Feynman-Kac formula for perturbations of order $\leqslant 1$, and noncommutative geometry

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Abstract. Let Q be a differential operator of order $\leqslant 1$ on a complex metric vector bundle $\mathscr{E} \to \mathscr{M}$ with metric connection ∇ over a possibly noncompact Riemannian manifold $\mathscr{M}$. Under very mild regularity assumptions on Q that guarantee that $\nabla^{\dagger}\nabla/2 + Q$ generates a holomorphic semigroup $\mathrm{e}^{-zH_Q^{\nabla}}$ in $\Gamma_{L^2}(\mathscr{M}, \mathscr{E})$ (where z runs through a complex sector which contains $[0, \infty)$), we prove an explicit Feynman-Kac type formula for $\mathrm{e}^{-tH_Q^{\nabla}}$, $t > 0$, generalizing the standard self-adjoint theory where Q is a self-adjoint zeroth order operator. For compact $\mathscr{M}$'s we combine this formula with Berezin integration to derive a Feynman-Kac type formula for an operator trace of the form

$$\mathrm{Tr}\left(\widetilde{V}\int_0^t \mathrm{e}^{-sH_V^{\nabla}} P \mathrm{e}^{-(t-s)H_V^{\nabla}} \mathrm{d}s\right),$$

where $V, \widetilde{V}$ are of zeroth order and P is of order $\leqslant 1$. These formulae are then used to obtain a probabilistic representations of the lower order terms of the equivariant Chern character (a differential graded extension of the JLO-cocycle) of a compact even-dimensional Riemannian spin manifold, which in combination with cyclic homology play a crucial role in the context of the Duistermaat-Heckmann localization formula on the loop space of such a manifold.

1. Introduction

The classical Feynman-Kac formula states that given a real-valued (for simplicity) smooth potential $V : \mathscr{M} \to \mathbb{R}$ on a possibly noncompact Riemannian manifold $\mathscr{M}$ such that the symmetric Schrödinger operator $\Delta/2 + V$ is semibounded from below in $L^2(\mathscr{M})$ (defined initially on smooth compactly supported functions), one has

$$\mathrm{e}^{-tH_V}\Psi(x) = \mathbb{E}\left[1_{\{t<\zeta^x\}}\mathrm{e}^{-\int_0^t V(\mathsf{b}_s^x)\mathrm{d}s}\Psi(\mathsf{b}_t^x)\right] \quad \text{for all } \Psi \in L^2(\mathscr{M}),\ t > 0, \text{ a.e. } x \in \mathscr{M},$$

whenever the expectation value is well-defined. Here

- H_V denotes the Friedrichs realization[1] of $\Delta/2 + V$, taking into account that in general $\Delta/2 + V$ need not have a unique self-adjoint realization, and e^{-tH_V} is defined via spectral calculus,
- b^x is an arbitrary Brownian motion on $\mathscr{M}$ starting from x with lifetime $\zeta^x > 0$, taking into account that $\mathscr{M}$ need not be stochastically complete.

[1] which corresponds to Dirichlet boundary conditions

Covariant versions of this formula have played a crucial role in mathematical physics through the Feynman-Kac-Itô formula [S05, BHL00] and in geometry through probabilistic proofs of the Atiyah-Singer index theorem [B84, H02]. In this context, one replaces Δ with $\nabla^{\dagger}\nabla$, where

$$\nabla : \Gamma_{C^{\infty}}(\mathscr{M},\mathscr{E}) \longrightarrow \Gamma_{C^{\infty}}(\mathscr{M},T^{*}\mathscr{M}\otimes\mathscr{E})$$

is a metric connection on a metric vector bundle $\mathscr{E}\to\mathscr{M}$, and the potential with a smooth pointwise self-adjoint section V of $\mathrm{End}(\mathscr{E})\to\mathscr{M}$. In other words, V is a self-adjoint zeroth order operator. Assuming now that the symmetric covariant Schrödinger type operator $\nabla^{\dagger}\nabla/2+V$ in the space of square integrable sections $\Gamma_{L^2}(\mathscr{M},\mathscr{E})$ is bounded from below, one can prove that

$$\mathrm{e}^{-tH_V^{\nabla}}\Psi(x)=\mathbb{E}\left[1_{\{t<\zeta^x\}}\mathcal{V}^x_{\nabla}(t)/\!\!/^x_{\nabla}(t)^{-1}\Psi(\mathsf{b}^x_t)\right]\quad\text{for all }\Psi\in\Gamma_{L^2}(\mathscr{M},\mathscr{E}),\ t>0,\text{ a.e. }x\in\mathscr{M},\tag{1.1}$$

whenever the expectation is well-defined. Here

- H_V^{∇} is the Friedrichs realization of $\nabla^{\dagger}\nabla/2+V$,
- $/\!\!/^x_{\nabla}$ denotes the stochastic parallel transport along the paths of b^x (cf. section 2 below for the precise definition),
- $\mathcal{V}^x_{\nabla}$ denotes the solution of the following pathwise given ordinary differential equation in $\mathrm{End}(\mathscr{E}_x)$,

$$(\mathrm{d}/\mathrm{d}t)\mathcal{V}^x_{\nabla}(t)=-\mathcal{V}^x_{\nabla}(t)/\!\!/^x_{\nabla}(t)^{-1}V(\mathsf{b}^x_t)/\!\!/^x_{\nabla}(t),\quad \mathcal{V}^x_{\nabla}(0)=1.$$

These facts are well-established (cf. the appendix of [BD01]). Note that a classical assumption on the negative part V^- of V that guarantees that $\nabla^{\dagger}\nabla/2+V$ is semibounded from below and that one has the uniform square-integrability

$$\sup_{x\in\mathscr{M}}\mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathcal{V}^x_{\nabla}(t)|^2\right]<\infty\quad\text{for all }t>0$$

(so that by Cauchy-Schwarz the Feynman-Kac formula holds [G12] for *all* $f\in\Gamma_{L^2}(\mathscr{M},\mathscr{E})$) is given by $|V^-|\in\mathcal{K}(\mathscr{M})$, the Kato class of $\mathscr{M}$ (cf. Definition 2.4). Since bounded functions are always Kato, and since it is possible to find large (possibly weighted) L^p+L^{∞}-type subspaces of $\mathcal{K}(\mathscr{M})$ under very weak assumptions on the geometry of $\mathscr{M}$ (cf. Proposition 2.5), the Kato class becomes very convenient in the context of Feynman-Kac formulae and their applications.

In contrast to the self-adjoint case, very little seems to be known concerning Feynman-Kac formulae in the situation where one replaces the self-adjoint zeroth order operator V by an arbitrary differential operator Q of order $\leqslant 1$, a situation that naturally leads to a non self-adjoint theory. The aim of this paper is to provide a systematic treatement of this problem, dealing with all probabilistic and functional analytic problems that arise naturally in this context, mainly from the noncompactness of $\mathscr{M}$. Our essential insight here, which allows to detect the new probabilistic pieces of the Feynman-Kac formula *explicitly* and which allows to deal with some of the functional analytic problems using perturbation theory, is to decompose Q canonically in the form

$$Q=Q_{\nabla}+\sigma_1(Q)\nabla,$$

where

$$\sigma_1(Q)\in\Gamma_{C^{\infty}}\big(\mathscr{M},\mathrm{Hom}(T^{*}\mathscr{M}\otimes\mathscr{E},\mathscr{E})\big)$$

denotes the first order principal symbol of Q, so that $Q_{\nabla}:=Q-\sigma_1(Q)$ is zeroth order. Since now $\nabla^{\dagger}\nabla+Q$ will typically not be symmetric in $\Gamma_{L^2}(\mathscr{M},\mathscr{E})$, we cannot use the Friedrichs construction to get a self-adjoint operator. Instead, we use Kato's theory of sectorial forms and operators (cf. appendix for the basics of sectorial forms/operators and holomorphic semigroups): to this end, we assume that $\nabla^{\dagger}\nabla/2+Q$ is *sectorial*. It then follows from abstract results that this operator canonically induces a sectorial operator H_Q^{∇} which generates a semigroup of bounded operators $\mathrm{e}^{-zH_Q^{\nabla}}$ in $\Gamma_{L^2}(\mathscr{M},\mathscr{E})$ which is holomorphic for z running through some sector of the complex plane which contains $[0,\infty)$. For fixed $x \in \mathscr{M}$ let now $\mathscr{Q}^x_\nabla$ denote the solution to the Itô equation

$$\mathrm{d}\mathscr{Q}^x_\nabla(t) = -\mathscr{Q}^x_\nabla(t) /\!\!/^x_\nabla(t)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_t) + Q_\nabla(\mathsf{b}^x_t)\mathrm{d}t\big) /\!\!/^x_\nabla(t), \quad \mathscr{Q}^x_\nabla(0) = 1,$$

noting that one can give sense to the underlying Itô differential $\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_t)$ using the Levi-Civita connection on $\mathscr{M}$ (cf. Section 2). With these preparations, our main result, Theorem 2.2 below, reads as follows:

Let $\nabla^\dagger\nabla + Q$ be sectorial and let

$$\sup_{x\in K} \mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x_\nabla(t)|^2\right] < \infty \quad \textit{for all } K \subset \mathscr{M} \textit{ compact, } t > 0. \tag{1.2}$$

Then for all $t > 0$, $\Psi \in \Gamma_{L^2}(\mathscr{M},\mathscr{E})$, $x \in \mathscr{M}$, one has

$$\mathrm{e}^{-tH^\nabla_Q}\Psi(x) = \mathbb{E}\left[1_{\{t<\zeta^x\}}\mathscr{Q}^x_\nabla(t) /\!\!/^x_\nabla(t)^{-1}\Psi(\mathsf{b}^x_t)\right]. \tag{1.3}$$

Let us note that the locally uniform L^2-assumption (1.2) serves two purposes: firstly, it decouples the validity of the Feynman-Kac formula from Ψ (as in the above self-adjoint Kato situation). Secondly and more importantly, it allows us to conclude that the smooth representative of $\mathrm{e}^{-tH^\nabla_Q}\Psi$, which exists by local parabolic regularity, is in fact *pointwise equal* to the right hand side of (1.3), and not only almost everywhere. This is achieved by first proving the formula on relatively compact subsets of $\mathscr{M}$ using Itô-calculus, and then letting these local formulae run through an exhaustion of $\mathscr{M}$, using a recent result for monotone convergence of nondensely defined sectorial forms (this procedure is, up to additional technical difficulties, somewhat analogous to the self-adjoint case) with a parabolic maximum principle for the heat equation (the use of which in this form being new even in the self-adjoint case). To the best of our knowledge, this pointwise identification of the smooth representative is new for stochastically incomplete $\mathscr{M}$'s even in the self-adjoint case.

