{V}_s^{x,-1} \mathscr{V}_t^x //_t^{x,-1} f\left(B_t(x)\right)\right],
\end{aligned} \tag{172}$$

where we have used (168) for the second equality, lemma 6.3.1 in [37] as in the proof of part a) together with (165) for the third equality, and the flow property of (164) for the last equality.

■

Next, we will prove:

Proposition 8.15 *Let M be geodesically complete with Ricci curvature bounded from below and a positive injectivity radius. Then for any $t > 0$ and any*

$$f \in \Gamma_{\mathrm{L}^\infty}(M, E) \cap \Gamma_{\mathrm{L}^2}(M, E),$$

the section given by

$$M \longrightarrow E,\ \ x \longmapsto Q_t^0 f(x) = \mathbb{E}\left[//_t^{x,-1} f(B_t(x))\right] \in E_x \tag{173}$$

is smooth.

Proof. We owe Anton Thalmaier the crucial idea (which is to use formula (175) below) for the following proof. Let $Q_t f := Q_t^0 f$. By remark 8.9, $\mathrm{e}^{-tH(0)} f$ has a smooth representative which is given by

$$f_t(\bullet) := \int_M \mathrm{e}^{-t\widetilde{H}(0)}(\bullet, y) f(y) \mathrm{vol}(\mathrm{d}y).$$

Furthermore, the map $(t, x) \mapsto f_t(x)$ is smooth with $f_t \to f$ in the sense of $\|\bullet\|$ as $t \searrow 0$ and

$$\partial_t f_t(x) = -\frac{1}{2} \nabla^* \nabla f_t(x) \ \text{ for all } t > 0,\ x \in M. \tag{174}$$

We fix arbitrary $x \in M$ and $t > 0$ now. Then (75) and (174) give

$$\begin{aligned}
&\mathrm{d}_s\Big(//_s^{x,-1} f_{t-s}(B_s(x))\Big) \\
&= //_s^{x,-1} \sum_{j=1}^{l} (\nabla_{A_j} f_{t-s})(B_s(x)) \mathrm{d}W_s^j - \frac{1}{2} //_s^{x,-1} \nabla^* \nabla f_{t-s}(B(x)) \mathrm{d}s \\
&\quad + //_s^{x,-1} \partial_s f_{t-s}(B_s(x)) \mathrm{d}s \\
&= //_s^{x,-1} \sum_{j=1}^{l} (\nabla_{A_j} f_{t-s})(B_s(x)) \mathrm{d}W_s^j,
\end{aligned} \tag{175}$$

so that the process

$$N : [0, t) \times \Omega \longrightarrow E_x,\ \ N_s := //_s^{x,-1} f_{t-s}(B_s(x))$$

is a continuous local martingale. It is in fact a martingale: For any $0 \leq s < t$ the following inequalities hold $\mathbb{P}$-a.s.,

$$
\begin{aligned}
\|N_s\|_x &\leq \|f_{t-s}(B_s(x))\|_{B_s(x)} \\
&= \left\| \int_M \mathrm{e}^{-(t-s)\tilde{H}(0)}(B_s(x), y) f(y) \mathrm{vol}(\mathrm{d}y) \right\|_{B_s(x)} \\
&\leq \|f\|_\infty \int_M \left\| \mathrm{e}^{-(t-s)\tilde{H}(0)}(B_s(x), y) \right\|_{y, B_s(x)} \mathrm{vol}(\mathrm{d}y) \\
&\leq d \|f\|_\infty \int_M p_{t-s}(B_s(x), y) \mathrm{vol}(\mathrm{d}y) = d \|f\|_\infty , \qquad (176)
\end{aligned}
$$

where we have used remark 8.9, so the martingale property of N follows from a standard criterion (see for example p.129 in [66]). This shows that for all $0 \leq s < t$,

$$f_t(x) = \mathbb{E}[N_0] = \mathbb{E}[N_s],$$

so that the proposition follows from dominated convergence, in view of (176) and $N_s \to //_t^{x,-1} f(B_t(x))$ as $s \nearrow t$, $\mathbb{P}$-a.s.

■

The next proposition is concerned with the first exit time of $B(x)$ from open geodesic balls, where x runs through a compact set. Although the arguments of the (alternative) proof that we are going to present are certainly well-known from proofs of stochastic completeness, the result itself has not yet appeared in the literature, as far as we know.

Proposition 8.16 *Let M be geodesically complete with Ricci curvature bounded from below. Let $K \neq \emptyset$ be a compact subset of M, fix some origin $x_0 \in K$, and for any $t > 0$, $x \in K$, $r > 0$ let*

$$\chi(r, t, x) := 1_{\{t < \zeta(r,x)\}},$$

where $\zeta(r, x)$ stands for the first exit time of $B(x)$ from $\mathrm{K}_r(x_0)$. Then one has

$$\lim_{r \to \infty} \sup_{x \in K} \mathbb{E}[1 - \chi(r, t, x)] = 0 \quad \text{for any } t > 0. \qquad (177)$$

Remark 8.17 If M is the Euclidean $\mathbb{R}^m$, then (177) follows from Lévy's maximal inequality. This has been pointed out in [13].

□

Proposition 8.16 can be seen as follows: Theorem 5.40 in [79] (the corresponding proof is purely analytical) implies the existence of constants

$$C_1(M) > 0, \ \ C_2(t) > 0, \ \ C_3(M,t) > 0$$

such that for any $x \in M$ and any $r > 0$ one has

$$\mathbb{E}\left[1 - \chi(r,t,x)\right] \leq C_1(M) \mathrm{e}^{C_2(t)\mathrm{d}(x_0,x)} \mathrm{e}^{-C_3(M,t)r^2}. \tag{178}$$

Bounding $\mathrm{d}(x_0, \bullet)$ on K, this obviously implies (177).
As we have already mentioned, we are going to give a stochastic analysis proof of proposition 8.16:

Alternative proof of proposition 8.16. Note that (177) is nothing but

$$\lim_{r\to\infty} \inf_{x\in K} \mathbb{P}\{t < \zeta(r,x)\} = 1 \quad \text{for any } t > 0.$$

Since M is stochastically complete, we can assume $K \neq \{x_0\}$. Let $R(x) := \mathrm{d}(x_0, x)$. Then R is a smooth function on the open set

$$\tilde{M} := M \setminus (\mathrm{Cut}(x_0) \cup \{x_0\}).$$

If $C > 0$ is such that the Ricci curvature of M is bounded from below by $-C$, then by a standard argument of differential geometry, which uses the index lemma and the second variation formula, one can easily deduce [19]

$$\Delta R(x) \leq h(R(x)) \quad \text{for all } x \in \tilde{M}, \tag{179}$$

where

$$h : (0,\infty) \longrightarrow (0,\infty), \ \ h(r) := \frac{m-1}{r} + \frac{C}{3} r.$$

Furthermore, although $R \notin \mathrm{C}^\infty(M)$, the process $R(B(x))$ is a continuous semi-martingale (this is a highly nontrivial result [50]) which satisfies

$$R(B_t(x)) - R(x) = Z^x_t + \int_0^t \Delta R(B_s(x))\mathrm{d}s - L^x_t \quad \mathbb{P}\text{-a.s.} \tag{180}$$

for any $t \geq 0$, $x \in M$, where Z^x is a Brownian motion which starts in 0, L^x is a continuous nondecreasing process which starts in 0, and where the integral can be defined since $B(x)$ does not spend time in $M \setminus \tilde{M}$ (this follows from

[19] For example, (179) can be seen by replacing $K_i(s)$ in the proof of theorem 3.4.3 in [41] with $(sX_i)/r$. This result also follows directly from setting $\varphi(t) := t/r$ in inequality (2.3) of [62].

the well-known fact that $\mathrm{Cut}(x_0) \cup \{x_0\}$ is a null set; see p.527 in [37] for details). We set

$$C_K := \max_{y \in K} R(y) \ \ (> 0).$$

For any $x \in M$ let Y^x be the uniquely determined maximal solution of

$$\mathrm{d}Y^x = \mathrm{d}Z^x + h(Y^x)\mathrm{d}t, \ \ Y_0^x = C_K. \tag{181}$$

The Feller explosion test as formulated in proposition 4.2.2 in [41] can be checked with elementary estimates to prove that Y^x is nonexplosive. Furthermore, (179), (180) and a classical comparison theorem (theorem 1.1 in [44]) imply

$$R(B_t(x)) \leq Y_t^x \quad \mathbb{P}\text{-a.s. for any } t \geq 0,\ x \in K. \tag{182}$$

Now (182) shows the following uniform estimate in x: For any $t \geq 0$ and $r > 0$,

$$\begin{aligned}
\inf_{x \in K} \mathbb{P}\{t < \zeta(r,x)\} &= \inf_{x \in K} \mathbb{P}\left\{R(B_s(x)) < r \text{ for all } s \in [0,t]\right\} \\
&\geq \inf_{x \in K} \mathbb{P}\left\{Y_s^x < r \text{ for all } s \in [0,t]\right\} \\
&= \inf_{x \in K} \mathbb{P}\left\{Y_s^{x_0} < r \text{ for all } s \in [0,t]\right\} \\
&= \mathbb{P}\left\{Y_s^{x_0} < r \text{ for all } s \in [0,t]\right\},
\end{aligned} \tag{183}$$

where we have used that the smoothness of h implies uniqueness in law for the pair $(1,h)$ (theorem 1.1.10 in [41]). Since Y^{x_0} does not explode, the last term in (183) goes to 1 as $r \to \infty$, and the proof is complete.

