The Chern character of ϑ-summable Fredholm modules over DGA's and localization on loop space (arXiv:1901.04721)

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Alvarez-Gaumé/Atiyah/Bismut/Witten (1980's):

With X a cpt., even-dim. Riem. spin MF, there (formally!) exists a canonically given even linear map

$$I : \widehat{\Omega}(LX) := \prod_{j=0}^{\infty} \Omega^j(LX) \longrightarrow \mathbb{C}, \quad \xi \longmapsto \int_{LX} e^{-E+\omega} \wedge \xi,$$

the *supersymmetric path integral*, such that

i) I is supersymmetric (or equivariantly co-closed):

$$I[(d+\iota_K)\xi] = 0 \quad \text{for all } \xi \in \widehat{\Omega}_{S^1}(LX) := \widehat{\Omega}(LX) \cap \{\zeta : \iota_K\zeta = 0\}.$$

Above, K is the generator of $S^1 \circlearrowright LX$, so $d + \iota_K$ turns $\widehat{\Omega}_{S^1}(LX)$ into a super complex ($\rightsquigarrow$ equivariant homology).

ii) For all $\xi \in \widehat{\Omega}_{S^1}(LX)$ with $(d+\iota_K)\xi = 0$, one has the *Duistermaat-Heckmann localization formula*

$$I[\xi] = \int_{X=\mathrm{Fix}(S^1 \circlearrowright LX)} \widehat{A}(X) \wedge \xi|_X. \qquad (1)$$

iii) Given $\mathscr{E} = (E, \nabla) \to X$, there exists a canonically given $\mathrm{Bch}(\mathscr{E}) \in \widehat{\Omega}^+_{S^1}(LX)$ such that

- $\mathrm{Bch}(\mathscr{E})|_X = \mathrm{Ch}(\mathscr{E})$,
- $(d+\iota_K)\mathrm{Bch}(\mathscr{E}) = 0$,
- $\mathrm{ind}(D^{\mathscr{E}}) = I[\mathrm{Bch}(\mathscr{E})]$.

Why all this?

$$\mathrm{ind}(D_{\mathscr{E}}) = I[\mathrm{Bch}(\mathscr{E})] = \int_X \widehat{A}(X) \wedge \mathrm{Bch}(\mathscr{E})|_X = \int_X \widehat{A}(X) \wedge \mathrm{Ch}(\mathscr{E}).$$

Two very serious (and obviously connected) mathematical problems:

- definition of I: $K(\gamma) = \dot{\gamma} \to\leftarrow$ Brownian motion
- right choice of observables: growth conditions $\to\leftarrow$ sufficiently large to carry $\mathrm{Bch}(\mathscr{E})$.

Program: construct a natural map $\rho_\epsilon : \mathsf{C}^\epsilon(\Omega(X)_\mathbb{T}) \to \widehat{\Omega}(LX)$ of super complexes, and a linear functional $\underline{I} : \mathsf{C}^\epsilon(\Omega(X)_\mathbb{T}) \to \mathbb{C}$ which descends to a linear functional

$$I : \Omega_{\mathrm{int}}(LX) := \mathrm{im}(\rho_\epsilon) \longrightarrow \mathbb{C}, \quad \text{so that } I \text{ does the job.}$$

Surprising fact: Using cyclic homology, $\underline{I}$ can be constructed in a very general framework, namely, it is the *Chern character of a ϑ-summable Fredholm over a locally convex DGA....*

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Def. With Ω a LC-DGA (locally convex DGA), an (even-dimensional) ϑ - *summable Fredholm module* over Ω is given by a triple $\mathscr{M} = (\mathcal{H}, \mathbf{c}, Q)$, such that

- $\mathcal{H}$ is a super Hilbert space,
- $\mathbf{c} : \Omega \to \mathscr{L}(\mathcal{H})$ is an even bounded linear map,
- Q an odd self-adjoint (unbounded) linear operator on $\mathcal{H}$ with e^{-tQ^2} trace class for all $t > 0$,

with

$$[Q, \mathbf{c}(f)] = \mathbf{c}(df), \quad \text{and} \quad \mathbf{c}(f\theta) = \mathbf{c}(f)\mathbf{c}(\theta), \quad \mathbf{c}(\theta f) = \mathbf{c}(\theta)\mathbf{c}(f),$$

for all $f \in \Omega^0$, $\theta \in \Omega$. **New:** $\mathbf{c}$ is not a representation (only on Ω^0)!