Making contact with Kato type assumptions, in Proposition 2.6 we prove:

Assume either

- $|\Re(\sigma_1(Q))| \in L^\infty(\mathscr{M})$,
- $\Re(Q_\nabla)$ *is bounded from below by a constant* $\kappa \in \mathbb{R}$,
- $|\Im(Q_\nabla)| \in \mathcal{K}(\mathscr{M})$,

or

- $\sigma_1(Q)$ *is anti-selfadjoint and* $|\sigma_1(Q)| \in L^\infty(\mathscr{M})$,
- $|\Re(Q_\nabla)^-| \in \mathcal{K}(\mathscr{M})$,
- $|\Im(Q_\nabla)| \in \mathcal{K}(\mathscr{M})$.

Then $\nabla^\dagger\nabla + Q$ is sectorial, and one has

$$\sup_{x\in\mathscr{M}} \mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x_\nabla(t)|^2\right] < \infty \quad \textit{for all } t > 0. \tag{1.4}$$

In particular, (1.3) holds true.

Note that above $\Re(A)$ and $\Im(A)$ denote, respectively, the fiberwise defined real part and imaginary part of any zeroth order operator. Since these are self-adjoint zeroth order operators, one can define their positive/negative parts using the spectral calculus fiberwise. Note that, while in the self-adjoint case one can control $|\mathscr{Q}^x_\nabla(t)|$ pathwise using Gronwall's inequality, in the situation of Theorem 2.2 and Proposition 2.6 one has to estimate the solution of a covariant Itô-equation, which in combination with the noncompactness of $\mathscr{M}$ leads to several technical difficulties. Although the present formulation of Proposition 2.6 should cover most applications, it would be natural to replace any (lower) boundedness assumption in Proposition 2.6 with an appropriate Kato-type assumption. Although we tried hard, we have not been able to do that. It would also be very interesting to obtain non self-adjoint variants of semigroup domination [B86, BD01, O99, IS97] (also called 'Kato-Simon inequality' in [G17]) using the Feynman-Kac formula in the above setting, keeping in mind that such estimates play a crucial role in geometric analysis (see e.g. [GP15, BG20]) and in mathematical physics (where they are called 'diamagnetic inequalities' [S77, BHL00]). In the self-adjoint case these estimates take the form

$$|\mathrm{e}^{-tH_V^{\nabla}}\Psi(x)| \leqslant \mathrm{e}^{-tH_v}|\Psi|(x),$$

where $v : \mathscr{M} \to \mathbb{R}$ is any scalar potential such that for all $x \in \mathscr{M}$ every eigenvalue of $V(x)$ is $\geqslant v(x)$.
It should also be noted that, if one ignores functional analytic problems, it is somewhat natural that *some* probabilistic representation of $\mathrm{e}^{-tH_Q^{\nabla}}$ must exist: $\nabla^{\dagger}\nabla + Q$ has a scalar second order principal symbol, and any such operator can be uniquely written in the form $\widetilde{\nabla}^{\dagger}\widetilde{\nabla} + \widetilde{Q}$, where $\widetilde{\nabla}$ is another connection and $\widetilde{Q}$ is of *zeroth* order. However, the assignment $(\nabla, Q) \mapsto (\widetilde{\nabla}, \widetilde{Q})$ is by no means explicit (cf. Proposition 2.5 in [BGV92]), and $\widetilde{\nabla}$ need not be metric, even if even ∇ is so. From this point of view, we believe that our formulation of the Feynman-Kac formula is optimal from the point of view of explicitness and accessibility to perturbation theoretic results such as Proposition 2.6.

Our next main result is the following trace formula (cf. Theorem 2.9):

Assume $\mathscr{M}$ is compact, and let P be of order $\leqslant 1$, and let $V, \widetilde{V}$ be of zeroth order. Then for all $t > 0$ one has
(1.5)
$$\begin{aligned}&\mathrm{Tr}\left(\widetilde{V}\int_0^t \mathrm{e}^{-sH_V^{\nabla}} P \mathrm{e}^{-(t-s)H_V^{\nabla}}\mathrm{d}s\right)\\ &= -\int_{\mathscr{M}} \widetilde{V}(x)\mathrm{e}^{-tH}(x,x)\mathbb{E}_t^{x,x}\left[\mathcal{V}_{\nabla}^x(t)\int_0^t /\!\!/_{\nabla}^x(s)^{-1}\big(\sigma_1(P)^{\flat}(\mathrm{d}\mathsf{b}_s^x) + P_{\nabla}(\mathsf{b}_s^x)\mathrm{d}s\big)/\!\!/_{\nabla}^x(s)/\!\!/_{\nabla}^x(t)^{-1}\right]\mathrm{d}\mu(x),\end{aligned}$$
where $\mathrm{e}^{-tH}(x,y)$ denotes the integral kernel of the Friedrichs realization of Δ (in other words, the heat kernel on $\mathscr{M}$), and $\mathbb{E}_t^{x,x}$ denotes the exppectation with respect to the Brownian bridge starting in x and ending in x at the time t.

The proof of this result is in fact reduced to (1.3) using Berezin integration, a trick which has been communicated to the authors by Shu Shen. It would be very interesting to see, if at least for certain P's it is possible to obtain (1.5) using the very general Bismut derivative formulae from [BD01] in combination with the Markov property of Brownian motion. We have not worked into this direction.

Finally, we use (1.5) together with a new commutation formula for spin-Dirac operators (cf. formula (3.4) below) to establish a probabilistic formula for the 'first order' part of the equivariant Chern-Character $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ of a compact even-dimensional Riemannian spin manifold $\mathscr{M}$, where $\mathbb{T} := S^1$. We refer the reader to Section 3 for the definition of $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ and concentrate here only the probabilistic side of the formula: to this end, note that every element α of the space $\Omega_{\mathbb{T}}(\mathscr{M})$ of $\mathbb{T}$-invariant differential forms on $\mathscr{M} \times \mathbb{T}$ can be uniquely written in the form $\alpha = \alpha' + \alpha''\mathrm{d}t$ with $\mathrm{d}t$ the volume form on $\mathbb{T}$. Then $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ becomes a complex linear functional on the space

$$\mathsf{C}_{\mathbb{T}}(\mathscr{M}) := \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(\mathscr{M})^{\otimes(N+1)}.$$

In Theorem 3.1 we prove:

For all $\alpha_0, \alpha_1 \in \Omega_{\mathbb{T}}(\mathscr{M})$, $t > 0$ one has

$$\begin{aligned}&\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0 \otimes \alpha_1)\\ &= \int_{\mathscr{M}} \mathrm{e}^{-tH}(x,x)\mathrm{Str}_x \left(c(\alpha_0')(x)\mathbb{E}_t^{x,x} \left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x)\mathrm{d}s} \int_0^t /\!\!/_{\nabla}^{x}(s)^{-1} \Big(2c(*\mathrm{d}\mathsf{b}_s^x \lrcorner \alpha_1') - c(\alpha_1'')(\mathsf{b}_s^x)\mathrm{d}s \Big) /\!\!/_{\nabla}^{x}(s) /\!\!/_{\nabla}^{x}(t)^{-1} \right] |_{t=2} \right) \mathrm{d}\mu(x),\end{aligned}$$

where

- Str_x *denotes the $\mathbb{Z}_2$-graded trace on* $\mathrm{End}(\mathscr{S}_x)$, *with $\mathscr{S} \to \mathscr{M}$ the spin bundle,*
- $/\!\!/_{\nabla}^{x}$ *denotes the stochastic parallel transport $\mathscr{S} \to \mathscr{M}$,*
- $c : \Omega_{C^\infty}(\mathscr{M}) \to \Gamma_{C^\infty}(\mathscr{M}, \mathrm{End}(\mathscr{S}))$ *denotes Clifford multiplication,*
- $c(*\mathrm{d}\mathsf{b}_s^x \lrcorner \alpha)$ *denotes a* Stratonovic *differential with respect to the* $\mathrm{End}(\mathscr{S})$*-valued* 1*-form* $v \mapsto c(v \lrcorner \alpha)$,
- $\mathbb{E}_t^{x,x}$ *denotes the expectation with respect to the Brownian bridge starting x and ending at the time t in x.*

We remark that $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ has been introduced in [GL19] in the abstract setting of ϑ-summable Fredholm modules over locally convex differential graded algebras and is in fact a differential-graded refinement of the JLO-cocycle [JLO88] for ungraded algebras. When applied to a compact even dimensional Riemannian spin-manifold, this construction provides via Chen integrals an algebraic model for Duistermaat-Heckman localization on the space of smooth loops, allowing a proof of the Atiyah-Singer index theorem for twisted spin-Dirac operators in the spirit of Atiyah [A83] and Bismut [B85]. We refer the reader to the introduction of [GL19] for a detailed explanation of these results. Obtaining a probabilistic formula for the higher order pieces of the equivarant Chern character remains an open problem at this point.

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2. Main results

Let $\mathscr{M}$ be a connected Riemannian manifold of dimension m, where we work exclusively in the catogory of smooth manifolds without boundary. As such it is equipped with its Levi-Civita connection and its volume measure μ. We denote the open geodesic balls with $B(x,r) \subset \mathscr{M}$. Any fiberwise metric on a vector bundle will simply be denoted with $(\bullet, \bullet)$, with $|\bullet| := \sqrt{(\bullet, \bullet)}$. If $\mathscr{E} \to \mathscr{M}$ is a metric vector bundle and $p \in [1, \infty]$, then the norm on the complex Banach space of L^p-sections is denoted with

$$\|\Psi\|_p := \left(\int |\Psi|^p \mathrm{d}\mu \right)^{1/p}.$$

(with the obvious replacement for $p = \infty$). The scalar product in the Hilbert space $\Gamma_{L^2}(\mathscr{M}, \mathscr{E})$ is denoted by

$$\langle \Psi_1, \Psi_2 \rangle = \int (\Psi_1, \Psi_2) \mathrm{d}\mu.$$

- Given another metric vector bundle $\mathscr{F} \to \mathscr{M}$ and a differential operator

$$P : \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathscr{F})$$

of order $\leqslant k$ with smooth coefficients, its formal adjoint

$$P^\dagger : \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathscr{F})$$

is the uniquely determined differential operator of order $\leqslant k$ with smooth coefficients, which satisfies

$$\langle P\Psi_1, \Psi_2 \rangle = \langle \Psi_1, P^\dagger \Psi_2 \rangle \quad \text{for all } \Psi_1 \in \Gamma_{C_c^\infty}(\mathscr{M}, \mathscr{E}),\ \Psi_2 \in \Gamma_{C_c^\infty}(\mathscr{M}, \mathscr{E}).$$