■

We will use proposition 8.16 to prove part b) of:

Proposition 8.18 a) *Let M be stochastically complete and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathcal{K}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then one has

$$\lim_{t \searrow 0} \sup_{x \in M} \mathbb{E}\left[\left\|\mathbf{1} - \mathscr{V}_t^x\right\|_x^2\right] = 0. \tag{184}$$

b) *Let M be geodesically complete with Ricci curvature bounded from below and let V be a potential with*

$$C_V \mathbf{1} \leq V \in \Gamma_{\mathcal{K}_{\mathrm{loc}}}(M, \mathrm{End}(E)) \quad \textit{for some } C_V \in \mathbb{R}.$$

Then for all compact $K \subset M$ one has

$$\lim_{t \searrow 0} \sup_{x \in K} \mathbb{E}\left[\|\mathbf{1} - \mathscr{V}_t^x\|_x^2\right] = 0. \tag{185}$$

Proof. a) Let $f_1, \dots, f_d$ be a global orthonormal frame for E. Of course, these sections in E cannot be chosen smooth in general, but they can always be chosen measurable, so

$$f_1, \dots, f_d \in \Gamma_{\mathrm{L}_0}(M, E).$$

For any $x \in M$ we define a $\mathrm{Mat}(\mathbb{C}^d)$-valued process $\mathscr{A}(x, \bullet)$ by setting

$$\mathscr{A}_j^i(x,t) := \langle f_i(x), \mathscr{V}_t^x f_j(x) \rangle_x .$$

Then it is sufficient to prove that for all $i, j = 1, \dots, d$ one has

$$\lim_{t \searrow 0} \sup_{x \in M} \mathbb{E}\left[\left|\delta_j^i - \mathscr{A}_j^i(x,t)\right|^2\right] = 0. \tag{186}$$

The following (in)equalities are all valid $\mathbb{P}$-a.s. One has

$$\frac{\mathrm{d}}{\mathrm{d}t} \left|\delta_j^i - \mathscr{A}_j^i(x,t)\right|^2 = 2\mathrm{Re}\left\{ \left(\delta_j^i - \mathscr{A}_j^i(x,t)\right) \overline{\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_j^i - \mathscr{A}_j^i(x,t)\right)} \right\},$$

so that integrating this equality and using the differential equation for $\mathscr{V}^x$ we get

$$\begin{aligned} &\left|\delta_j^i - \mathscr{A}_j^i(x,t)\right|^2 \\ &= \int_0^t 2\mathrm{Re}\left\{ \left(\delta_j^i - \mathscr{A}_j^i(x,s)\right) \overline{\left\langle f_i(x), \mathscr{V}_s^x //_s^{x,-1} V(B_s(x)) //_s^x f_j(x) \right\rangle_x} \right\} \mathrm{d}s. \end{aligned}$$

This implies

$$\begin{aligned} &\left|\delta_j^i - \mathscr{A}_j^i(x,t)\right|^2 \\ &\leq 2 \int_0^t \left|\delta_j^i - \mathscr{A}_j^i(x,s)\right| \left|\left\langle f_i(x), \mathscr{V}_s^x //_s^{x,-1} V(B_s(x)) //_s^x f_j(x) \right\rangle_x\right| \mathrm{d}s, \end{aligned}$$

so that noting that for $0 \leq s \leq t$,

$$\|\mathscr{V}_s^x\|_x \leq C_t := \max\left\{\mathrm{e}^{-C_V t}, 1\right\}, \ \left|\mathscr{A}_j^i(x,s)\right| \leq C_t,$$

we arrive at

$$\sup_{x \in M} \mathbb{E}\left[\left|\delta_j^i - \mathscr{A}_j^i(x,t)\right|^2\right] \leq 2\left(1 + C_t\right) C_t \sup_{x \in M} \mathbb{E}\left[\int_0^t \|V(B_s(x))\|_{B_s(x)} \, \mathrm{d}s\right]. \tag{187}$$

The last expression tends to 0 as $t \searrow 0$ by the definition of the Kato class.
b) We use a standard localization procedure. Let K, x_0 and $\chi(r,t,x)$ be as in proposition 8.16. Then we have

$$\begin{aligned} &\sup_{x\in K} \mathbb{E}\left[\Big(1-\chi(r,t,x)+\chi(r,t,x)\Big)\,\|\mathbf{1}-\mathscr{V}_t^x\|_x^2\right] \\ &\leq C_t \sup_{x\in K}\mathbb{E}\left[1-\chi(r,t,x)\right]+\sup_{x\in K}\mathbb{E}\left[\chi(r,t,x)\,\|\mathbf{1}-\mathscr{V}_t^x\|_x^2\right], \end{aligned} \tag{188}$$

where

$$C_t := 2\mathrm{e}^{-2C_V t}+2,$$

so that in view of (177) it is sufficient to prove that for all $r>0$,

$$\lim_{t\searrow 0}\sup_{x\in M}\mathbb{E}\left[\chi(r,t,x)\,\|\mathbf{1}-\mathscr{V}_t^x\|_x^2\right]=0. \tag{189}$$

To this end, let $t>0$, $r>0$ and take a nonnegative $\Psi\in \mathrm{C}_0^\infty(M)$ such that $\Psi=1$ in $\mathrm{K}_r(x_0)$. We denote with $\mathscr{V}^{\Psi,x}$ the solution of (124) with V replaced with ΨV and remark that

$$\Psi V\in\Gamma_{\mathcal{K}}(M,\mathrm{End}(E)).$$

Since in $\{\chi(r,t,x)\neq 0\}$ one has

$$//_s^{x,-1}V(B_s(x))//_s^x = //_s^{x,-1}\Psi(B_s(x))V(B_s(x))//_s^x \quad \text{for any } 0\leq s\leq t,$$

expanding $\mathscr{V}^x$ and $\mathscr{V}^{\Psi,x}$ in path ordered exponentials as in (78) shows

$$\mathbb{E}\left[\chi(r,t,x)\,\|\mathbf{1}-\mathscr{V}_t^x\|_x^2\right]=\mathbb{E}\left[\chi(r,t,x)\left\|\mathbf{1}-\mathscr{V}_t^{\Psi,x}\right\|_x^2\right],$$

and (189) follows from part a).

■

Now we are prepared to prove theorem 8.12.

Proof of theorem 8.12. Boundedness: We estimate as follows,

$$\begin{aligned} &\sup_{x\in M}\left\|\mathbb{E}\left[\mathscr{V}_t^x//_t^{x,-1}f(B_t(x))\right]\right\|_x \\ &\leq \sup_{x\in M}\left(\mathbb{E}\left[\|\mathscr{V}_t^x\|_x^2\right]^{\frac12}\mathbb{E}\left[\|f(B_t(x))\|_{B_t(x)}^2\right]^{\frac12}\right) \\ &\leq \mathrm{e}^{-C_V t}\left(\sup_{x,y\in M}p_t(x,y)\right)^{\frac12}\|f\|<\infty, \end{aligned} \tag{190}$$

where we have used the Cauchy-Schwarz inequality and (114).

Continuity: It follows from remark 8.13 that it is sufficient to prove that for any compact $K \subset M$ and any

$$f \in \Gamma_{\mathrm{L}^\infty}(M,E) \cap \Gamma_{\mathrm{L}^2}(M,E)$$

one has

$$\lim_{s \searrow 0} \sup_{x \in K} \left\| Q^0_s Q^V_{t-s} f(x) - Q^V_t f(x) \right\|_x = 0. \tag{191}$$

By (161), one has

$$\begin{aligned} &\left\| Q^0_s Q^V_{t-s} f(x) - Q^V_t f(x) \right\|_x \\ &= \left\| \mathbb{E}\left[\left(\mathscr{V}^{x,-1}_s \mathscr{V}^x_t - \mathscr{V}^x_t \right) //^{x,-1}_t f(B_t(x)) \right] \right\|_x \\ &= \left\| \mathbb{E}\left[\left(\mathbf{1} - \mathscr{V}^x_s \right) \mathscr{V}^{x,-1}_s \mathscr{V}^x_t //^{x,-1}_t f(B_t(x)) \right] \right\|_x . \end{aligned} \tag{192}$$

Since we have

$$\left\| \mathscr{V}^{x,-1}_s \mathscr{V}^x_t \right\|_x \leq \mathrm{e}^{-C_V(t-s)} \quad \mathbb{P}\text{-a.s.} \tag{193}$$

for some lower bound C_V of V by proposition A.1, and furthermore

$$\mathbb{E}\left[\| f(B_t(x)) \|^2_{B_t(x)} \right] = \int_M p_t(x,y) \, \| f(y) \|^2_y \operatorname{vol}(\mathrm{d}y) \leq \| f \|^2_\infty ,$$

we get

$$\begin{aligned} &\sup_{x \in K} \left\| Q^0_s Q^V_{t-s} f(x) - Q^V_t f(x) \right\|_x \\ &\leq \mathrm{e}^{-C_V(t-s)} \sup_{x \in K} \left(\mathbb{E}\left[\| \mathbf{1} - \mathscr{V}^x_s \|^2_x \right]^{1/2} \mathbb{E}\left[\| f(B_t(x)) \|^2_{B_t(x)} \right]^{1/2} \right) \\ &\leq \mathrm{e}^{-C_V(t-s)} \| f \|_\infty \sup_{x \in K} \mathbb{E}\left[\| \mathbf{1} - \mathscr{V}^x_s \|^2_x \right]^{1/2} \\ &\quad \to 0 \quad \text{as } s \searrow 0, \end{aligned} \tag{194}$$

where we have used the Cauchy-Schwarz inequality and proposition 8.18.

■

8.4 Some specific remarks on trivial vector bundles

In this section, we would like to specify some of the results from the previous sections to trivial vector bundles and to apply these results to operators that have direct applications in theoretical physics, such as magnetic Schrödinger operators or Yang-Mills Hamiltonians.