Ex. (spin case): We can take $\Omega = \Omega(X)$, $\mathcal{H} = L^2(X, \Sigma)$ with $\Sigma \to X$ spin bundle, $Q = D$ the Dirac operator in $L^2(X, \Sigma)$ and

$$\mathbf{c} : \Omega(X) \longrightarrow \mathscr{L}(L^2(X, \Sigma)), \quad \mathbf{c}(\alpha)\Psi(x) := c(\alpha(x))\Psi(x).$$

Def.: The *acyclic extension* of Ω is the LC-DGA given by $\Omega_{\mathbb{T}} := \Omega[\sigma]$ with σ a formal variable of degree -1 with $\sigma^2 = 0$, where on $\theta = \theta' + \sigma\theta'' \in \Omega_{\mathbb{T}}$ the differential is $d_{\mathbb{T}} = d - \iota$ with

$$d\theta = d\theta' - \sigma d\theta'' \quad \text{and} \quad \iota\theta = \theta''.$$

Ex. (spin case): Here we have $\Omega(X)_{\mathbb{T}} \cong \Omega(X \times S^1)^{S^1}$ with

$$\iota \leftrightarrow \iota_{\partial_t}.$$

Def. For each cont. seminorm ν on $\Omega_{\mathbb{T}}$ define a seminorm ϵ_ν on

$$\mathsf{C}(\Omega_{\mathbb{T}}) := \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}} \otimes \Omega_{\mathbb{T}}[1]^{\otimes N}$$

by

$$\epsilon_\nu(\theta) := \sum_{N=0}^{\infty} \frac{\nu(\theta_N)}{N!}, \qquad \theta = \sum_{N=0}^{\infty} \theta_N \in \mathsf{C}(\Omega_{\mathbb{T}}).$$

The completion of $\mathsf{C}(\Omega_{\mathbb{T}})$ with respect to ϵ_ν's is denoted by $\mathsf{C}^{\epsilon}(\Omega_{\mathbb{T}})$ and called the *space of entire chains*.

$$\mathsf{C}^{\epsilon}_{+}(\Omega_{\mathbb{T}}) \xrightarrow{b+B} \mathsf{C}^{\epsilon}_{-}(\Omega_{\mathbb{T}}) \xrightarrow{b+B} \mathsf{C}^{\epsilon}_{+}(\Omega_{\mathbb{T}}) \quad \text{(cyclic homology).}$$

Recall that we have fixed a module $\mathscr{M} = (\mathcal{H}, \mathbf{c}, Q)$ over Ω. Define for each $N \in \mathbb{N}_0$ a linear map

$$F_{\mathscr{M}} : \mathsf{C}^{\epsilon}(\Omega_{\mathbb{T}}) \longrightarrow \{\text{closed operators in } \mathcal{H}\}$$

by

$$\begin{aligned}
&F^{(0)}_{\mathscr{M}} = Q^2,\\
&F^{(1)}_{\mathscr{M}}(\theta) = \mathbf{c}(d\theta') - [Q, \mathbf{c}(\theta')] - \mathbf{c}(\theta''),\\
&F^{(2)}_{\mathscr{M}}(\theta_1 \otimes \theta_2) = (-1)^{|\theta_1'|}\big(\mathbf{c}(\theta_1'\theta_2') - \mathbf{c}(\theta_1')\mathbf{c}(\theta_2')\big),\\
&F^{(N)}_{\mathscr{M}} = 0 \quad \text{for all } N \geqslant 3.
\end{aligned}$$

For $M \leqslant N$ denote with $\mathscr{P}_{M,N}$ all tuples $I = (I_1, \dots, I_M)$ of subsets of $\{1 \dots, N\}$ with $I_1 \cup \cdots \cup I_M = \{1 \dots, N\}$ and with each element of I_a smaller than each element of I_b whenever $a < b$.

Given $\theta_1 \otimes \cdots\cdots \theta_N \in \Omega_{\mathbb{T}}^{\otimes N}$ and $I = (I_1, \ldots, I_M) \in \mathscr{P}_{M,N}$, $1 \leqslant a \leqslant M$ set $\theta_{I_a} := (\theta_{i+1}, \ldots, \theta_{i+m})$, if $I_a = \{j \mid i < j \leqslant i+m\}$ for some i, m.