Assume from now on that $\mathscr{E}\to\mathscr{M}$ is a metric vector bundle with a smooth metric connection

$$\nabla:\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})\longrightarrow\Gamma_{C^\infty}(\mathscr{M},T^*\mathscr{M}\otimes\mathscr{E})$$

Given a differential operator

$$Q:\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})$$

of order $\leqslant 1$, then with its first order principal symbol

$$\sigma_1(Q)\in\Gamma_{C^\infty}\big(\mathscr{M},\mathrm{Hom}(T^*\mathscr{M},\mathrm{End}(\mathscr{E}))\big)=\Gamma_{C^\infty}\big(\mathscr{M},\mathrm{Hom}(T^*\mathscr{M}\otimes\mathscr{E},\mathscr{E})\big),$$

the operator

$$Q_\nabla:=Q-\sigma_1(Q)\nabla\quad\text{is zeroth order,}$$

thus

$$Q_\nabla\in\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{E})),\quad Q=Q_\nabla+\sigma_1(Q)\nabla.$$

Assume that for every $x\in\mathscr{M}$ we are given a maximally defined Brownian motion

$$\mathsf{b}^x:[0,\zeta^x)\times\Omega\longrightarrow\mathscr{M}$$

on $\mathscr{M}$ with starting point x and explosion time $\zeta^x>0$, which is defined on a fixed filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_*,\mathbb{P})$ that satisfies the usual assumptions. Let

$$/\!/^x_\nabla:[0,\zeta^x)\times\Omega\longrightarrow\mathscr{E}\boxtimes\mathscr{E}^\dagger$$

be the corresponding stochastic parallel transport with respect to the fixed metric connection, where $\mathscr{E}\boxtimes\mathscr{E}^\dagger\to\mathscr{M}\times\mathscr{M}$ denotes the vector bundle whose fiber at (a,b) is $\mathrm{Hom}(\mathscr{E}_a,\mathscr{E}_b)$. This is the uniquely determined continuous semimartingale such that [N92] for all $t\in[0,\zeta^x)$,

- one has $/\!/^x_\nabla(t):\mathscr{E}_x\to\mathscr{E}_{\mathsf{b}_t(x)}$ unitarily,
- for all $\Psi\in\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})$ one has

$$/\!/^x_\nabla(t)^{-1}\Psi(\mathsf{b}^x_t)=/\!/^x_\nabla(t)^{-1}\nabla(*\mathrm{d}\mathsf{b}^x_t)\Psi(\mathsf{b}^x_t),\quad /\!/^x_\nabla(0)=1.\tag{2.1}$$

Above and in the sequel, $*\mathrm{d}$ stands for Stratonovic integration, while d will denote Itô integration. Note that one can integrate 1-forms in the Stratonovic sense on any manifold along any continuous semimartingale, while one can integrate 1-forms on $\mathscr{M}$ along b^x also in the Itô sense, using the Levi-Civita connection on $\mathscr{M}$.
Define the process

$$\mathscr{Q}^x_\nabla:[0,\zeta^x)\times\Omega\longrightarrow\mathrm{End}(\mathscr{E}_x)$$

as the unique solution to the Itô equation

$$\mathrm{d}\mathscr{Q}^x_\nabla(t)=-\mathscr{Q}^x_\nabla(t)/\!/^x_\nabla(t)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_t)+Q_\nabla(\mathsf{b}^x_t)\mathrm{d}t\big)/\!/^x_\nabla(t),\quad \mathscr{Q}^x_\nabla(0)=1.$$

Written out explicitly, the above equation means that for all $t\geqslant 0$ one has

$$\mathscr{Q}^x_\nabla(t)=1-\int_0^t\mathscr{Q}^x_\nabla(s)/\!/^x_\nabla(s)^{-1}\sigma_1(Q)^\flat(U^x_se_j)/\!/^x_\nabla(s)\mathrm{d}W^{x,j}_s+\int_0^t/\!/^x_\nabla(s)^{-1}Q_\nabla(\mathsf{b}^x_s)/\!/^x_\nabla(s)\mathrm{d}s,$$

a.s. on $\{t<\zeta^x\}$, where $e_1,\dots,e_m$ is the standard basis of $\mathbb{R}^m$,

$$U^x:[0,\zeta^x)\times\Omega\longrightarrow O(\mathscr{M})=\bigcup_{x\in\mathscr{M}}O(\mathbb{R}^m,T_x\mathscr{M})$$

is a horizontal lift of b^x with respect to the Levi-Civita connection on $\mathscr{M}$ to the principal fiber bundle of orthonormal frames $O(\mathscr{M})\to\mathscr{M}$, and

$$W^x:=\int_0^\bullet\varpi(*\mathrm{d}U^x_s):[0,\zeta^x)\times\Omega\longrightarrow\mathbb{R}^m$$

is the $\mathbb{R}^m$-representation of b^x (in particular, W^x is a Euclidean Brownian motion), with

$$\varpi\in\Omega^1_{C^\infty}(O(\mathscr{M}),\mathbb{R}^m),\quad\varpi_u(A):=u^{-1}(T\pi(A_u)),\quad A_u\in T_uO(\mathscr{M}),\quad u\in O(\mathscr{M}),$$

the solder 1-form of $\pi : O(\mathscr{M}) \to \mathscr{M}$. These constructions do not depend on the initial value $U_0^x \in O(\mathbb{R}^m, T_x\mathscr{M})$.

It is often useful to know for estimates that the processes of the form $\mathscr{Q}_\nabla^x$ factor as follows:

Remark 2.1. Let $\alpha \in \Omega^1_{C^\infty}(\mathscr{M}, \mathrm{End}(\mathscr{E}))$, $V, W \in \Gamma_{C^\infty}(\mathscr{M}, \mathrm{End}(\mathscr{E}))$ and let

$$C : [0, \zeta^x) \times \Omega \longrightarrow \mathrm{End}(\mathscr{E}_x)$$

be the solution to

$$\mathrm{d}C(t) = -C(t) /\!\!/_\nabla^x(t)^{-1} \big(V(\mathsf{b}_t^x) + \alpha(\mathrm{d}\mathsf{b}_t^x) + W(\mathsf{b}_t^x)\mathrm{d}t\big) /\!\!/_\nabla^x(t), \quad C(0) = 1.$$

Such a C factors as follows: let

$$A : [0, \zeta^x) \times \Omega \longrightarrow \mathrm{End}(\mathscr{E}_x)$$

be the solution to

$$\mathrm{d}A(t) = -A(t) /\!\!/_\nabla^x(t)^{-1} \big(\alpha(\mathrm{d}\mathsf{b}_t^x) + W(\mathsf{b}_t^x)\mathrm{d}t\big) /\!\!/_\nabla^x(t), \quad A(0) = 1.$$

Then A is invertible and

$$A^{-1} : [0, \zeta^x) \times \Omega \longrightarrow \mathrm{End}(\mathscr{E}_x)$$

is the solution to

$$\mathrm{d}A(t)^{-1} = /\!\!/_\nabla^x(t)^{-1} \big(\alpha(\mathrm{d}\mathsf{b}_t^x) + W(\mathsf{b}_t^x)\mathrm{d}t\big) /\!\!/_\nabla^x(t) A(t)^{-1}, \quad A(0)^{-1} = 1.$$

Let B be the solution to

$$\mathrm{d}B(t) = -B(t)A(t) /\!\!/_\nabla^x(t)^{-1} V(\mathsf{b}_t^x) /\!\!/_\nabla^x(t) A(t)^{-1}\mathrm{d}t, \quad B(0) = 1.$$

Then by the Itô product rule we have

$$\begin{aligned}
\mathrm{d}(B(t)A(t)) &= (\mathrm{d}B(t))A(t) + B(t)\mathrm{d}A(t) + \mathrm{d}B(t)\mathrm{d}A(t)\\
&= -B(t)A(t) /\!\!/_\nabla^x(t)^{-1} V(\mathsf{b}_t^x) /\!\!/_\nabla^x(t) A(t)^{-1}\mathrm{d}t A(t)\\
&\quad - B(t)A(t) /\!\!/_\nabla^x(t)^{-1} \big(\alpha(\mathrm{d}\mathsf{b}_t^x) + W(\mathsf{b}_t^x)\mathrm{d}t\big) /\!\!/_\nabla^x(t)\\
&\quad + \quad \text{summands containing } \mathrm{d}t \text{ and } \mathrm{d}\mathsf{b}_t^x, \text{ or } \mathrm{d}t \text{ and } \mathrm{d}t,
\end{aligned}$$

so that by uniqueness $C = AB$.

b) As a particular case of the above situation, Let

$$\mathscr{Q}_{1,\nabla}^x : [0, \zeta^x) \times \Omega \longrightarrow \mathrm{End}(\mathscr{E}_x)$$

be the solution to

$$\mathrm{d}\mathscr{Q}_{1,\nabla}^x(t) = -\mathscr{Q}_{1,\nabla}^x(t) /\!\!/_\nabla^x(t)^{-1} \sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_t^x) /\!\!/_\nabla^x(t), \quad \mathscr{Q}_{1,\nabla}^x(0) = 1,$$

and let $\mathscr{Q}_{2,\nabla}^x$ be the solution to

$$\mathrm{d}\mathscr{Q}_{2,\nabla}^x(t) = -\mathscr{Q}_{2,\nabla}^x(t)\mathscr{Q}_{1,\nabla}^x(t) /\!\!/_\nabla^x(t)^{-1} Q_\nabla(\mathsf{b}_t^x) /\!\!/_\nabla^x(t) \mathscr{Q}_{1,\nabla}^x(t)^{-1}\mathrm{d}t, \quad \mathscr{Q}_{2,\nabla}^x(t) = 1.$$

Then we have

$$\mathscr{Q}_\nabla^x(t) = \mathscr{Q}_{2,\nabla}^x(t)\mathscr{Q}_{1,\nabla}^x(t). \tag{2.2}$$

Any differential operator

$$Q : \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E})$$

induces a densely defined sesqui-linear form

$$\Gamma_{C^\infty_c}(\mathscr{M}, \mathscr{E}) \times \Gamma_{C^\infty_c}(\mathscr{M}, \mathscr{E}) \ni (\Psi_1, \Psi_2) \longmapsto h_Q^\nabla(\Psi_1, \Psi_2) := \big\langle (\nabla^\dagger\nabla/2 + Q)\Psi_1, \Psi_2 \big\rangle \in \mathbb{C} \tag{2.3}$$

in $\Gamma_{L^2}(\mathscr{M}, \mathscr{E})$. In case this form is sectorial it is automatically closable (stemming from a sectorial operator), and we denote the closed operator in $\Gamma_{L^2}(\mathscr{M}, \mathscr{E})$ induced by the closure of

h^{∇}_Q with H^{∇}_Q in the sense of Theorem A.2 from the appendix. It follows that H^{∇}_Q generates a holomorphic semigroup (cf. appendix)

$$(\mathrm{e}^{-zH^{\nabla}_Q})_{z\in\Sigma_{0,\beta}}\subset\mathscr{L}(\Gamma_{L^2}(\mathscr{M},\mathscr{E})),$$

which is defined on some sector of the form

$$\Sigma_{0,\beta}=\{r\mathrm{e}^{\sqrt{-1}\alpha}:r\geqslant 0,\alpha\in(-\beta,\beta)\}\quad\text{ for some }\beta\in(0,\pi/2].$$

In the situation of a trivial complex line bundle with its trivial connection (identifying sections with functions) we will ommit the dependence on the connection in the notation. In particular, $H\geqslant 0$ stands for the Friedrichs realization of the scalar Laplace-Beltrami operator $\Delta/2$ in $L^2(\mathscr{M})$.