Throughout this section, let M be geodesically and stochastically complete, let $E = M \times \mathbb{C}^d$ with its standard Hermitian structure, and let

$$\alpha \in \Omega^1(M, \mathscr{U}(d)).$$

We consider a potential V with

$$C_V \mathbf{1} \leq V \in \mathrm{L}^2_{\mathrm{loc}}(M, \mathrm{Mat}(\mathbb{C}^d)) \quad \text{for some } C_V \in \mathbb{R},$$

and the self-adjoint realization of $\frac{1}{2}(\mathrm{d} + \alpha)^*(\mathrm{d} + \alpha) + V$ in $\mathrm{L}^2(M, \mathbb{C}^d)$ will be denoted with $H(\alpha, V)$. Here, d and d^* act componentwise and α is considered as a zeroth order differential operator

$$\alpha : \mathrm{C}^\infty(M, \mathbb{C}^d) \longrightarrow \Omega^1(M, \mathbb{C}^d), \;\; \alpha(\Psi)^j = \sum_{k=1}^m \alpha_{jk} \psi^k.$$

The formal adjoint α^* of α is the zeroth order differential operator given by

$$\alpha^* : \Omega^1(M, \mathbb{C}^d) \longrightarrow \mathrm{C}^\infty(M, \mathbb{C}^d), \;\; \alpha^*(\beta)^j = \sum_{k=1}^m g(\overline{\alpha}_{kj}, \beta^k),$$

where $g(\overline{\alpha}_{kj}, \beta^k)$ stands for the pairing of $\overline{\alpha}_{kj}$ and β^k with respect to the (complexified) Riemannian structure g on M. In accordance to our previous notation for operators on functions, we will write $H_0(V) = H(0, V)$. Let us first derive a formula for $H(\alpha, V) \mid_{\mathrm{C}_0^\infty(M, \mathbb{C}^d)}$:

Lemma 8.19 *The following formula holds for any $\Psi \in \mathrm{C}_0^\infty(M, \mathbb{C}^d)$,*

$$\begin{aligned} H(\alpha, V)\Psi = &-\frac{1}{2}\Delta\Psi - \frac{1}{2}\sum_{j=1}^l \alpha(A_j)^2 \cdot \Psi - \frac{1}{2}\sum_{j=1}^l A_j(\alpha(A_j)) \cdot \Psi \\ &- \sum_{j=1}^l \alpha(A_j) \cdot A_j(\Psi) + V \cdot \Psi, \end{aligned} \tag{195}$$

where the number $m \leq l \in \mathbb{N}$ and the vector fields

$$A_1 \ldots, A_l \in \Gamma_{\mathrm{C}^\infty}(M, \mathrm{T}M)$$

have been defined by the Nash embedding theorem in section 4.

Proof. Let the morphism of vector bundles $A : M \times \mathbb{R}^l \longrightarrow \mathrm{T}M$ be as in section 4, let $\nu, \xi \in \Omega(M)$ and let $\# : \mathrm{T}^*M \to \mathrm{T}M$ be the isomorphism of vector bundles that is induced by the metric g. Clearly, one has $\mathrm{d}^*\mathrm{d}\Psi = -\Delta\Psi$. Furthermore,

$$g_x(v,w) = \left\langle A(x)^* v^\#, A(x)^* w^\# \right\rangle_{\mathbb{R}^l} \text{ for any } x \in M \text{ and } v, w \in \mathrm{T}_x^*M$$

and $A_j(x) = A(x)e_j$ imply

$$g(\nu,\xi) = \sum_{j=1}^{l} \nu(A_j)\xi(A_j). \tag{196}$$

Next, let $x \in M$ be arbitrary and let $w_1, \ldots, w_m$ be a local orthonormal frame for $\mathrm{T}M$ in a neighbourhood of x with

$$\nabla^{\mathrm{T}M}_{w_j} w_k \mid_x = 0 \quad \text{for } j,k = 1, \ldots, m.$$

Then

$$\mathrm{d}^*\nu \mid_x = -\mathrm{div}(\nu^\#) \mid_x = -\sum_{j=1}^{m} g\left(\nabla^{\mathrm{T}M}_{w_j} \nu^\#, w_j\right) \mid_x = -\sum_{j=1}^{m} w_j\Big(g\left(\nu^\#, w_j\right)\Big) \mid_x .$$

If we take $v_{m+1}, \ldots, v_l \in \mathbb{R}^l$ such that $v_1, \ldots, v_l$ is an orthonormal basis of $\mathbb{R}^l$, where $v_j := A(x)^* w_j(x)$ for $j = 1, \ldots, m$, then it follows that

$$\begin{aligned}
\mathrm{d}^*\nu \mid_x &= -\sum_{j=1}^{m} \Big(A(\bullet)A(\bullet)^* w_j\Big)\Big(g\left(\nu^\#, A(\bullet)A(\bullet)^* w_j\right)\Big) \mid_x \\
&= -\sum_{j=1}^{l} \Big(A(\bullet)v_j\Big)\Big(g\left(\nu^\#, A(\bullet)v_j\right)\Big) \mid_x \\
&= -\sum_{j=1}^{l} \Big(A(\bullet)e_j\Big)\Big(g\left(\nu^\#, A(\bullet)e_j\right)\Big) \mid_x \\
&= -\sum_{j=1}^{l} A_j(\nu(A_j)) \mid_x .
\end{aligned} \tag{197}$$

The assertion now follows easily from the formulae (196) and (197). ∎

The starting point for the results of this section is the following Feynman-Kac formula:

Theorem 8.20 a) *The following formula holds for any $t \geq 0$, $f \in \mathrm{L}^2(M, \mathbb{C}^d)$ and a.e. $x \in M$,*

$$\mathrm{e}^{-tH(\alpha,V)} f(x) = \mathbb{E}\left[\mathscr{A}_t^{\alpha,V}(x) f(B_t(x))\right], \tag{198}$$

where for a.e. $x \in M$, the process

$$\mathscr{A}^{\alpha,V}(x) : [0,\infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d)$$

is defined as the maximal solution of

$$\mathrm{d}\mathscr{A}_t^{\alpha,V}(x) = \mathscr{A}_t^{\alpha,V}(x)\Big(\alpha(\underline{\mathrm{d}}B_t(x)) - V(B_t(x))\mathrm{d}t\Big), \;\; \mathscr{A}_0^{\alpha,V}(x) = \mathbf{1}.$$

b) *Let M be geodesically complete with Ricci curvature bounded from below and positive injectivity radius, and let $t > 0$. Then $\mathrm{e}^{-tH(\alpha,V)}$ is an integral operator with an a.e. well-defined and essentially bounded integral kernel given by*

$$\begin{aligned} &\mathrm{e}^{-tH(\alpha,V)}(\bullet,\bullet) : M \times M \longrightarrow \mathrm{Mat}(\mathbb{C}^d) \\ &\mathrm{e}^{-tH(\alpha,V)}(x,y) := p_t(x,y)\mathbb{E}_t^{x,y}\left[\mathscr{A}_t^{\alpha,V}(x)\right] \in \mathrm{Mat}(\mathbb{C}^d). \end{aligned} \tag{199}$$

Remark 8.21 A geometric interpretation of $\mathscr{A}^{\alpha,0}$ as a stochastic parallel transport can be found in remark 2.22. Indeed, with standard identifications in trivial vector bundles it is possible to deduce theorem 8.20 somewhat directly from theorem 5.5. Nevertheless, we find it instructive to explain the "calculational" origin of formula (198), namely formula (200) below. The approximation arguments that follow the latter formula are the same ones as in the proof of theorem 5.5. □

Proof of theorem 8.20. a) We set $\mathscr{A}(x) := \mathscr{A}^{\alpha,V}(x)$. Let us first assume that V is continuous and bounded, let $\Psi \in \mathrm{C}_0^\infty(M, \mathbb{C}^d)$ and fix an arbitrary $x \in M$. Then using (56) and (73), one finds that for all $t \geq 0$,

$$\begin{aligned} &\mathrm{d}\Big(\mathscr{A}_t(x)\Psi(B_t(x))\Big) = \mathrm{d}\Big(\text{ a martingale which starts in zero }\Big) \\ &+ \mathscr{A}_t(x)\left(\frac{1}{2}\Delta\Psi(B_t(x)) - V\cdot\Psi(B_t(x))\right)\mathrm{d}t \\ &+ \frac{1}{2}\mathscr{A}_t(x)\sum_{j=1}^{l}\alpha(A_j)^2\cdot\Psi\mid_{B_t(x)}\mathrm{d}t + \frac{1}{2}\mathscr{A}_t(x)\sum_{j=1}^{l}A_j(\alpha(A_j))\cdot\Psi\mid_{B_t(x)}\mathrm{d}t \\ &+ \mathscr{A}_t(x)\sum_{j=1}^{l}\alpha(A_j)\cdot A_j(\Psi)\mid_{B_t(x)}\mathrm{d}t \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{200}$$

Using lemma 8.19, this implies

$$\mathrm{d}\Big(\mathscr{A}_t(x)\Psi(B_t(x))\Big) = \mathrm{d}\Big(\text{ a martingale which starts in zero }\Big) \\ - \mathscr{A}_t(x)H(\alpha, V)\Psi(B_t(x))\mathrm{d}t \tag{201}$$

and (198) for continuous and bounded V can be deduced as in the first part of the proof of theorem 5.3.

For general V one proceed as follows: Firstly, we remark that the process $\mathscr{A}(x)$ indeed exists for a.e. $x \in M$, as follows from proposition 5.6. Since we can decompose $\mathscr{A}(x)$ into a unitary process and a path ordered exponential (see part c) and d) of proposition C.29) and since one has

$$\|\mathscr{A}_t(x)\|_{\mathrm{Mat}(\mathbb{C}^d)} \leq \mathrm{e}^{-C_V t} \ \mathbb{P}\text{-a.s. for any } t \geq 0$$

for some lower bound C_V of V (which is implied by this decomposition and proposition A.1 b)), one can directly follow the approximation arguments of the second part of the proof of theorem 5.3 and of the proof of theorem 5.5 to deduce (198).

b) This assertion follows with standard arguments from part a).

■

We have collected some simple consequences of formula (198) in the following corollary. In particular, we obtain an extension of the Euclidean Feynman-Kac-Itô formula to the setting of Riemannian manifolds.

Corollary 8.22 *Let $v : M \to \mathbb{R}$ be locally square integrable and bounded from below and let $\beta \in \Omega^1_{\mathbb{R}}(M)$.*

a) *For any $f \in \mathrm{L}^2(M)$, $t \geq 0$ and a.e. $x \in M$ one has*

$$\mathrm{e}^{-tH(\mathrm{i}\beta,v)}f(x) = \mathbb{E}\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s + \mathrm{i}\int_0^t \beta(\underline{\mathrm{d}}B_s(x))} f(B_t(x))\right]. \tag{202}$$

b) *The operator $\mathrm{e}^{-tH_0(v)}$ is positivity improving for any $t \geq 0$.*

c) *If the ground state energy $\lambda := \inf \sigma(H_0(v))$ is an eigenvalue of $H_0(v)$, then λ is simple and the corresponding ground state eigenvector can be chosen strictly positive.*

Proof. a) This follows easily from theorem 8.20.

b) This is obviously implied by the Feynman-Kac-Itô formula with $\beta = 0$.

c) This follows from b) and theorem XIII.44 in [65].

■

Next, let us add some specific remarks about Schrödinger operators with magnetic fields.