Thm (G.-Ludewig) *There exists a unique continuous linear functional*

$$\mathrm{Ch}(\mathscr{M}) : \mathsf{C}^{\epsilon}(\Omega_{\mathbb{T}}) \longrightarrow \mathbb{C},$$

the Chern Character of $\mathscr{M}$, such that

$$\langle \mathrm{Ch}(\mathscr{M}), \theta_0 \otimes \cdots \otimes \theta_N \rangle = \sum_{M=1}^{N} (-1)^M \sum_{I \in \mathscr{P}_{M,N}} \int_{\Delta_M} \mathrm{Str}\Big(\mathbf{c}(\theta_0) e^{-\tau_1 Q^2} F_{\mathscr{M}}(\theta_{I_1}) \times$$

$$\times e^{-(\tau_2-\tau_1)Q^2} F_{\mathscr{M}}(\theta_{I_2}) \cdots e^{-(\tau_M-\tau_{M-1})Q^2} F_{\mathscr{M}}(\theta_{I_M}) e^{-(1-\tau_M)Q^2} \Big)\, d\tau.$$

On the properties of $\mathrm{Ch}(\mathscr{M})$:

Thm (G.-Ludewig) $\mathrm{Ch}(\mathscr{M})$ *is even, co-closed, invariant under homotopies of* $\mathscr{M}$*, Chen-normalized, and leads to a noncommutative index theorem of the form*

$$\mathrm{ind}_{\mathbf{c}(p)\mathcal{H}}(\mathbf{c}(p)D\mathbf{c}(p)) = \langle \mathrm{Ch}(\mathscr{M}), \mathrm{Ch}(p) \rangle,$$

for all $p = p^2 \in \mathrm{Mat}_n(\Omega^0)$*. Here,* $\mathrm{Ch}(p) \in \mathsf{C}_+^{\epsilon}(\Omega_{\mathbb{T}})$ *is the Bismut-Chern character of* p*.*

Consider now $\mathscr{M}_X = (L^2(X, \Sigma), \mathbf{c}, D)$ over $\Omega(X)$. The extended Chen integral map

$$\rho_\epsilon : \mathsf{C}^\epsilon\big(\Omega(X)_{\mathbb{T}}\big) \longrightarrow \widehat{\Omega}(\mathsf{L}X),$$

$$\rho(\theta_0, \ldots, \theta_N) = \mathbf{A}_{S^1} \int_{\Delta_N} \theta_0'(0) \wedge \big(\iota_K \theta_1'(\tau_1) + \theta_1''(\tau_1)\big) \wedge \cdots \wedge \big(\iota_K \theta_N'(\tau_N) + \theta_N''(\tau_N)\big) d\tau$$

is a continuous map of super complexes (G.-Cacciatori).

Set

$$\underline{I} := \mathrm{Ch}(\mathscr{M}_X) : \mathsf{C}^{\epsilon}\big(\Omega(X)_{\mathbb{T}}\big) \longrightarrow \mathbb{C}, \quad \Omega_{\mathrm{int}}(LX) := \mathrm{im}(\rho_\epsilon) \subset \widehat{\Omega}(LX).$$

Thm (G.-Ludewig) *There exists a unique linear functional*

$$I : \Omega_{\mathrm{int}}(LX) \longrightarrow \mathbb{C} \quad \textit{with } I \circ \rho_\epsilon = \underline{I}.$$

Moreover, I is even and equiv. co-closed, and for all $\xi \in \Omega_{\mathrm{int}}(LX)$ with $(d + \iota_K)\xi = 0$ one has the localization formula

$$I[\xi] = \int_X \widehat{A}(X) \wedge \xi|_X. \tag{2}$$

Finally, for all $\mathscr{E} \to X$ one has $I[\mathrm{Bch}(\mathscr{E})] = \mathrm{ind}(D_{\mathscr{E}})$.

Proof:

i) I is well-defined, even, equiv. co-closed $\leftrightarrow$ $\underline{I}$ Chen normalized, even, co-closed;

ii) $I[\mathrm{Bch}(\mathscr{E})] = \mathrm{ind}(D_{\mathscr{E}})$ $\leftrightarrow$ non. index theorem and $\rho_\epsilon(\mathrm{Ch}(p)) = \mathrm{Bch}(\mathrm{im}(p))$;

iii) localizaton formula $\leftrightarrow$ $\underline{I}$ homotopy invariant + heat kernel methods à la Getzler.

Thank you very much for your attention!