Theorem 2.2. *Let*

$$Q:\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})$$

be a differential operator of order $\leqslant 1$. Assume that h^{∇}_Q is sectorial and that

$$\sup_{x\in K}\mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x_{\nabla}(t)|^2\right]<\infty\quad\text{for all } K\subset\mathscr{M}\text{ compact, } t>0. \tag{2.4}$$

Then for all $t>0$, $\Psi\in\Gamma_{L^2}(\mathscr{M},\mathscr{E})$, $x\in\mathscr{M}$, one has

$$\mathrm{e}^{-tH^{\nabla}_Q}\Psi(x)=\mathbb{E}\left[1_{\{t<\zeta^x\}}\mathscr{Q}^x_{\nabla}(t)/\!\!/^x_{\nabla}(t)^{-1}\Psi(\mathsf{b}^x_t)\right]. \tag{2.5}$$

Remark 2.3. By local parabolic regularity, the time dependent section $(t,x)\mapsto\mathrm{e}^{-tH^{\nabla}_Q}\Psi(x)$ has a representative which is smooth on $(0,\infty)\times\mathscr{M}$, and (2.5) means that the RHS of this equation is precisely this smooth representative. This pointwise identification, which is based on the locally uniform integrability assumption (2.4), is highly nontrivial in the stochastically incomplete case and even slightly improves the existing results in the 'usual' Feynman-Kac setting ($\sigma_1(Q)=0$ and Q_∇ self-adjoint), where so far only an μ-almost everywhere equality has been established.

Proof of Theorem 2.2: We omit the dependence on ∇ of several data in the notation, whenever there is no danger of confusion. Fix $x\in\mathscr{M}$, $t>0$ and pick an exhaustion $(U_l)_{l\in\mathbb{N}}$ of $\mathscr{M}$ with open connected relatively compact subsets having a smooth boundary. Let $H_{Q,l}$ be defined with $\mathscr{M}$ replaced by U_l (note that this corresponds to Dirichlet boundary conditions). It suffices to show that (with an obvious notation) for all $\Psi\in\Gamma_{C^\infty_c}(\mathscr{M},\mathscr{E})$ and all l large enough such that Ψ is supported in U_l one has

$$\mathrm{e}^{-tH_{Q,l}}\Psi(x)=\mathbb{E}\left[1_{\{t<\zeta^l_x\}}\mathscr{Q}^x(t)/\!\!/^x(t)^{-1}\Psi(\mathsf{b}^x_t)\right]. \tag{2.6}$$

Indeed, we have

$$\lim_{l\to\infty}\left\|\mathrm{e}^{-tH_{Q,l}}\Psi-\mathrm{e}^{-tH_Q}\Psi\right\|_2=0 \tag{2.7}$$

by an abstract monotone convergence theorem for nondensely defined sectorial forms (Theorem 3.7 in [CtE18]), and furthermore for every compact set $K\subset\mathscr{M}$ with $x\in K$ we have

$$\begin{aligned}&\sup_{y\in K}\left|\mathbb{E}\left[(1_{\{t<\zeta^y\}}-1_{\{t<\zeta^y_l\}})\mathscr{Q}^y(t)/\!\!/^y(t)^{-1}\Psi(\mathsf{b}^y_t)\right]\right|\\
&\leqslant\sup_{y\in K}\|\Psi\|_\infty\,\mathbb{E}\left[1_{\{t<\zeta^y\}}-1_{\{t<\zeta^y_l\}}\right]^{1/2}\mathbb{E}\left[1_{\{t<\zeta^y\}}|\mathscr{Q}^y(t)|^2\right]^{1/2}\\
&\leqslant\sup_{y\in K}\mathbb{E}\left[1_{\{t<\zeta^y\}}|\mathscr{Q}^y(t)|^2\right]^{1/2}2^{1/2}\sup_{y\in K}\|\Psi\|_\infty\,(\mathrm{e}^{-tH}1(y)-\mathrm{e}^{-tH_l}1(y))^{1/2}.\end{aligned}$$

The latter expression converges to zero as $l\to\infty$ by a maximum principle for the heat equation of Dodziuk [D83], which shows that the RHS of (2.5) is continuous in x, and that in view of

(2.7) one has (2.5) for μ-a.e $x \in \mathscr{M}$. A posteriori this equality holds for *all* x, as both sides are continuous in x. If Ψ is only square integrable, we can pick a sequence of smooth compactly supported sections $(\Psi_n)_{n\in\mathbb{N}}$ with $\|\Psi_n - \Psi\|_2 \to 0$. Given an open relatively compact subset $U \subset \mathscr{M}$ with $x \in U$, we have

$$\mathrm{e}^{-tH_{Q,l}} : \Gamma_{L^2}(\mathscr{M}, \mathscr{E}) \longrightarrow \Gamma_{C_b}(U, \mathscr{E})$$

algebraically by elliptic regularity (where $\Gamma_{C_b}(U, \mathscr{E})$ denotes the Banach space of continuous bounded sections of $\mathscr{E}|_U \to U$ equipped with the uniform norm), and a posteriori *continuously* by the closed graph theorem, we then have

$$\lim_{n\to\infty} \mathrm{e}^{-tH_Q}\Psi_n(x) = \mathrm{e}^{-tH_Q}\Psi(x),$$

and

$$\begin{aligned}
&\left|\mathbb{E}\left[1_{\{t<\zeta^x\}}\mathscr{Q}^x(t)/\!\!/^x(t)^{-1}(\Psi_n(\mathsf{b}_t^x) - \Psi(\mathsf{b}_t^x))\right]\right| \\
&\leqslant \mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x(t)|^2\right]^{1/2} \mathbb{E}\left[1_{\{t<\zeta^x\}}|\Psi_n(\mathsf{b}_t^x) - \Psi(\mathsf{b}_t^x)|^2\right]^{1/2} \\
&= \mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x(t)|^2\right]^{1/2} \left(\int \mathrm{e}^{-tH}(x,y)|\Psi_n(y) - \Psi(y)|^2 \mathrm{d}\mu(y)\right)^{1/2} \\
&\leqslant \mathbb{E}\left[1_{\{t<\zeta^x\}}|\mathscr{Q}^x(t)|^2\right]^{1/2} \left(\sup_{y\in\mathscr{M}} \mathrm{e}^{-tH}(x,y)\right)^{1/2} \|\Psi_n - \Psi\|_2,
\end{aligned}$$

which tends to 0 as $n \to \infty$ and proves (2.5) again.
It remains to show (2.6): By parabolic regularity, the time dependent section

$$\Psi_s(y) := \mathrm{e}^{-(t-s)H_{Q,l}}\Psi(y)$$

of $\mathscr{E}|_{U_l} \to U_l$ extends smoothly to $[0,t] \times \overline{U_l}$ and Ψ_s vanishes in ∂U_l for all $s \in [0,t)$. Define a continuous semimartingale by

$$N : [0, t \wedge \zeta_l^x] \times \Omega \longrightarrow \mathscr{E}_x, \quad N_s := \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\Psi_s(\mathsf{b}_s^x).$$

Then we have

$$
\begin{aligned}
&\mathrm{d}N_s = (\mathrm{d}\mathscr{Q}^x(s))/\!\!/^x(s)^{-1}\Psi_s(\mathsf{b}_s^x) + \mathscr{Q}^x(s)\mathrm{d}/\!\!/^x(s)^{-1}\Psi_s(\mathsf{b}_s^x) + \mathrm{d}\mathscr{Q}^x(s)\mathrm{d}/\!\!/^x(s)^{-1}\Psi_s(\mathsf{b}_s^x)\\
&= -\mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_s^x) + Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\big)\Psi_s(\mathsf{b}_s^x)\\
&\quad + \mathscr{Q}^x(s)\left(/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\quad - \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_s^x) + Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\big)/\!\!/^x(s)\\
&\qquad \times\left(/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\equiv -\mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\big(Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\big)\Psi_s(\mathsf{b}_s^x)\\
&\quad + \mathscr{Q}^x(s)\left(/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) - \frac{1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x)\mathrm{d}s + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\quad - \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_s^x) + Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\big)/\!\!/^x(s)\\
&\qquad \times\left(/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + \frac{1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x)\mathrm{d}s + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\equiv -\mathscr{Q}^x(s)/\!\!/^x(s)^{-1}Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\Psi_s(\mathsf{b}_s^x) + \mathscr{Q}^x(s)\left(\frac{-1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x)\mathrm{d}s + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\quad - \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\big(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_s^x) + Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\big)/\!\!/^x(s)\\
&\qquad \times\left(/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + \frac{-1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x)\mathrm{d}s + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&= -\mathscr{Q}^x(s)/\!\!/^x(s)^{-1}Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\Psi_s(\mathsf{b}_s^x) + \mathscr{Q}^x(s)\left(\frac{-1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x) + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\quad - \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_s^x)\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x)\\
&= -\mathscr{Q}^x(s)/\!\!/^x(s)^{-1}Q_\nabla(\mathsf{b}_s^x)\mathrm{d}s\Psi_s(\mathsf{b}_s^x) + \mathscr{Q}^x(s)\left(\frac{-1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x) + /\!\!/^x(s)^{-1}\partial_s\Psi_s(\mathsf{b}_s^x)\mathrm{d}s\right)\\
&\quad - \mathscr{Q}^x(s)/\!\!/^x(s)^{-1}\sigma_1(Q)\nabla\Psi_s(\mathsf{b}_s^x)\\
&= 0,
\end{aligned}
$$

where $\equiv$ stands for equality up to continuous local martingales. In the above calculation, we have used the Itô product rule, the differential equation for $\mathscr{Q}^x$, the formula

$$\mathrm{d}/\!\!/^x(s)^{-1}\Psi_s(\mathsf{b}_s^x) = /\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + /\!\!/^x(s)_s^{-1}\partial_s\Psi_s(\mathsf{b}_s^x),$$

which follows from applying (2.1) to the metric connection $\pi^*\nabla$ on the metric vector bundle $\pi^*\mathscr{E} \to \mathscr{M} \times [0,\infty)$ with the projection $\pi : \mathscr{M} \times [0,\infty) \to \mathscr{M}$, the covariant Stratonovic-to-Itô formula

$$/\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) = /\!\!/^x(s)^{-1}\nabla\Psi_s(*\mathrm{d}\mathsf{b}_s^x)(\mathsf{b}_s^x) + \frac{1}{2}/\!\!/^x(s)^{-1}\nabla^\dagger\nabla\Psi_s(\mathsf{b}_s^x)\mathrm{d}s,$$

and

$$\partial_s\Psi_s = ((1/2)\nabla^\dagger\nabla + \sigma_1(Q)\nabla + Q_\nabla)\Psi_s.$$