The energy of a nonrelativistic spinless quantum mechanical particle with mass and charge equal to 1 which lives on a "nice" Riemannian manifold (for example, one may assume that M has a bounded geometry for the following arguments) under the influence of a electrical potential

$$C_v \leq v \in \mathrm{L}^2_{\mathrm{loc}}(M)$$

is described [15] by the spectrum of the Schrödinger operator $H_0(v)$ in $\mathrm{L}^2(M)$. Switching on a magnetic field with potential $\beta \in \Omega^1_{\mathbb{R}}(M)$ corresponds to changing $H_0(v)$ to $H(\mathrm{i}\beta, v)$. Note that if $\Psi \in \mathrm{C}^\infty_0(M)$, then one has

$$H(\mathrm{i}\beta, v)\Psi = -\frac{1}{2}\Delta\Psi + \frac{\mathrm{i}}{2}\Big(\mathrm{d}^*(\beta\Psi) - \beta^*(\mathrm{d}\Psi)\Big) + \frac{1}{2}\beta^*(\beta\Psi) + v\Psi,$$

where $\beta^*(\alpha) \in \mathrm{C}^\infty_0(M)$ stands for the pairing of β and $\alpha \in \Omega^1_0(M)$ with respect to the Riemannian structure of M. It is expected from classical physics that in some sense the energy of the quantum system given by $H(\mathrm{i}\beta, v)$ should be greater than (or equal to) the energy of the quantum system given by $H_0(v)$. This statement can be justified mathematically in three ways, with the probabilistic methods that have been developed so far: Firstly, one has the following diamagnetic inequality,

$$\left|\mathrm{e}^{-tH(\mathrm{i}\beta,v)}(x, y)\right| \leq \mathrm{e}^{-tH_0(v)}(x, y). \tag{203}$$

Secondly, one has

$$\mathrm{tr}\left(\mathrm{e}^{-tH(\mathrm{i}\beta,v)}\right) \leq \mathrm{tr}\left(\mathrm{e}^{-tH_0(v)}\right), \tag{204}$$

and finally

$$\inf \sigma(H(\mathrm{i}\beta, v)) \geq \inf \sigma(H_0(v)). \tag{205}$$

We find it instructive to explain how elementary the proofs of these inequalites become in this scalar setting: From conditioning the Feynman-Kac-Itô formula, one finds,

$$\begin{aligned}\left|\mathrm{e}^{-tH(\mathrm{i}\beta,v)}(x, y)\right| &= p_t(x, y)\left|\mathbb{E}^{x,y}_t\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s + \mathrm{i}\int_0^t \beta(\underline{\mathrm{d}}B_s(x))}\right]\right| \\ &\leq p_t(x, y)\mathbb{E}^{x,y}_t\left[\mathrm{e}^{-\int_0^t v(B_s(x))\mathrm{d}s}\right] = \mathrm{e}^{-tH_0(v)}(x, y).\end{aligned} \tag{206}$$

The inequality (204) now follows from

$$\mathrm{tr}\left(\mathrm{e}^{-tH(\mathrm{i}\beta,v)}\right) = \int_M\int_M \overline{\mathrm{e}^{-\frac{t}{2}H(\mathrm{i}\beta,v)}(x, y)}\mathrm{e}^{-\frac{t}{2}H(\mathrm{i}\beta,v)}(x, y)\mathrm{vol}(\mathrm{d}x)\mathrm{vol}(\mathrm{d}y).$$

If $f \in \mathrm{L}^2(M)$, then multiplying (206) with $|f(y)|$ and integrating with respect to $\int_M (\bullet)\mathrm{vol}(\mathrm{d}y)$ gives

$$\left|\mathrm{e}^{-tH(\mathrm{i}\beta,v)} f(x)\right| \leq \mathrm{e}^{-tH_0(v)} \left|f(x)\right|. \tag{207}$$

The latter inequality implies

$$\left\langle \mathrm{e}^{-tH(\mathrm{i}\beta,v)} f, f\right\rangle \leq \left\langle \mathrm{e}^{-tH_0(v)} \left|f\right|, \left|f\right|\right\rangle, \tag{208}$$

from which (205) can easily be deduced with the help of the following two abstract facts: If H is a self-adjoint operator in a Hilbert space $(\mathscr{H}, \langle \bullet, \bullet\rangle_{\mathscr{H}})$ such that H is semi-bounded from below, then by the spectral calculus one has ([87], p.322)

$$\sup \sigma\left(\mathrm{e}^{-H}\right) = \mathrm{e}^{-\inf \sigma(H)},$$

and furthermore the variational principle (see for example [84], theorem 2.19) gives

$$\sup \sigma\left(\mathrm{e}^{-H}\right) = \sup \left\{ \left\langle \mathrm{e}^{-H}\Psi, \Psi\right\rangle_{\mathscr{H}} \,\middle|\, \Psi \in \mathscr{H}, \|\Psi\|_{\mathscr{H}} = 1\right\}.$$

Alternatively, one can of course also use the somewhat more sophisticated arguments from the proof of theorem 8.1 (that have been used there to prove the inclusion iii)) to deduce (205) from (208).

We close this section with a measure theoretic proof of a Goldon-Thompson-Symanzik type bound and a phase space bound for Yang-Mills Hamiltonians in the Euclidean $\mathbb{R}^m$. Let $\alpha \in \Omega(\mathbb{R}^m, \mathscr{U}(d))$ and let V be a potential with

$$C_V \mathbf{1} \leq V \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^m, \mathrm{Mat}(\mathbb{C}^d)) \cap \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^m, \mathrm{Mat}(\mathbb{C}^d)) \quad \text{for some } C_V \in \mathbb{R}.$$

We are interested in the operator $H(\alpha, V)$ in $\mathrm{L}^2(\mathbb{R}^m, \mathbb{C}^d)$. Note that in this case we have

$$H(\alpha, V)\Psi = -\frac{1}{2}\Delta\Psi - \frac{1}{2}\sum_{j=1}^{m}\Big((\partial_j \alpha_j) + 2\alpha_j\partial_j + \alpha_j^2\Big)\Psi + V\Psi$$

for any $\Psi \in \mathrm{C}_0^\infty(\mathbb{R}^m, \mathbb{C}^d)$. Differential operators of this type arise in quantum mechanics, if one wants to describe the energy of nonrelativistic Yang-Mills particles [40] [20] (with internal symmetries that are modelled by a subgroup of $\mathrm{U}(d)$), which live on $\mathbb{R}^m$ under the influence of the "electrical" potential V. We want to prove an extension of (152) and the phase space bound (153) to this setting. As we have already mentioned, by using the linear structure of $\mathbb{R}^m$ explicitely, the proof of these inequalities becomes elementary (in the sense that no operator theoretic methods are needed), when compared with the proof of theorem 8.10. To this end, let $\underline{V} : \mathbb{R}^m \to \mathbb{R}$ be given by

$\underline{V}(x) = \min \sigma(V(x))$ and let $t > 0$. In view of (199) and proposition C.29, one can use the same arguments as in the proof of theorem 8.8 to see that

$$\mathrm{tr}\left(\mathrm{e}^{-tH(\alpha,V)}\right) \leq d\ \mathrm{tr}\left(\mathrm{e}^{-tH_0(\underline{V})}\right),$$

where $H_0(\underline{V})$ denotes the self-adjoint realization of $-\Delta/2 + \underline{V}$ in $\mathrm{L}^2(\mathbb{R}^m)$. In this situation, the map

$$(x,y) \longmapsto \mathrm{e}^{-tH_0(\underline{V})}(x,y) = \frac{1}{(2\pi t)^{\frac{m}{2}}} \mathrm{e}^{-\frac{\|x-y\|^2_{\mathbb{R}^m}}{2t}} \mathbb{E}_t^{x,y}\left[\mathrm{e}^{-\int_0^t \underline{V}(B_s(x))\mathrm{d}s}\right]$$

is continuous [13], so that we may interpret

$$\begin{aligned}
\mathrm{tr}\left(\mathrm{e}^{-tH_0(\underline{V})}\right) &= \int_{\mathbb{R}^m} \mathrm{e}^{-tH_0(\underline{V})}(y,y)\mathrm{d}y \\
&= \frac{1}{(2\pi t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathbb{E}_t^{y,y}\left[\mathrm{e}^{-\int_0^t \underline{V}(B_s(y))\mathrm{d}s}\right]\mathrm{d}y \\
&= \frac{1}{(2\pi t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathbb{E}_t^{0,0}\left[\mathrm{e}^{-\int_0^t \underline{V}(B_s(0)+y)\mathrm{d}s}\right]\mathrm{d}y
\end{aligned}$$

literally. Using Fubini's theorem twice together with Jensen's inequality (applied to the probability measure $\mu(\mathrm{d}s) = \mathrm{d}s/t$ on $[0,t]$ and the exponential function), one gets

$$\begin{aligned}
&\frac{1}{(2\pi t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathbb{E}_t^{0,0}\left[\mathrm{e}^{-\int_0^t \underline{V}(B_s(0)+y)\mathrm{d}s}\right]\mathrm{d}y \\
&\leq \frac{1}{(2\pi t)^{\frac{m}{2}}} \int_0^t \mathbb{E}_t^{0,0}\left[\int_{\mathbb{R}^m} \mathrm{e}^{-t\underline{V}(B_s(0)+y)}\mathrm{d}y\right]\frac{\mathrm{d}s}{t}.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^m} \mathrm{e}^{-t\underline{V}(B_s(0)(\omega)+y)}\mathrm{d}y = \int_{\mathbb{R}^m} \mathrm{e}^{-t\underline{V}(y)}\mathrm{d}y$$

for any $\omega \in \Omega$ and

$$\frac{1}{(2\pi t)^{\frac{m}{2}}} = \int_{\mathbb{R}^m} \mathrm{e}^{-\frac{1}{2}t\|x\|^2_{\mathbb{R}^m}}\mathrm{d}x,$$

we arrive at the desired bounds

$$\mathrm{tr}\left(\mathrm{e}^{-tH(\alpha,V)}\right) \leq \frac{d}{(2\pi t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \mathrm{e}^{-t\underline{V}(y)}\mathrm{d}y \tag{209}$$

$$= d \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \mathrm{e}^{-t\left(\frac{1}{2}\|x\|^2_{\mathbb{R}^m} + \underline{V}(y)\right)}\mathrm{d}x\mathrm{d}y. \tag{210}$$

A Appendix: Some inequalities for operator valued differential equations

Let $\mathscr{H}$ be a d-dimensional, $d < \infty$, complex or real Hilbert space with scalar product $\langle \bullet, \bullet \rangle$ and the corresponding norm $\|\bullet\|$. The induced operator norm will be denoted with the same symbol. If $a \geq 0$, $F \in \mathrm{L}^1_{\mathrm{loc}}([a,\infty), \mathscr{L}(\mathscr{H}))$, then a standard use of the Banach fixed point theorem shows that there is a unique weak (= locally absolutely continuous) solution $Y : [a, \infty) \to \mathscr{L}(\mathscr{H})$ of the ordinary initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}s} Y(s) = Y(s)F(s), \;\; Y(a) = \mathbf{1}.$$

We want to prove the following two propositions in this section:

Proposition A.1 a) *For any $t \geq a$,*

$$\|Y(t)\| \leq \mathrm{e}^{\int_a^t \|F(s)\| \mathrm{d}s}.$$

b) *Let $t \geq a$, assume that $F(s)$ is Hermitian for a.e. $s \in [a, t]$ and that there exists a real-valued function $c \in \mathrm{L}^1[a, t]$ such that for all $v \in \mathscr{H}$ one has*

$$\langle F(s)v, v \rangle \leq c(s) \|v\|^2 \quad \textit{for a.e. } s \in [a, t].$$

Then one has

$$\|Y(t)\| \leq \mathrm{e}^{\int_a^t c(s) \mathrm{d}s}.$$

Proof. a) This becomes a direct consequence of the Gronwall lemma, if one integrates the differential equation and takes norms.
b) Let $f_1, \dots, f_d$ be an orthonormal basis of $\mathscr{H}$. Since $\|Y^*\| = \|Y\|$, we can assume that

$$\frac{\mathrm{d}}{\mathrm{d}s} Y(s) f_j = F(s) Y(s) f_j, \;\; Y(a) = \mathbf{1},$$

so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \|Y(s) f_j\|^2 &= 2 \langle F(s)(Y(s) f_j), Y(s) f_j \rangle \\ &\leq 2c(s) \|Y(s) f_j\|^2 \quad \text{for a.e. } s \in [a, t], \end{aligned} \tag{211}$$

and the assertion again follows from the Gronwall lemma.

∎

Proposition A.2 *Let* $F_1,\ F_2 \in \mathrm{L}^1_{\mathrm{loc}}([a,\infty),\mathscr{L}(\mathscr{H}))$ *and let*

$$Y_1, Y_2 : [a,\infty) \longrightarrow \mathscr{L}(\mathscr{H})$$

be the unique solutions of the ordinary initial value problems

$$\frac{\mathrm{d}}{\mathrm{d}s} Y_j(s) = Y_j(s) F_j(s),\ \ Y_j(a) = \mathbf{1} \quad \textit{for } j = 1, 2.$$

The following inequality holds for all $t \geq a$*,*

$$\|Y_1(t) - Y_2(t)\| \leq \mathrm{e}^{2\int_a^t \|F_1(s)\|\mathrm{d}s + \int_a^t \|F_2(s)\|\mathrm{d}s} \int_a^t \|F_1(s) - F_2(s)\|\,\mathrm{d}s.$$

Proof. $Y_1(s)$ and $Y_2(s)$ are invertible for any $s \geq a$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} Y_j^{-1}(s) = -F_j(s) Y_j^{-1}(s).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(Y_1^{-1}(s) Y_2(s)\Big) = Y_1^{-1}(s)(F_2(s) - F_1(s)) Y_2(s) \quad \text{for a.e. } s \geq a,$$

one obtains the following equality (after integration and multiplication with $Y_1(t)$):

$$Y_2(t) = Y_1(t) + Y_1(t) \int_a^t Y_1^{-1}(s)(F_2(s) - F_1(s)) Y_2(s) \mathrm{d}s.$$

Thus,

$$\|Y_1(t) - Y_2(t)\| \leq \|Y_1(t)\| \int_a^t \left\|Y_1^{-1}(s)\right\| \|F_2(s) - F_1(s)\| \, \|Y_2(s)\|\,\mathrm{d}s. \quad (212)$$

The claim follows from observing that

$$\|Y_j(s)\| \leq \mathrm{e}^{\int_a^t \|F_j(r)\|\mathrm{d}r},\ \ \left\|Y_j^{-1}(s)\right\| \leq \mathrm{e}^{\int_a^t \|F_j(r)\|\mathrm{d}r}$$

by proposition A.1.

■

We remark that this result is similar to inequality III, [68], p.53 (which could be adjusted to our situation by applying it to step functions and taking limits then). Our proof follows the strategy of theorem 5.1, p.33 in [22].

B Appendix: Riemannian manifolds with bounded geometry

A Riemannian manifold $M = (M, g)$ is called a manifold with *bounded geometry*, if

(i) $\inf_{x\in M} r_{\mathrm{inj}}(x) > 0$, with $r_{\mathrm{inj}}(x)$ the injectivity radius of M at x, and

(ii) all covariant derivatives of the Riemannian curvature tensor of M are bounded.

Examples of manifolds with bounded geometry are homogeneous Riemannian manifolds with an invariant metric, covering manifolds of compact Riemannian manifolds and leaves of a foliation on a compact Riemannian manifold (the latter two with the induced Riemannian structures) [73], [26].

Let M have a bounded geometry now. This property always implies geodesic completeness [26]. Furthermore, by lemma 4.8 in [73], there is a $C > 0$ such that for all $x \in M$ and all $r > 0$ one has

$$\mathrm{vol}(x, r) \leq \mathrm{e}^{Cr}, \tag{213}$$

where $\mathrm{vol}(x, r)$ stands for the volume of the open geodesic ball with radius r around x. This inequality can be used to prove that M is stochastically complete. Next, we list some heat kernel bounds for M: Let $m := \dim M$, and let $p_t(x, y)$ be the minimal heat kernel of M. Firstly, one has the following global upper bound [35]: There is a constant $C > 0$ (which depends on the Riemannian structure of M) such that for all $t > 0$, $x, y \in M$,

$$p_t(x, y) \leq \frac{C}{\min\{t^{\frac{m}{2}}, 1\}} \left(1 + \frac{\mathrm{d}(x, y)^2}{t}\right)^{\frac{m}{2}+1} \mathrm{e}^{-\frac{\mathrm{d}(x,y)^2}{4t} - t \inf \sigma(-\Delta)}. \tag{214}$$

Secondly, one has the following bounds, which are local in t: For any $t > 0$ there are $A_t, B_t, C_t, D_t > 0$ such that for all $0 < s \leq t$ and all $x, y \in M$ one has

$$\frac{A_t \mathrm{e}^{-B_t \frac{\mathrm{d}(x,y)^2}{s}}}{s^{m/2}} \leq p_s(x, y) \leq \frac{C_t \mathrm{e}^{-D_t \frac{\mathrm{d}(x,y)^2}{s}}}{s^{m/2}}. \tag{215}$$

These estimates are included in lemma 7.1.

Finally, we remark the following bound for the gradient of the logarithmic heat kernel: Let ∇ be the Levi-Civita connection. Then for any $t > 0$ there is a $C_t > 0$ such that for all $0 < s \leq t$ and all $x, y \in M$ one has

$$\|\nabla_y \log p_s(x,y)\|_y \leq C_t \left(\frac{\mathrm{d}(x,y)}{s} + \frac{1}{\sqrt{s}} \right). \tag{216}$$

Proof. Since M is geodesically complete and has Ricci curvature bounded from below, the main result of [30] shows the following estimate: If there exist $r_0 > 0$ and $v_0 > 0$ such that for any $x \in M$ one has $\mathrm{vol}(x, r_0) \geq v_0$, then there are constants $C_{r_0}, \tilde{C}_{r_0} > 0$ such that for all $0 < s \leq \tilde{C}_{r_0}$ and all $x, y \in M$ one has

$$\|\nabla_y \log p_s(x,y)\|_y \leq C_{r_0} \left(\frac{\mathrm{d}(x,y)}{s} + \frac{1}{\sqrt{s}} \right). \tag{217}$$

The proof (see p.527 in [30]) shows that one can actually take $\tilde{C}_{r_0} = 2r_0^2$. Since one has ([15], p.784)

$$\inf_{x \in M} \mathrm{vol}(x, r) > 0 \quad \text{for all } r > 0$$

on manifolds with bounded geometry, the inequality (216) follows from these considerations.

■

C Appendix: Stochastic differential equations in $\mathbb{R}^m$

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Whenever it is necessary[20], we will assume that $\mathscr{F}_* := (\mathscr{F}_t)_{t\geq 0}$ satisfies the *usual hypthothesis*, that is, the pair $(\mathbb{P}, \mathscr{F}_t)$ is complete in the usual measure theoretic sense for all $t \geq 0$, and $\mathscr{F}_*$ is *right-continuous*: $\mathscr{F}_t = \mathscr{F}_{t+} := \cap_{s>t} \mathscr{F}_s$ for any $t \geq 0$. This property can always be achieved in a canonic way: If $\mathscr{F}_*$ is not complete, then one can augment $\mathscr{F}_*$ [36], and if $\mathscr{F}_*$ is not continuous, then one way replace $\mathscr{F}_*$ with $(\mathscr{F}_{t+})_{t\geq 0}$.
In the following, let M be a topological space, canonically equipped with the Borel-σ-algebra $\mathcal{B}(M)$. If $X : \Omega \to M$ is a random variable (= a $\mathscr{F}$-measurable map), then the *law* $\mathbb{P}^X$ *of* X is defined as the pushforward measure of $\mathbb{P}$ with respect to X, that is, $\mathbb{P}^X$ is the probability measure on $\mathcal{B}(M)$

[20] This is in particular the case, if one considers stochastic integrals.

which is defined by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{B}(M) \arrow[r, "\mathbb{P}^X"] \arrow[d, "X^{-1}"'] & {[0,1]} \\
\mathscr{F} \arrow[ur, "\mathbb{P}"'] &
\end{tikzcd}$$

A *M-valued process* is a map

$$X : [0, \infty) \times \Omega \longrightarrow M, \;\; (t, \omega) \longmapsto X_t(\omega) \tag{218}$$

such that $X_t : \Omega \to M$ is a random variable for any $t \geq 0$. The process (218) is called *adapted* (to $\mathscr{F}_*$), if X_t is $\mathscr{F}_t$-measurable for any $t \geq 0$, and *continuous*, if X has this property pathwise, that is, if the sample paths $X(\omega)$ are continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$. Analogously, if $M = \mathbb{R}$, then X is said to be *of finite variation* (o.f.v.), if the sample paths are locally o.f.v., and X is called *increasing*, if its sample paths are increasing. We use the notation $\mathbb{E}\left[\bullet\right] := \int_\Omega (\bullet) \mathrm{d}\mathbb{P}$ for the expectation value with respect to $\mathbb{P}$.