This shows that N is a continuous local martingale. Since U_l is relatively compact, N is in fact a martingale: indeed, a.s., for all $s > 0$ we have in $\{s < \zeta^x\}$ from the differential equation for $\mathscr{Q}^x$ and Jenßen's inequality

$$|\mathscr{Q}^x(s)|^2 \leqslant C + C\left|\int_0^s \mathscr{Q}^x(r)/\!\!/^x(r)^{-1}\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}_r^x)/\!\!/^x(r)\right|^2 + Cs\int_0^s |\mathscr{Q}^x(r)|^2|Q(\mathsf{b}_r^x)|^2\mathrm{d}r,$$

so that by the Burkholder-Davis-Gundy inequality, with

$$\vartheta_n := \inf\{r \geqslant 0 : |\mathcal{Q}^x(r)| > n\}, \quad n \in \mathbb{N},$$

one has

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s\wedge\vartheta_n)|^2\Big] \\
&\leqslant C' + C'\mathbb{E}\left[\int_0^{t\wedge\zeta_l^x} |\mathcal{Q}^x(r\wedge\vartheta_n)|^2|\sigma_1(Q)^\flat(\mathsf{b}_r^x)|^2\mathrm{d}r\right] + t\mathbb{E}\left[\int_0^{t\wedge\zeta_l^x} |\mathcal{Q}^x(r\wedge\vartheta_n)|^2|Q_\nabla(\mathsf{b}_r^x)|^2\mathrm{d}r\right] \\
&\leqslant C' + C'\left(\sup_{y\in U_l}|\sigma_1(Q)^\flat(y)|^2\right)\mathbb{E}\left[\int_0^{t\wedge\zeta_l^x} |\mathcal{Q}^x(r\wedge\vartheta_n)|^2\mathrm{d}r\right] \\
&\qquad + t\left(\sup_{y\in U_l}|Q_\nabla(y)|^2\right)\mathbb{E}\left[\int_0^{t\wedge\zeta_l^x} |\mathcal{Q}^x(r\wedge\vartheta_n)|^2\mathrm{d}r\right] \\
&\leqslant C_{Q,l} + (C_{Q,l} + tC_{Q,l})\mathbb{E}\left[\int_0^{t\wedge\zeta_l^x} |\mathcal{Q}^x(r\wedge\vartheta_n)|^2\mathrm{d}r\right] \\
&\leqslant C_{Q,l} + (C_{Q,l} + tC_{Q,l})\mathbb{E}\left[\int_0^{t} \sup_{s\leqslant r\wedge\zeta_l^x}|\mathcal{Q}^x(s\wedge\vartheta_n)|^2\mathrm{d}r\right],
\end{aligned}$$

where C, C' are universal constants, and $C_{Q,l}$ depends only on $\|Q_\nabla|_{U_l}\|_\infty$ and $\|\sigma_1(Q)|_{U_l}\|_\infty$. As a consequence, for all $T > 0$ with $t \leqslant T$, Gronwall's inequality gives

$$\mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s\wedge\vartheta_n)|^2\right] \leqslant C_{Q,l}\mathrm{e}^{C_{Q,l,T}t},$$

where $C_{Q,l,T}$ only depends on Q, l, T, and so

$$\mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s)|^2\right] = \mathbb{E}\left[\max_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s)|^2\right] = \mathbb{E}\left[\lim_n \max_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s\wedge\vartheta_n)|^2\right] \tag{2.8}$$

$$\leqslant \liminf_n \mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s\wedge\vartheta_n)|^2\right] \leqslant C_{Q,l}\mathrm{e}^{C_{Q,l,T}t} < \infty \tag{2.9}$$

by Fatou's lemma. We arrive at

$$\mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |N_s|^2\right] \leqslant \left(\sup_{s\in[0,t],y\in U_l}|\Psi_s(y)|^2\right)\mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |\mathcal{Q}^x(s)|^2\right] < \infty,$$

so that

$$\mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |N_s|\right] \leqslant \mathbb{E}\left[\sup_{s\leqslant t\wedge\zeta_l^x} |N_s|^2\right]^{1/2} < \infty,$$

which shows that N is a martingale, as claimed.
We thus have

$$\begin{aligned}
\mathrm{e}^{-tH^l_Q}\Psi(x) &= \mathbb{E}[N_0] = \mathbb{E}[N_{t\wedge\zeta^x_l}]\\
&= \mathbb{E}\left[\mathcal{Q}^x(t\wedge\zeta^x_l)/\!/^x(t\wedge\zeta^x_l)^{-1}\Psi_{t\wedge\zeta^x_l}(\mathsf{b}^x_{t\wedge\zeta^x_l})\right]\\
&= \mathbb{E}\left[(1_{\{t<\zeta^l_x\}}+1_{\{t\geqslant\zeta^l_x\}})\mathcal{Q}^x(t\wedge\zeta^x_l)/\!/^x(t\wedge\zeta^x_l)^{-1}\Psi_{t\wedge\zeta^x_l}(\mathsf{b}^x_{t\wedge\zeta^x_l})\right]\\
&= \mathbb{E}\left[1_{\{t<\zeta^l_x\}}\mathcal{Q}^x(t)/\!/^x(t)^{-1}\Psi_{t\wedge\zeta^x_l}(\mathsf{b}^x_t)\right]+\mathbb{E}\left[1_{\{t\geqslant\zeta^l_x\}}\mathcal{Q}^x(\zeta^x_l)/\!/^x(\zeta^x_l)^{-1}\Psi_{t\wedge\zeta^x_l}(\mathsf{b}^x_{\zeta^x_l})\right]\\
&= \mathbb{E}\left[1_{\{t<\zeta^l_x\}}\mathcal{Q}^x(t)/\!/^x(t)^{-1}\Psi(\mathsf{b}^x_t)\right].
\end{aligned}$$

This completes the proof. ∎

In order to evaluate the somewhat abstract assumptions from Theorem 2.2, we recall the definition of the Kato class (referring the reader to [G17, SV96, AS, S82, G12, S93] and the refernces therein for some fundamental results concerning this class):

Definition 2.4. A Borel function $w:\mathscr{M}\to\mathbb{R}$ is said to be in the Kato class $\mathcal{K}(\mathscr{M})$ of $\mathscr{M}$, if

$$\lim_{t\to 0+}\sup_{x\in\mathscr{M}}\int_0^t\mathbb{E}\left[1_{\{s<\zeta^x\}}|w(\mathsf{b}^x_s)|\right]\mathrm{d}s=0.$$

By Khashminskii's lemma [G17], $w\in\mathcal{K}(\mathscr{M})$ implies

$$\sup_{x\in\mathscr{M}}\mathbb{E}\left[1_{\{t<\zeta^x\}}\mathrm{e}^{p\int_0^t|w(\mathsf{b}^x_s)|\mathrm{d}s}\right]<\infty\quad\text{for all } t>0,\ p\in[1,\infty).$$

One trivially always has $L^\infty(\mathscr{M})\subset\mathcal{K}(\mathscr{M})$, and under a mild control on the geometry one has L^p+L^∞-type subspaces of the Kato class. For example, one has (cf. Chapter VI in [G17] and the appendix of [BrG]):

Proposition 2.5. *a) Assuming there exists of a Borel function $\theta:\mathscr{M}\to(0,\infty)$ with*

$$\sup_{x\in\mathscr{M}}\mathrm{e}^{-tH}(x,y)\leqslant\theta(y)t^{-m/2}\quad\text{for all } 0<t<1,\ y\in\mathscr{M}.$$

Then one has

$$L^p_\theta(\mathscr{M})+L^\infty(\mathscr{M})\subset\mathcal{K}(\mathscr{M}),\quad\text{for all } p\geqslant 1 \text{ if } m=1, \text{ and all } p>m/2 \text{ if } m\geqslant 2,$$

where $L^p_\theta(\mathscr{M})$ denotes the weighted L^p-space of all equivalence classes of Borel functions f on $\mathscr{M}$ such that $\int|f|^p\theta\mathrm{d}\mu<\infty$.
b) If $\mathscr{M}$ is geodesically complete and quasi-isometric to a Riemannian manifold with Ricci curvature bounded from below by a constant, then one has

$$L^p_{1/\mu(B(\cdot,1))}(\mathscr{M})+L^\infty(\mathscr{M})\subset\mathcal{K}(\mathscr{M}),\quad\text{for all } p\geqslant 1 \text{ if } m=1, \text{ and all } p>m/2 \text{ if } m\geqslant 2.$$

Given an endomorphism A on a metric vector bundle, we denote with

$$\Re(A):=(1/2)(A+A^\dagger)$$

its real part and with

$$\Im(A):=-\sqrt{-1}(A-\Re(A))$$

its imaginary part, so that $A=\Re(A)+\sqrt{-1}\Im(A)$, where $\Re(A)$ and $\Im(A)$ are self-adjoint (and then, for example, the positive and negative parts $\Re(A)^\pm\geqslant 0$ are defined via the fiberwise spectral calculus, giving $\Re(A)=\Re(A)^+-\Re(A)^-$). Note also that $\Re(A)=\Re(A^\dagger)$, and that $\Re(A)=U\Re(B)U^\dagger$ if $A=UBU^\dagger$ for some unitary U. .