Remark C.1 If we don't specify M, then $M = \mathbb{R}$ in the following.

$\square$

Martingales are defined as follows:

Definition C.2 a) *Let X be an integrable process (that is, X_t is in* $\mathrm{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ *for all $t \geq 0$). Then X is called a ($\mathscr{F}_*$-)martingale, if*

$$X_s = \mathbb{E}\left[X_t | \mathscr{F}_s\right] \quad \mathbb{P}\text{-}a.s. \text{ for all } 0 \leq s \leq t \; .$$

b) *A process $X = (X^1, \ldots, X^m)^t$ with values in $\mathbb{R}^m$ is called a martingale, if X_j is a martingale in the sense of a) for any $j = 1, \ldots, m$.*

Here,

$$\mathbb{E}\left[\bullet | \mathscr{F}_t\right] : \mathrm{L}^1(\Omega, \mathscr{F}, \mathbb{P}) \longrightarrow \mathrm{L}^1(\Omega, \mathscr{F}_s, \mathbb{P})$$

stands for the *conditional expectation of X_t given (the information) $\mathscr{F}_s$*. This morphism of Banach spaces is uniquely determined by the following fact: If $f \in \mathrm{L}^2(\Omega, \mathscr{F}, \mathbb{P})$, then $\mathbb{E}\left[f | \mathscr{F}_s\right]$ is the Hilbert space projection of f onto $\mathrm{L}^2(\Omega, \mathscr{F}_s, \mathbb{P})$ ([77], p.22).

Note that any martingale X is adapted and that the expectation value of X is constant: $\mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_0\right]$ for all $t \geq 0$.

Next, we remark the following important *martingale convergence theorem* (theorem 3.15 in [46]):

Theorem C.3 *Let X be a continuous martingale with*

$$\sup_{t \geq 0} \mathbb{E}\left[|X_t|\right] < \infty.$$

Then there is an integrable random variable $\tilde{X}$ with $X_t \to \tilde{X}$ as $t \to \infty$ pointwise $\mathbb{P}$-a.s.

Theorem C.3 particularly applies, if X is a nonnegative continuous martingale.

Definition C.4 a) *A ($\mathscr{F}_*$-)stopping time is a map $\zeta : \Omega \to [0, \infty]$ such that*

$$\left\{\zeta \leq t\right\} \in \mathscr{F}_t \quad \textit{for all } t \geq 0.$$

b) *A stopping time ζ is called predictable, if there is a sequence of stopping times (ζ_n) with $\zeta_n \nearrow \zeta$ $\mathbb{P}$-a.s. such that*

$$\zeta_n < \zeta \quad \mathbb{P}\text{-}a.s. \ \textit{in} \ \left\{\zeta > 0\right\}.$$

Such a sequence is said to announce ζ.

If ζ_1 and ζ_2 are stopping times, then so are $\zeta_1 \wedge \zeta_2 := \min\{\zeta_1, \zeta_2\}$ and $\zeta_1 \vee \zeta_2 := \max\{\zeta_1, \zeta_2\}$, and if (ζ_n) is a sequence of stopping times, then $\sup_{n \in \mathbb{N}} \zeta_n$ is again a stopping time.

Example C.5 An important class of stopping times is given by first exit times: If N is a seperable metrizable topological space, if X is a continuous adapted process with values in N and if and $U \subset N$ is open, then the map

$$\begin{aligned} \Omega &\longrightarrow [0, \infty] \\ \omega &\longmapsto \begin{cases} \infty, & \text{if } X_t(\omega) \in U \text{ for all } t \geq 0, \\ \inf\left\{t \middle| \ X_t(\omega) \notin U\right\}, & \text{else} \end{cases} \end{aligned}$$

is $\mathbb{P}$-a.s. equal to a stopping time $\zeta_{X,U}$, called the *first exit time* of X from N ([28], p.41). In particular, if $N = \hat{M}$ is the Alexandroff compactification of a seperable metrizable space M, then $\zeta_X := \zeta_{X,M}$ is a stopping time, called the *explosion time* or *lifetime* of X.

□

For any process X with values in M and any stopping time ζ, let X^ζ be the process given by $X_t^\zeta(\omega) := X_{\zeta(\omega) \wedge t}(\omega)$.

Proposition C.6 *Let ζ be a stopping time and let X be a martingale. Then the process X^ζ is again a martingale.*

Proof. [66], corollary 3.6, p.71.

■

Whenever necessary, we will consider $\mathbb{R}^m$ as a smooth Riemannian manifold with its standard Euclidean metric. Let

$$p : (0,\infty) \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow (0,\infty), \quad (t,x,y) \longmapsto \frac{1}{(2\pi t)^{\frac{m}{2}}} \mathrm{e}^{-\frac{\|x-y\|^2_{\mathbb{R}^m}}{2t}}$$

be the heat kernel of $\mathbb{R}^m$. Of fundamental importance is:

Definition C.7 a) *A process W with values in $\mathbb{R}^m$ is called a Brownian motion in $\mathbb{R}^m$ with starting point* [21] *$x \in \mathbb{R}^m$, if W is continuous and for any finite sequence of times $0 =: t_0 < t_1 \leq \cdots \leq t_n$ and $E_1, \ldots, E_n \in \mathcal{B}(\mathbb{R}^m)$ one has*

$$\begin{aligned} &\mathbb{P}\Big\{W_{t_1} \in E_1, \ldots, W_{t_n} \in E_n\Big\} \\ &= \int_{E_1} \cdots \int_{E_n} p_{\delta_0}(x, x_1) \cdots p_{\delta_{n-1}}(x_{n-1}, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_m, \end{aligned} \tag{219}$$

with $\delta_j := t_{j+1} - t_j$ for $j = 0, \ldots, n-1$.
b) *A Brownian motion W in $\mathbb{R}^m$ is called $\mathscr{F}_*$-compatible, if W is adapted to $\mathscr{F}_*$ and $W_{t+s} - W_t$ is independent of $\mathscr{F}_t$ for any $t \geq 0$ and $s > 0$.*

If W is a Brownian motion in $\mathbb{R}^m$, then the components of W are (independent) Brownian motions in $\mathbb{R}^1$, and if $\mathscr{F}^W_*$ is the filtration which is generated by W, then W is $\mathscr{F}^W_*$-compatible, so that from now on we will always assume that Brownian motions are compatible with the given filtration. In this sense, Brownian motions are square integrable continuous martingales ([43], theorem 7.2, p. 42) and they can be constructed as follows:

Example C.8 By setting

$$\mathrm{d}(\omega_1, \omega_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} \min\left(\|\omega_1(t) - \omega_2(t)\|_{\mathbb{R}^m}, 1\right),$$

$\mathrm{C}([0,\infty), \mathbb{R}^m)$ becomes a metric space and we denote with $\mathscr{F}^m$ the corresponding Borel-σ-algebra and with $\mathscr{F}^m_*$ the filtration

$$\mathscr{F}^m_t := \sigma\{W_s | 0 \leq s \leq t\},$$

[21] It follows from (219) that the law of W_t is equal to $\int_\bullet p_t(x,y)\mathrm{d}y$ for any $t > 0$, so that indeed $W_0 = x$ $\mathbb{P}$-a.s.

with the coordinate process $W_t(\omega) := \omega(t)$. By using Kolmogorov's extension theorem and Kolmogorov's continuity criterion, one finds that for any $x \in \mathbb{R}^m$ there is a unique probability measure $\mathbb{P}^x$ on $(\mathrm{C}([0,\infty),\mathbb{R}^m), \mathscr{F}^m)$ such that formula (219) holds. In this setting, the coordinate process W on $(\mathrm{C}([0,\infty),\mathbb{R}^m), \mathscr{F}^m, \mathscr{F}^m_*, \mathbb{P}^x)$ is a $\mathscr{F}^m_*$-compatible Brownian motion in $\mathbb{R}^m$ which starts in x. We refer the reader to chapter 2 in [46] for details.

$\square$

Our next aim in this section is to explain, given two processes X and F satisfying certain assumptions, how one can define a new process $I(F,X)$, which deserves the notation ("stochastic integral")

$$I(F,X)_t = \int_0^t F_s \mathrm{d}X_s.$$

We shall assume from now on that the given filtered probability space satisfies the usual assumptions. Let

$$\mathcal{A} := \Big\{ X \Big| \;\; X \text{ is cont., adapted, o.f.v and } X_0 = 0 \ \mathbb{P}\text{-a.s.} \;\Big\}$$

and

$$\mathcal{A}^+ := \Big\{ X \Big| \;\; X \text{ is cont., adapted, increasing and } X_0 = 0 \ \mathbb{P}\text{-a.s.} \;\Big\}.$$

If $X \in \mathcal{A}$, then the definition of $I(F,X)$ is straighforward by the pathwise use of the usual Lebesgue-Stieltjes calculus. But there is a no-go theorem:

Theorem C.9 . *If a continuous martingale X is an element of $\mathcal{A}$, then $X_t = 0$ $\mathbb{P}$-a.s. for any $t \geq 0$.*

Proof. [66], Proposition 1.2, p.120.

$\blacksquare$

Since one is especially interested in the case where X is equal to a Brownian motion, theorem C.9 actually shows that one has to use a different concept in order to define stochastic integrals, namely, the concept of quadratic variations, which will be explained below.