Proposition 2.6. *Let*

$$Q : \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathscr{E})$$

be a differential operator of order $\leqslant 1$.
a) Assume

- $|\Re(\sigma_1(Q))| \in L^\infty(\mathscr{M})$,
- $\Re(Q_\nabla)$ *is bounded from below by a constant* $\kappa \in \mathbb{R}$,
- $|\Im(Q_\nabla)| \in \mathcal{K}(\mathscr{M})$.

Then h_Q^∇ is sectorial and

$$\sup_{x \in \mathscr{M}} \mathbb{E}\left[1_{\{t<\zeta^x\}} |\mathscr{Q}^x_\nabla(t)|^2\right] < \infty \quad \textit{for all } t > 0, \tag{2.10}$$

in particular, (2.5) holds true.
b) Assume

- $\sigma_1(Q)$ *is anti-selfadjoint and* $|\sigma_1(Q)| \in L^\infty(\mathscr{M})$,
- $|\Re(Q_\nabla)^-| \in \mathcal{K}(\mathscr{M})$,
- $|\Im(Q_\nabla)| \in \mathcal{K}(\mathscr{M})$.

Then h_Q^∇ is sectorial and one has (2.10), in particular, (2.5) holds true.

Proof: We have

$$h_Q^\nabla = h_a + h_b + h_c + h_d + h_e,$$

where

$$\begin{aligned} &h_a(\Psi_1, \Psi_2) := (1/2)\left\langle \nabla\Psi_1, \nabla\Psi_2\right\rangle, \quad h_b(\Psi_1, \Psi_2) := \left\langle \sigma_1(Q)\nabla\Psi_1, \Psi_2\right\rangle, \\ &h_c(\Psi_1, \Psi_2) := \left\langle \Re(Q_\nabla)^+\Psi_1, \Psi_2\right\rangle, \quad h_d(\Psi_1, \Psi_2) := \left\langle \Re(Q_\nabla)^-\Psi_1, \Psi_2\right\rangle, \\ &h_e(\Psi_1, \Psi_2) := \left\langle \Im(Q_\nabla)\Psi_1, \Psi_2\right\rangle. \end{aligned}$$

a) We have

$$|h_b(\Psi, \Psi)| \leqslant \|\sigma_1(Q)\|_\infty \|\nabla\Psi\| \|\Psi\| \leqslant \|\sigma_1(Q)\|_\infty \left(C_\epsilon \|\Psi\|^2 + \epsilon h_a(\Psi, \Psi)\right), \tag{2.11}$$

and (as Kato perturbations of Bochner-Laplacians are infinitesimally form small; cf. Lemma VII.4 in [G17])

$$|h_e(\Psi, \Psi)| \leqslant \left(C_\epsilon \|\Psi\|^2 + \epsilon h_a(\Psi, \Psi)\right),$$

which shows that $h_a + h_b + h_e$ is sectorial, as h_a is so (cf. Theorem A.1 in the appendix). Moreover,

$$h_c(\Psi, \Psi) + h_d(\Psi, \Psi) = \langle \Re(Q_\nabla)\Psi, \Psi\rangle$$

is bounded from below, so that the sum

$$h = h_a + h_b + h_e + h_c + h_d$$

of sectorial forms is sectorial, too.
Let $v \in \mathscr{E}_x$. Almost surely, for all $s > 0$, we have in $\{s < \zeta^x\}$ by the Itô product rule,

$$\begin{aligned}
\mathrm{d}\left|\mathcal{Q}^x_\nabla(s)^\dagger v\right|^2 &= 2\Re\left(\mathrm{d}\mathcal{Q}^x_\nabla(s)^\dagger v, \mathcal{Q}^x_\nabla(s)^\dagger v\right) + \left(\mathrm{d}\mathcal{Q}^x_\nabla(s)^\dagger v, \mathrm{d}\mathcal{Q}^x_\nabla(s)^\dagger v\right)\\
&\leqslant -2\big(/\!\!/^x_\nabla(s)^{-1}\Re(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_s))/\!\!/^x_\nabla(t)\mathcal{Q}^x_\nabla(s)^\dagger v, \mathcal{Q}^x_\nabla(s)^\dagger v\big)\\
&\quad - 2\left(/\!\!/^x_\nabla(s)^{-1}\Re(Q_\nabla(\mathsf{b}^x_s))/\!\!/^x_\nabla(s)\mathcal{Q}^x_\nabla(s)^\dagger v, \mathcal{Q}^x_\nabla(s)^\dagger v\right)\mathrm{d}s\\
&\quad + |\sigma_1(Q)^\flat(\mathsf{b}^x_s)|^2|Q_\nabla(s)^\dagger v|^2\mathrm{d}s\\
&\leqslant -2\left(/\!\!/^x_\nabla(s)^{-1}\Re(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_s))/\!\!/^x_\nabla(s)\mathcal{Q}^x_\nabla(s)v, Q_\nabla(s)^\dagger v\right)\\
&\quad - 2\kappa|\mathcal{Q}^x_\nabla(s)^\dagger v|^2\mathrm{d}s\\
&\quad + \left\|\Re(\sigma_1(Q))\right\|^2_\infty|\mathcal{Q}^x_\nabla(s)^\dagger v|^2\mathrm{d}s.
\end{aligned}$$

With the sequences of stopping times ϑ_n and ζ^x_l as in the proof of Theorem 2.2, the Itô isometry an Jenßen's inequality imply that for all $t > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathcal{Q}^x_\nabla(t\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v\right|^2\right]\\
&\leqslant 1 + 2\mathbb{E}\left[\left|\int_0^t\left(/\!\!/^x_\nabla(r)^{-1}\Re(\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_r))/\!\!/^x_\nabla(r)\mathcal{Q}^x_\nabla(r)^\dagger v, Q^x_\nabla(r)^\dagger v\right)\right|^{2\frac{1}{2}}\Big|_{r=s\wedge\vartheta_n\wedge\zeta^x_l}\right]\\
&\qquad - 2\kappa\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s + \left\|\Re(\sigma_1(Q))\right\|^2_\infty\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s\\
&\leqslant 1 + 2\left\|\Re(\sigma_1(Q))\right\|_\infty\mathbb{E}\left[\int_0^t|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\mathrm{d}s\right]^{\frac{1}{2}}\\
&\qquad - 2\kappa\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s + \left\|\Re(\sigma_1(Q))\right\|^2_\infty\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s\\
&\leqslant 1 + 2\left\|\Re(\sigma_1(Q))\right\|_\infty\left(\mathbb{E}\left[\int_0^t|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\mathrm{d}s\right] + 1\right)\\
&\qquad - 2\kappa\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s + \left\|\Re(\sigma_1(Q))\right\|^2_\infty\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s\\
&\leqslant 1 + 2\left\|\Re(\sigma_1(Q))\right\|_\infty + 2\left\|\Re(\sigma_1(Q))\right\|_\infty\mathbb{E}\left[\int_0^t|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\mathrm{d}s\right]\\
&- 2\kappa\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s + \left\|\Re(\sigma_1(Q))\right\|^2_\infty\int_0^t\mathbb{E}\left[|\mathcal{Q}^x_\nabla(s\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v|^2\right]\mathrm{d}s.
\end{aligned}$$

By Gronwall's lemma and Fatou's lemma, this estimate implies

$$\begin{aligned}
&\mathbb{E}\left[1_{\{t<\zeta^x\}}\left|\mathcal{Q}^x_\nabla(t)^\dagger v\right|^2\right] \leqslant \lim_l\mathbb{E}\left[1_{\{t<\zeta^x_l\}}\left|\mathcal{Q}^x_\nabla(t)^\dagger v\right|^2\right]\\
&= \lim_l\mathbb{E}\left[1_{\{t<\zeta^x_l\}}\left|\mathcal{Q}^x_\nabla(t\wedge\zeta^x_l)^\dagger v\right|^2\right] \leqslant \lim_l\lim_n\mathbb{E}\left[1_{\{t<\zeta^x_l\}}\left|\mathcal{Q}^x_\nabla(t\wedge\vartheta_n\wedge\zeta^x_l)^\dagger v\right|^2\right]\\
&\leqslant C_Q\mathrm{e}^{tC_Q} < \infty,
\end{aligned}$$

uniformly in $x \in \mathscr{M}$.

b) As in the proof of part a),

$$|h_b(\Psi,\Psi)| \leqslant \left\|\sigma_1(Q)\right\|_\infty\left(C_\epsilon\left\|\Psi\right\|^2 + \epsilon h_a(\Psi,\Psi)\right),$$

and

$$|h_d(\Psi,\Psi)| \leqslant \left(C_\epsilon \|\Psi\|^2 + \epsilon h_a(\Psi,\Psi)\right),$$
$$|h_e(\Psi,\Psi)| \leqslant \left(C_\epsilon \|\Psi\|^2 + \epsilon h_a(\Psi,\Psi)\right),$$

which shows that $h_a + h_b + h_d + h_e$ is sectorial, and h_c is nonnegative so that h is sectorial. In the notation of Remark 2.1, a.s., for all $s > 0$ we have in $\{s < \zeta^x\}$,

$$\mathrm{d}\mathcal{Q}^x_{1,\nabla}(s)^{-1} = /\!/^x_\nabla(s)^{-1}\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_s)/\!/^x_\nabla(s)\mathcal{Q}^x_{1,\nabla}(s)^{-1}, \quad \mathcal{Q}^x_{1,\nabla}(0)^{-1} = 1,$$

and

$$\mathrm{d}\mathcal{Q}^x_{1,\nabla}(s)^* = -/\!/^x_\nabla(s)^{-1}\sigma_1(Q)^\flat(\mathrm{d}\mathsf{b}^x_s)^\dagger/\!/^x_\nabla(s)\mathcal{Q}^x_{1,\nabla}(s)^*, \quad \mathcal{Q}^x_{1,\nabla}(0)^* = 1,$$

which shows that $\mathcal{Q}^x_{1,\nabla}(s)$ is unitary, if $\sigma_1(Q)$ is anti-selfadjoint. Thus we have

$$|\mathcal{Q}^x_\nabla(s)| = |\mathcal{Q}^x_{2,\nabla}(s)\mathcal{Q}^x_{1,\nabla}(s)| \leqslant |\mathcal{Q}^x_{2,\nabla}(s)|.$$