For some applications, the class of continuous martingales is too small and it turns out that the following is an appropriate generalization ([37], p.159):

Definition C.10 *A continuous adapted process X is called a continuous local martingale, if there exists a sequence (ζ_n) of stopping times with $\zeta_n \nearrow \infty$ $\mathbb{P}$-a.s. as $n \to \infty$, such that for any $n \in \mathbb{N}$ the process $(X - X_0)^{\zeta_n}$ is a martingale.*

Definition C.10 extends in the obvious way to processes with values in $\mathbb{R}^m$. A sufficient condition for a continuous local martingale to be a martingale is:

Proposition C.11 *A continuous local martingale X is a martingale, if*

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t|\right] < \infty \quad \textit{for all } T > 0.$$

Proof. This follows from the first part of Satz 4.13 in [37].

■

Proposition and definition C.12 *For any continuous local martingale X there is a unique $[X] \in \mathcal{A}^+$ such that $X^2 - [X]$ is a continuous local martingale. The process $[X]$ is called the quadratic variation of X. Moreover, if X, Y are continuous local martingales, then*

$$[X,Y] := \frac{1}{4}\Big([X+Y] - [X-Y]\Big)$$

is the unique process in $\mathcal{A}$ such that $XY - [X,Y]$ is a continuous local martingale. $[X,Y]$ called the cross variation of X and Y.

Remark C.13 Here and in the following, uniqueness is understood in the sense that if $[\tilde{X}]$ is another process with the above properties, then $[X]_t = [\tilde{X}]_t$ $\mathbb{P}$-a.s. for any $t \ge 0$, that is, the processes are versions of each other. We recall the following simple fact in this context: If B, C are right-continuous processes with values in a Hausdorff space, then B and C are versions of each other, if and only if they are indistinguishable:

$$\mathbb{P}\Big\{B_t = C_t \text{ for all } t \ge 0\Big\} = 1 \iff \mathbb{P}\Big\{B_t = C_t\Big\} = 1 \text{ for all } t \ge 0.$$

□

The map $(X,Y) \mapsto [X,Y]$ is symmetric and bilinear and if ζ is a stopping time, then

$$[X^\zeta, Y^\zeta]_t = [X, Y^\zeta]_t = [X,Y]_t^\zeta \quad \mathbb{P}\text{-a.s. for any } t \ge 0. \tag{220}$$

Furthermore, one has $[X,X] = 0$, if and only if $X_t = X_0$ $\mathbb{P}$-a.s. for any $t \ge 0$. A proof of these facts and of proposition C.12 can be found in [66], pp.124–125.

The notion of quadratic variation characterizes Brownian motions with values in $\mathbb{R}^m$ by the following Lévy theorem:

Theorem C.14 *Let $X = (X^1, \ldots, X^m)^t$ be a continuous adapted process with values in $\mathbb{R}^m$ and a deterministic initial value. Then X is a Brownian motion, if and only if X is a local martingale with*

$$[X^j, X^k]_t = \delta^{jk} t \ \mathbb{P}\text{-a.s. for any } t \geq 0.$$

Proof. [66], theorem 3.6, p.150.

■

Now we are in the position to define stochastic integrals by a uniqueness property:

Theorem C.15 *For any continuous local martingale X and any continuous adapted process F there is a unique continuous local martingale $I(F,X)$ with $I(F,X)_0 = 0$ $\mathbb{P}$-a.s., written as*

$$(t,\omega) \longmapsto \left(\int_0^t F_s \mathrm{d}X_s \right)(\omega) := I(F,X)_t(\omega),$$

such that

$$[I(F,X), Z] = I(F, [X,Z]) \quad \text{for any continuous local martingale } Z.$$

Proof. [66], proposition 2.7, p.140.

■

The process $I(F,X)$ is called the *(Itô) stochastic integral* of F with respect to X. Stochastic integrals have the following properties ([66], p.140 and [37], Satz 4.35):

Proposition C.16 *Let X be a continuous local martingale and let F, G be continuous adapted processes. Then the following statements hold:*

a) One has

$$I(FG,X) = I(F, I(G,X)).$$

b) If ζ is a stopping time, then

$$I(F,X)^\zeta = I(F, X^\zeta) = I(F^\zeta, X^\zeta). \tag{221}$$

One can extend the definition of stochastic integrals further to continuous semi-martingales:

Proposition and definition C.17 a) *A process X is called a continuous semi-martingale, if there is a continuous local martingale X_1 and a continuous adapted process of finite variation X_2 such that $X = X_1 + X_2$. This decomposition is unique.*

b) *For continuous semi-martingales X, Y we define $[X, Y] := [X_1, Y_1]$, if $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ are the decompositions of X and Y in the sense of part a).*

Proof. The only thing that has to be proved is the asserted uniqueness, which follows easily from theorem C.9.

∎

Remark C.18 If X, Y, Z are continuous semi-martingales, then definition C.17 implies $[[X, Y], Z] = 0$.

□

Of course, definition C.17 can be extended to vector-valued processes again. If $X = X_1 + X_2$ is a continuous semi-martingale and if F is a continuous adapted process, then we can define a continuous semi-martingale $I(F, X)$ which starts in 0 by setting

$$I(F, X) := I(F, X_1) + I(F, X_2),$$

where $I(F, X_2) \in \mathcal{A}$ is defined as a pathwise Stieltjes integral. This definition could actually be extended to all locally bounded integrands (this has been carried out in [66], p.140), but we won't need this extension here. One has:

Proposition C.19 *Let X, Y be continuous semi-martingales and let F, G be continuous adapted processes.*

i) *If X is a continuous local martingale, then so is $I(F, X)$. If X is a continuous adapted process of finite variation, then so is $I(F, X)$.*

ii) *One has*
$$I(F, I(G, X)) = I(FG, X).$$

iii) *One has*
$$[I(F, X), I(G, Y)] = I(FG, [X, Y]).$$

iv) *If ζ is a stopping time, then one has the stopping rule*
$$I(F, X)^\zeta = I(F, X^\zeta) = I(F^\zeta, X^\zeta). \tag{222}$$

Proof. i) follows from the definition, ii) and iii) can be found in [37], p.194, and iv) follows from (221) and the stopping rule for pathwise defined Stieltjes integrals ([37], Satz 4.7).

■

In the case of Brownian motions as integrators, one has the following characterization:

Proposition C.20 *Let B be a Brownian motion and let F be a continuous adapted process with*

$$\mathbb{E}\left[\int_0^t F_s^2 \mathrm{d}s\right] < \infty \quad \textit{for any } t \geq 0.$$

Then $I(F, B)$ is a square integrable continuous martingale.

Proof. This follows from the basic construction of the Itô integral with respect to Brownian motions as integrators, which can be found for example in [43], definition 1.5 on p.49.

■

We can now state the Itô formula:

Theorem C.21 *Let $X = (X^1, \ldots, X^m)^t$ be a continuous semi-martingale with values in $\mathbb{R}^m$ and let $f \in \mathrm{C}^2(\mathbb{R}^m)$ be real-valued. Then the process $f(X)$ is a continuous semi-martingale and one has*

$$f(X_t) = f(X_0) + \sum_{k=1}^{m} \int_0^t \partial_k f(X_s) \mathrm{d}X_s^k + \frac{1}{2} \sum_{k,l=1}^{m} \int_0^t \partial_k \partial_l f(X_s) \mathrm{d}[X^k, X^l]_s \tag{223}$$

$\mathbb{P}$-a.s. for any $t \geq 0$.

Proof. [66], theorem 3.3, p.147.

■

In the symbolic differential notation of stochastic integrals, formula (223) is also often written as

$$\mathrm{d}f(X) = \sum_{k=1}^{m} \partial_k f(X) \mathrm{d}X^k + \frac{1}{2} \sum_{k,l=1}^{m} \partial_k \partial_l f(X) \mathrm{d}[X^k, X^l],$$

and we will use this notation, whenever it is convenient. Note that taking $f(x^1, x^2) := x^1 x^2$, the Itô formula implies the following product rule,

$$\mathrm{d}(X^1 X^2) = X^2 \mathrm{d}X^1 + X^1 \mathrm{d}X^2 + \mathrm{d}[X^1, X^2]. \tag{224}$$

It turns out that the introduction of the following stochastic integral is convenient for local stochastic calculus on manifolds. This statement is mainly motivated by theorem C.24, which shows that this calculus behaves well under changes of coordinate systems:

Definition C.22 *For any two continuous semi-martingales X, Y the continuous semi-martingale given by*

$$\int X \underline{\mathrm{d}} Y := \int X \mathrm{d} Y + \frac{1}{2}[X, Y]$$

is called the Stratonovic stochastic integral of Y with respect to the integrator X.

In the situation of definition C.22, one has (where we write $X(s)$ and $Y(s)$ instead of X_s and Y_s, respectively):

$$\int_0^t X(s) \mathrm{d} Y(s) = \underset{n \to \infty}{\text{l.i.p.}} \sum_{k=1}^{n} X\left(\frac{(k-1)t}{n}\right)\left(-Y\left(\frac{(k-1)t}{n}\right) + Y\left(\frac{kt}{n}\right)\right), \tag{225}$$

whereas

$$\begin{aligned} \int_0^t X(s) \underline{\mathrm{d}} Y(s) = \; & \underset{n \to \infty}{\text{l.i.p.}} \sum_{k=1}^{n} \frac{1}{2}\left(X\left(\frac{(k-1)t}{n}\right) + X\left(\frac{kt}{n}\right)\right) \\ & \times \left(-Y\left(\frac{(k-1)t}{n}\right) + Y\left(\frac{kt}{n}\right)\right). \end{aligned} \tag{226}$$

Here, l.i.p. stands for the limit in probability with respect to $\mathbb{P}$. In particular, there is a subsequence which converges $\mathbb{P}$-a.s. The proof of this convergence can be found in [43]. We remark that the symmetric definition of Stratonovic integrals correspond to the Hermitian symmetry of the integral kernels corresponding to magnetic Schrödinger operators (see [77], p.160).

In analogy to proposition C.19, we get the following

Proposition C.23 *Let W, X, Y, Z be continuous semi-martinagales.*

i) *One has*

$$\int X \underline{\mathrm{d}} \int Y \underline{\mathrm{d}} Z = \int XY \underline{\mathrm{d}} Z.$$

ii) *One has*

$$\left[\int X \underline{\mathrm{d}} Y, \int Z \underline{\mathrm{d}} W\right] = \int XZ \underline{\mathrm{d}}[Y, W] = \int XZ \mathrm{d}[Y, W].$$

iii) *If ζ is a stopping time, then one has the Stratonovic stopping rule*

$$\int_0^{t\wedge\zeta} X_s \underline{\mathrm{d}} Y_s = \int_0^t X_s \underline{\mathrm{d}} Y_s^\zeta = \int_0^t X_s^\zeta \underline{\mathrm{d}} Y_s^\zeta \quad \mathbb{P}\text{-a.s. for any } t \geq 0.$$

Proof. i) and ii) can be found in [37], p.200-201, and iii) follows from (220) and (222).

■

The Itô formula takes the following form with the notion of Stratonovic integrals:

Theorem C.24 *Let X be a continuous semi-martingale with values in $\mathbb{R}^m$ and let $f \in \mathrm{C}^3(\mathbb{R}^m)$. Then the process $f(X)$ is a continuous semi-martingale and one has*

$$f(X_t) = f(X_0) + \sum_{k=1}^m \int_0^t \partial_k f(X_s) \underline{\mathrm{d}} X_s^k \quad \mathbb{P}\text{-a.s. for any } t \geq 0.$$

Proof. One has

$$\begin{aligned}
\sum_{k=1}^m \partial_k f(X) \underline{\mathrm{d}} X^k &= \sum_{k=1}^m \partial_k f(X) \mathrm{d} X^k + \frac{1}{2} \sum_{k=1}^m \mathrm{d}[\partial_k f(X), X^k] \\
&= \sum_{k=1}^m \partial_k f(X) \mathrm{d} X^k + \frac{1}{2} \sum_{j,k=1}^m \partial_j \partial_k f(X) \mathrm{d}[X^j, X^k] \\
&\qquad + \frac{1}{2} \sum_{j,k,l=1}^m \partial_l \partial_j \partial_k f(X) \mathrm{d}[[X^j, X^l], X^k] \\
&= \sum_{k=1}^m \partial_k f(X) \mathrm{d} X^k + \frac{1}{2} \sum_{j,k=1}^m \partial_j \partial_k f(X) \mathrm{d}[X^j, X^k] \\
&= \mathrm{d} f(X),
\end{aligned}$$

where we have used the usual Itô formula and proposition C.19 iii) for the second equality, remark C.18 for the third equality and the usual Itô formula again for the last equality.

■

Theorem C.24 shows that Stratonovic differentials behave like ordinary differentials in the sense of the symbolic notation

$$\mathrm{d} f(X) = \mathrm{grad}(f) \underline{\mathrm{d}} X.$$

The product rule (224) now takes the form

$$\mathrm{d}(X^1X^2) = X^2\underline{\mathrm{d}}X^1 + X^1\underline{\mathrm{d}}X^2 \tag{227}$$

and, more generally, if B and C are continuous semi-martingales with values in $\mathrm{Mat}(\mathbb{C}^d)$ (with the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$), then a short calculation shows that

$$\mathrm{d}(BC) = (\underline{\mathrm{d}}B)C + B(\underline{\mathrm{d}}C). \tag{228}$$

Finally, we briefly explain the concept of stochastic differential equations in $\mathbb{R}^m$.

Definition C.25 *Let Z be a continuous semi-martingale with values in $\mathbb{R}^l$ and let*

$$A = (A_i^j)_{1\le i\le l}^{1\le j\le m} : \mathbb{R}^m \longrightarrow \mathrm{Mat}_{\mathbb{R}}(m\times l)$$

be continuous. A solution of the Itô stochastic differential equation determined by (A, Z) is a $\mathbb{R}^m$-valued continuous adapted process X such that for all $j = 1, \ldots, m$ one has

$$X_t^j = X_0^j + \sum_{i=1}^{l} \int_0^t A_i^j(X)\mathrm{d}Z_s^i \ \mathbb{P}\text{-}a.s. \ for\ any\ t \ge 0. \tag{229}$$

In particular, this definition implies that the solutions of Itô stochastic differential equations are continuous semi-martingales. As above, one usually uses the symbolic notation $\mathrm{d}X = A(X)\mathrm{d}Z$ instead of (229). Under the assumption of globally Lipschitz continuous coefficients one has:

Theorem C.26 *Let Z be a continuous semi-martingale with values in $\mathbb{R}^l$ and let*

$$A : \mathbb{R}^m \longrightarrow \mathrm{Mat}_{\mathbb{R}}(m\times l)$$

be globally Lipschitz continuous. Then for any $\mathscr{F}_0$-measurable $x_0 : \Omega \to \mathbb{R}^m$ there is a unique solution X of the Itô equation $\mathrm{d}X = A(X)\mathrm{d}Z$ with $X_0 = x_0$ $\mathbb{P}$-a.s.

A *proof* of theorem C.26 can be found in [37] (Satz 6.15). We now turn to Stratonovic stochastic differential equations.

Definition C.27 *Let Z be a continuous semi-martingale with values in $\mathbb{R}^l$ and let*

$$A : \mathbb{R}^m \longrightarrow \mathrm{Mat}_{\mathbb{R}}(m\times l)$$

be smooth. We say that a $\mathbb{R}^m$-valued continuous semi-martingale X is a solution of the Stratonovic stochastic differential equation determined by (A, Z), if for all $j = 1, \dots, m$ one has

$$X_t^j = X_0^j + \sum_{i=1}^{l} \int_0^t A_i^j(X) \underline{\mathrm{d}} Z_s^i \ \mathbb{P}\text{-}a.s.\ \textit{for any } t \geq 0. \tag{230}$$

Symbolically, one writes $\mathrm{d}X = A(X)\underline{\mathrm{d}}Z$. Let us state some simple basic facts that motivate the definition of stochastic differential equations on smooth finite dimensional manifolds: Let (A, Z) be as in definition C.27 and let X be a continuous semi-martingale in $\mathbb{R}^m$. The map A defines smooth vector fields $A_1, \dots, A_l$ on $\mathbb{R}^m$ by setting $A_i(f) := \sum_{j=1}^m A_i^j \partial_j f$. A simple calculation shows that X is a solution of $\mathrm{d}X = A(X)\underline{\mathrm{d}}Z$, if and only if for any real-valued $f \in \mathrm{C}^\infty(\mathbb{R}^m)$ one has

$$\begin{aligned}
\mathrm{d}f(X) &= \sum_{i=1}^{l} A_i(f)(X) \underline{\mathrm{d}} Z^i \qquad (231)\\
&= \sum_{i=1}^{l} A_i(f)(X) \mathrm{d}Z^i + \frac{1}{2} \sum_{i,k=1}^{l} A_i A_k(f)(X) \mathrm{d}[Z^i, Z^k],
\end{aligned}$$

which, by letting f go through the coordinate maps, easily implies that X is a solution of $\mathrm{d}X = A(X)\underline{\mathrm{d}}Z$, if and only if X solves the following Itô type equation:

$$\mathrm{d}X^j = \sum_{i=1}^{l} A_i^j(X) \mathrm{d}Z^i + \frac{1}{2} \sum_{i,k=1}^{l} \sum_{p=1}^{m} A_k^p(X) \partial_p A_i^j(X) \mathrm{d}[Z^i, Z^k].$$

In particular, one can use theorem C.26 to formulate a basic existence and uniqueness theorem for Stratonovic stochastic differential equations given by (A, Z) (for example, one may assume that A is globally Lipschitz).

Remark C.28 Assume that $A_1, \dots, A_l$ are smooth vector fields on a smooth finite dimensional manifold M. Then equation (231) makes sense, if one *defines* a continuous adapted process X on M to be a semi-martingale, if $f(X)$ is a real-valued semi-martingale for arbitrary real-valued $f \in \mathrm{C}^\infty(M)$. This observation leads to the usual definition of (solutions of) stochastic differential equations on manifolds.

□

We close this section with some specific remarks about matrix-valued linear Stratonovic equations: Let

$$B, C : [0, \infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d)$$

be adapted processes such that B is a continuous semi-martingale and such that $t \mapsto C_t$ is pathwise locally integrable. Then by the above considerations there is a uniquely determined solution

$$\mathscr{A}^{B,C} : [0, \infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d)$$

of

$$\mathrm{d}\mathscr{A}_t^{B,C} = \mathscr{A}_t^{B,C}\Big(\underline{\mathrm{d}}B_t - C_t\mathrm{d}t\Big), \;\; \mathscr{A}_0^{B,C} = \mathbf{1}.$$

We have collected some properties of $\mathscr{A}^{B,C}$ in the following proposition:

Proposition C.29 a) *$\mathscr{A}^{B,0,*}$ is uniquely determined as the solution of*

$$\mathrm{d}X_t = (\underline{\mathrm{d}}B_t^* - C_t^*\mathrm{d}t)X_t, \;\; X_0 = \mathbf{1}.$$

b) *$\mathscr{A}^{B,0}$ is invertible and $\mathscr{A}^{B,0,-1}$ is uniquely determined as the solution of*

$$\mathrm{d}X_t = -(\underline{\mathrm{d}}B - C_t\mathrm{d}t)X_t, \;\; X_0 = \mathbf{1}.$$

c) *If*

$$B : [0, \infty) \times \Omega \longrightarrow \mathscr{U}(d),$$

where $\mathscr{U}(d)$ stands for the anti-Hermitian elements of $\mathrm{Mat}(\mathbb{C}^d)$*, then $\mathscr{A}^{B,0}$ is unitary.*

d) *If*

$$B : [0, \infty) \times \Omega \longrightarrow \mathscr{U}(d),$$

and if

$$\tilde{\mathscr{A}}^{B,C} : [0, \infty) \times \Omega \longrightarrow \mathrm{Mat}(\mathbb{C}^d)$$

denotes the pathwise weak (= locally absolutely continuous) solution of

$$\mathrm{d}\tilde{\mathscr{A}}_t^{B,C} = -\tilde{\mathscr{A}}_t^{B,C}\left(\mathscr{A}_t^{B,0}C_t\mathscr{A}_t^{B,0,*}\right)\mathrm{d}t, \;\; \tilde{\mathscr{A}}_0^{B,C} = \mathbf{1},$$

then one has

$$\mathscr{A}^{B,C} = \tilde{\mathscr{A}}^{B,C}\mathscr{A}^{B,0}. \tag{232}$$

Proof. a) This is obvious.

b) This follows from applying the Stratonovic product rule (228) to $\mathscr{A}^{B,0}X$.

c) This follows from combining a) and b).

d) Formula (232) follows from part c) and the Stratonovic product rule.

■

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