For all $v \in \mathscr{E}_x$ (as both $\mathcal{Q}^x_{1,\nabla}(s)$ and the parallel transport are unitary),

$$\begin{aligned}
(\mathrm{d}/\mathrm{d}s)\left|\mathcal{Q}^x_{2,\nabla}(s)^\dagger v\right|^2 &= 2\Re\Big((\mathrm{d}/\mathrm{d}s)\mathcal{Q}^x_{2,\nabla}(s)^\dagger v, \mathcal{Q}^x_{2,\nabla}(s)^\dagger v\Big)\\
&= -2\Re\Big(\mathcal{Q}^x_{1,\nabla}(s)/\!/^x_\nabla(s)^{-1}Q_\nabla(\mathsf{b}^x_s)^\dagger/\!/^x_\nabla(s)\mathcal{Q}^x_{1,\nabla}(s)^{-1}\mathcal{Q}^x_{2,\nabla}(s)^\dagger v, \mathcal{Q}^x_{2,\nabla}(s)^\dagger v\Big)\\
&= -2\Big(\mathcal{Q}^x_{1,\nabla}(s)/\!/^x_\nabla(s)^{-1}\Re(Q_\nabla(\mathsf{b}^x_s))/\!/^x_\nabla(s)\mathcal{Q}^x_{1,\nabla}(s)^{-1}\mathcal{Q}^x_{2,\nabla}(s)^\dagger v, \mathcal{Q}^x_{2,\nabla}(s)^\dagger v\Big)\\
&\leqslant 2|\Re(Q_\nabla(\mathsf{b}^x_s))^-||\mathcal{Q}^x_{2,\nabla}(s)^\dagger v|^2
\end{aligned}$$

and so by Gronwall, a.s., for all $t > 0$ we have in $\{t < \zeta^x\}$,

$$|\mathcal{Q}^x_{2,\nabla}(t)|^2 = |\mathcal{Q}^x_{2,\nabla}(t)^\dagger|^2 \leqslant \mathrm{e}^{2\int_0^t |\Re(Q_\nabla(\mathsf{b}^x_s))^-|\mathrm{d}s}$$

and finally

$$\sup_{x\in\mathscr{M}} \mathbb{E}\left[1_{\{t<\zeta^x\}}\mathrm{e}^{2\int_0^t |\Re(Q_\nabla(\mathsf{b}^x_s))^-|\mathrm{d}s}\right] < \infty$$

by Khashiminskii's lemma. ∎

Given $x \in \mathscr{M}$, let $(\mathbb{P}^{x,y}_t)_{t>0,y\in\mathscr{M}}$ be the bridge measures associated with $\mathsf{b}(x)$: for all $t > 0$, $y \in \mathscr{M}$, the measure $\mathbb{P}^{x,y}_t$ is the uniquely determined probability measure (cf. [P03], p. 36) on $\{t < \zeta^x\}$ equipped with the sigma-algebra $\mathscr{F}_t^{\mathsf{b}^x|_{\{t<\zeta^x\}}}$ such that

$$\mathbb{P}^{x,y}_t(A) = \mathbb{E}\left[1_A\frac{p(t-s,\mathsf{b}^x_s,y)}{p(t,x,y)}\right] \quad \text{for all } 0 < s < t,\ A \in \mathscr{F}_s^{\mathsf{b}^x|_{\{s<\zeta^x\}}}.$$

This provides a pointwise disintegration of Brownian motion, in the sense that for all $t > 0$, $x, y \in \mathscr{M}$ one has

$$\mathbb{P}(A) = \int \mathrm{e}^{-tH}(x,y)\mathbb{P}^{x,y}_t(A)\mathrm{d}\mu(y) \quad \text{for all } A \in \mathscr{F}_t^{\mathsf{b}^x} \cap \{t < \zeta^x\},$$
$$\mathbb{P}^{x,y}_t(\mathsf{b}^x_t = y) = 1.$$

We remark that one has to *locally* complete these probability spaces so that $\mathcal{Q}^x_\nabla(t)$ and $/\!/^x_\nabla(t)$ become $\mathscr{F}_t^{\mathsf{b}^x|_{\{t<\zeta^x\}}}$-measurable (cf. p. 250 in [HT94] for a precise treatment of this issue.)

We immediately get the following consequence of Theorem 2.2:

Corollary 2.7. *In the situation of Theorem 2.2, for all $t > 0$, $x, y \in \mathscr{M}$ one has*

$$\mathrm{e}^{-tH^\nabla_Q}(x,y) = \int_M \mathrm{e}^{-tH}(x,y)\mathbb{E}^{x,y}_t\left[\mathcal{Q}^x_\nabla(t)/\!/^x_\nabla(t)^{-1}\right]. \tag{2.12}$$

Remark 2.8. The precise meaning of this result is as follows: there exists a unique jointly smooth map

$$(0,\infty)\times\mathscr{M}\times\mathscr{M}\ni(t,x,y)\longmapsto \mathrm{e}^{-tH^{\nabla}_{Q}}(x,y)\in \mathrm{Hom}(\mathscr{E}_y,\mathscr{E}_x)\in\mathscr{E}\boxtimes\mathscr{E}^{\dagger}$$

such that for all $t>0$, $x\in\mathscr{M}$, $\Psi\in\Gamma_{L^2}(\mathscr{M},\mathscr{E})$ one has

$$\int|\mathrm{e}^{-tH^{\nabla}_{Q}}(x,y)|^2\mathrm{d}\mu(y)<\infty,\quad \mathrm{e}^{-tH^{\nabla}_{Q}}\Psi(x)=\int\mathrm{e}^{-tH^{\nabla}_{Q}}(x,y)\Psi(y)\mathrm{d}\mu(y),$$

(this follows from the proof of Theorem II.1 in [G17], where the required self-adjointness and semiboundedness of the operator $\tilde{P}$ is only used to get a semigroup which is holomorphic in a sector of the complex plane which contains $(0,\infty)$), and Corollary 2.7 states this map is *pointwise* equal to the RHS of (2.12).

In the following result we assume for simplicity that $\mathscr{M}$ is compact, in order to not obscure the algebraic machinery behind its proof, and to guarantee the required trace class property:

Theorem 2.9. *Assume $\mathscr{M}$ is compact. Let $V\in\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{E}))$ (considered as a differential operator of order $\leqslant 1$ in $\mathscr{E}\to\mathscr{M}$) and let*

$$P:\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})$$

be a differential operator of order $\leqslant 1$ and denote its closure in $\Gamma_{L^2}(\mathscr{M},\mathscr{E})$, defined a priori on $\Gamma_{C^\infty}(\mathscr{M},\mathscr{E})$, with P again. Then for all $t>0$ the operator

$$\int_0^t \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\mathrm{d}s\in\mathscr{L}(\Gamma_{L^2}(\mathscr{M},\mathscr{E})),\tag{2.13}$$

is given for all $x,y\in\mathscr{M}$ by

$$\begin{aligned}&\int_0^t \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\mathrm{d}s\ (x,y)\\&=-\mathrm{e}^{-tH}(x,y)\mathbb{E}^{x,y}_t\left[\mathcal{V}^x_{\nabla}(t)\int_0^t /\!\!/^x_{\nabla}(s)^{-1}\big(\sigma_1(P)^{\flat}(\mathrm{d}\mathsf{b}^x_s)+P_{\nabla}(\mathsf{b}^x_s)\mathrm{d}s\big)/\!\!/^x_{\nabla}(s)/\!\!/^x_{\nabla}(t)^{-1}\right],\end{aligned}\tag{2.14}$$

in particular, for every $\widetilde{V}\in\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{E}))$ one has

$$\begin{aligned}&\mathrm{Tr}\left(\widetilde{V}\int_0^t \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\mathrm{d}s\right)\\&=-\int_{\mathscr{M}}\widetilde{V}(x)\mathrm{e}^{-tH}(x,x)\mathbb{E}^{x,x}_t\left[\mathcal{V}^x_{\nabla}(t)\int_0^t /\!\!/^x_{\nabla}(s)^{-1}\big(\sigma_1(P)^{\flat}(\mathrm{d}\mathsf{b}^x_s)+P_{\nabla}(\mathsf{b}^x_s)\mathrm{d}s\big)/\!\!/^x_{\nabla}(s)/\!\!/^x_{\nabla}(t)^{-1}\right]\mathrm{d}\mu(x).\end{aligned}$$

This result has to read as follows: By elliptic regularity, for all $t>0$, the function

$$[0,t]\ni s\longmapsto \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\Psi\in\Gamma_{L^2}(\mathscr{M},\mathscr{E})$$

is well-defined and continuous, so

$$\int_0^t \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\Psi\mathrm{d}s$$

is well-defined in the sense of $\Gamma_{L^2}(\mathscr{M},\mathscr{E})$-valued Riemann integrals. Furthermore,

$$\Gamma_{L^2}(\mathscr{M},\mathscr{E})\ni\Psi\longmapsto\int_0^t \mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(t-s)H^{\nabla}_{V}}\Psi\mathrm{d}s\in\Gamma_{L^2}(\mathscr{M},\mathscr{E})$$

is bounded, and our proof shows that $\int_0^{\bullet}\mathrm{e}^{-sH^{\nabla}_{V}}P\mathrm{e}^{-(\bullet-s)H^{\nabla}_{V}}\mathrm{d}s$ has a jointly smooth integral kernel in the sense of Remark 2.8, and that this smooth representative is pointwise equal to the RHS of (2.14). The asserted trace formula then follows from the fact that if an operator A_1 in

$\Gamma_{L^2}(\mathscr{M},\mathscr{E})$ has a smooth integral kernel and A_2 is zeroth order, then A_2A_1 has the smooth integral kernel $[A_2A_1](x,y) = A_2(x)A_1(x,y)$ and A_2A_1 is trace class (as $\mathscr{M}$ is compact) with

$$\mathrm{Tr}\,(A_2A_1) = \int_{\mathscr{M}} \mathrm{Tr}_x(A_2(x)A_1(x,x))\mathrm{d}\mu(x),$$

where Tr_x denotes the finite dimensional trace on $\mathrm{End}(\mathscr{E}_x)$.

Proof of Theorem 2.9: Denote with $\Lambda(\mathbb{R}) = \mathbb{R}\oplus\Lambda^1(\mathbb{R})$ the Grassmann algebra over $\mathbb{R}$, which is generated by $1\in\mathbb{R}$ and $\theta\in\Lambda^1(\mathbb{R})$. In particular, we have $\theta^2=0$. Given a linear space $\mathscr{A}$, the Berezin integral is the linear map

$$\int_{\Lambda(\mathbb{R})} : \mathscr{A}\otimes\Lambda(\mathbb{R}) \longrightarrow \mathscr{A}, \quad a+b\theta \longmapsto \int_{\Lambda(\mathbb{R})}(a+b\theta)\mathrm{d}\theta := b, \quad a,b\in\mathscr{A},$$

which picks the θ-coefficient. Note that if $\mathscr{A}$ is an associative algebra, then so is $\mathscr{A}\otimes\Lambda(\mathbb{R})$. With the differential operator

$$V+P^\theta := V+\theta P : \Gamma_{C^\infty}(\mathscr{M},\mathscr{E}\otimes\Lambda(\mathbb{R})) = \Gamma_{C^\infty}(\mathscr{M},\mathscr{E})\otimes\Lambda(\mathbb{R}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M},\mathscr{E}\otimes\Lambda(\mathbb{R})),$$

of order $\leqslant 1$, the operator $H^\nabla_{V+P^\theta}$ in

$$\Gamma_{L^2}(\mathscr{M},\mathscr{E}\otimes\Lambda(\mathbb{R})) = \Gamma_{L^2}(\mathscr{M},\mathscr{E})\otimes\Lambda(\mathbb{R})$$

is well-defined and in fact equal to the operator sum $H^\nabla_V + P^\theta$ (as $\mathscr{M}$ is compact). The perturbation series

$$\mathrm{e}^{-tH^\nabla_{V+P^\theta}} = 1 + \sum_{j=1}^\infty \int_{\{0<t_1<\cdots<t_j<t\}} \mathrm{e}^{-t_1H^\nabla_V}P^\theta\mathrm{e}^{-(t_2-t_1)H^\nabla_V}P^\theta\cdots\mathrm{e}^{-(t-t_j)H^\nabla_V}\mathrm{d}t_1\cdots\mathrm{d}t_n$$

cancels after $j\geqslant 2$ because of $\theta^2=0$, and we have

$$\int_{\Lambda(\mathbb{R})}\mathrm{e}^{-tH^\nabla_{V+P^\theta}}\mathrm{d}\theta = \int_0^t \mathrm{e}^{-sH^\nabla_V}P\mathrm{e}^{-(t-s)H^\nabla_V}\mathrm{d}s, \tag{2.15}$$

in particular, $\int_0^\bullet \mathrm{e}^{-sH^\nabla_V}P\mathrm{e}^{-(\bullet-s)H^\nabla_V}\mathrm{d}s$ has a jointly smooth integral kernel in the sense of Remark 2.8. By Corollary 2.7 and Remark 2.1 we have

$$\mathrm{e}^{-tH^\nabla_{V+P_\theta}}(x,y) = \mathrm{e}^{-tH}(x,y)\mathbb{E}^{x,y}_t\left[\mathcal{V}^x_\nabla(t)\mathcal{P}^x_{\theta,\nabla}(t)/\!/^x_\nabla(t)^{-1}\right],$$

where

$$\mathcal{P}^x_{\theta,\nabla} : [0,\zeta^x)\times\Omega \longrightarrow \mathrm{End}(\mathscr{E}_x\otimes\Lambda(\mathbb{R}))$$

denotes the unique solution of

$$\mathrm{d}\mathcal{P}^x_{\theta,\nabla}(t) = -\mathcal{P}^x_{\theta,\nabla}(t)/\!/^x_\nabla(t)^{-1}\big(\sigma_1(P^\theta)^\flat(\mathrm{d}\mathsf{b}^x_t) + P^\theta_\nabla(\mathsf{b}^x_t)\mathrm{d}t\big)/\!/^x_\nabla(t), \quad \mathcal{P}^x_{\theta,\nabla}(0)=1.$$

Because of $\theta^2=0$ the time ordered exponential series

$$\mathcal{P}^x_{\theta,\nabla}(t) = 1+\sum_{j=1}^\infty\int_{\{0\leqslant t_1\leqslant\cdots\leqslant t_j\leqslant t\}}\prod_{i=1}^j\theta\Big(-/\!/^x_\nabla(t_i)^{-1}\big(\sigma_1(P)^\flat(\mathrm{d}\mathsf{b}^x_{t_i}) + P_\nabla(\mathsf{b}^x_{t_i})\mathrm{d}t_i\big)/\!/^x_\nabla(t_i)\Big)$$

has only two summands, giving

$$\begin{aligned}&\int_{\Lambda(\mathbb{R})}\mathrm{e}^{-tH^\nabla_{V+P_\theta}}(x,y)\mathrm{d}\theta\\ &= -\mathrm{e}^{-tH}(x,y)\mathbb{E}^{x,y}_t\left[\mathcal{V}^x_\nabla(t)\int_0^t /\!/^x_\nabla(s)^{-1}\big(\sigma_1(P)^\flat(\mathrm{d}\mathsf{b}^x_s)+P_\nabla(\mathsf{b}^x_s)\mathrm{d}s\big)/\!/^x_\nabla(s)/\!/^x_\nabla(t)^{-1}\right],\end{aligned}$$

which in view of (2.15) is the claimed formula. ■

3. Applications to noncommutative geometry

In this section we present an application of Theorem 2.9 to recent results concerning an algebraic model given in [GL19] for Duistermaat-Heckman localization on the space of smooth loops in a compact Riemannian spin manifold. We refer the reader to [LM89] for details on spin geometry (noting that a brief introduction can also be found in [H02]).

Assume $\mathscr{M}$ is a compact Riemannian spin manifold of even dimension, with $\mathscr{S}\to\mathscr{M}$ its spin bundle, which is naturally $\mathbb{Z}_2$-graded by an endomorphism $\gamma\in\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{E}))$. The vector bundle $\mathscr{S}\to\mathscr{M}$ inherits a metric and a metric connection ∇ from the Riemannian metric and the Levi-Civita connection on $\mathscr{M}$. Let

$$D:\Gamma_{C^\infty}(\mathscr{M},\mathscr{S})\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathscr{S})$$

denote the induced Dirac operator and let

$$\begin{aligned}&c:\Omega_{C^\infty}(M)\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{S})),\quad c(\alpha_1\wedge\cdots\wedge\alpha_p)\Psi:=\frac{1}{p!}\alpha_1\cdots\alpha_p\cdot\Psi,\\ &\alpha_1,\dots,\alpha_p\in\Omega^1_{C^\infty}(M),\quad \Psi\in\Gamma_{C^\infty}(\mathscr{M},\mathscr{S}),\end{aligned}$$

denote the natural extension of the (dual) Clifford multiplication

$$\Omega^1_{C^\infty}(M)\longrightarrow\Gamma_{C^\infty}(\mathscr{M},\mathrm{End}(\mathscr{S})),\quad \alpha\longmapsto(\Psi\longmapsto\alpha\cdot\Psi)$$

from 1-forms to arbitrary differential forms. The operator D (defined a priori on $\Gamma_{C^\infty}(\mathscr{M},\mathscr{S})$) is essentially self-adjoint in $\Gamma_{L^2}(\mathscr{M},\mathscr{S})$, and its unique self-adjoint realization will be denoted with the same symbol again. With $\mathbb{T}:=S^1$ let

$$\Omega_{\mathbb{T}}(\mathscr{M}):=\Omega_{C^\infty}(\mathscr{M}\times\mathbb{T})^{\mathbb{T}}$$

denote the space of $\mathbb{T}$-invariant differential forms on $\mathscr{M}\times\mathbb{T}$. Each element α of $\Omega_{\mathbb{T}}(\mathscr{M})$ can be uniquely written in the form $\alpha=\alpha'+\alpha''\mathrm{d}t$ with $\mathrm{d}t$ the volume form on $\mathbb{T}$. Define a complex linear space by

$$\mathsf{C}_{\mathbb{T}}(\mathscr{M}):=\bigoplus_{N=0}^{\infty}\Omega_{\mathbb{T}}(\mathscr{M})^{\otimes(N+1)}.$$

Since, $\mathscr{M}$ is compact, e^{-tD^2} is trace class for all $t>0$. In this situation, the *Chern Character* $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ is a linear functional[2]

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M}):\mathsf{C}_{\mathbb{T}}(\mathscr{M})\longrightarrow\mathbb{C},$$

that has been introduced in [GL19]. The formula for $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ is given as follows: define

$$F_{\mathbb{T}}:\mathsf{C}_{\mathbb{T}}(\mathscr{M})\longrightarrow\{\text{differential operators of order}\leqslant 2\text{ in }\mathscr{S}\to\mathscr{M}\}$$

by

$$\begin{aligned}&F_{\mathbb{T}}(\alpha_0)=c(\mathrm{d}\alpha_0')-[D,c(\alpha_0')]-c(\alpha_0'')\\ &F_{\mathbb{T}}(\alpha_0\otimes\alpha_1)=(-1)^{|\alpha_0'|}\big(c(\alpha_0'\wedge\alpha_1')-c(\alpha_0')c(\alpha_1')\big),\\ &F_{\mathbb{T}}(\alpha_0\otimes\cdots\otimes\alpha_N)=0\quad\text{ for all }N\geqslant 3.\end{aligned}$$

Above, $[D,c(\alpha)]$ denotes a $\mathbb{Z}_2$-graded commutator (where differential forms are $\mathbb{Z}_2$-graded through even/odd form degrees). Explicitly, one has

$$[D,c(\alpha)]=Dc(\alpha)-(-1)^pc(\alpha)D,\quad \text{if }\alpha\in\Omega^p_{C^\infty}(\mathscr{M}).$$

[2]In fact, $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ extends continuously to a certain completion of $\mathsf{C}_{\mathbb{T}}(\mathscr{M})$, but we shall not be concerned with this fact here.

For natural numbers $L \leqslant N$ denote with $\mathsf{P}_{L,N}$ all tuples $I = (I_1, \dots, I_L)$ of subsets of $\{1 \dots, N\}$ with

$$I_1 \cup \cdots \cup I_L = \{1 \dots, N\}$$

and with each element of I_a smaller than each element of I_b whenever $a < b$. Given

- $\alpha_1 \otimes \cdots \otimes \alpha_N \in \Omega_{\mathbb{T}}(\mathscr{M})^{\otimes N}$,
- $I = (I_1, \dots, I_L) \in \mathsf{P}_{L,N}$,
- $1 \leqslant a \leqslant L$,

we set

$$\alpha_{I_a} := \alpha_{i+1} \otimes \cdots \otimes \alpha_{i+l}, \qquad \text{if } I_a = \{j \mid i < j \leqslant i+l\} \text{ for some } i, l.$$

Then with $\mathrm{Str}(\bullet) := \mathrm{Tr}(\gamma \bullet)$ the $\mathbb{Z}_2$-graded trace on $\mathscr{L}(\Gamma_{L^2}(\mathscr{M}, \mathscr{S}))$, one has

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0 \otimes \cdots \otimes \alpha_N) := \sum_{L=1}^{N} (-1)^L \sum_{I \in \mathsf{P}_{L,N}} \int_{\{0 \leqslant s_1 \leqslant \cdots \leqslant s_L \leqslant 1\}} \mathrm{Str}\Big(c(\alpha_0) \mathrm{e}^{-s_1 D^2} F_{\mathbb{T}}(\alpha_{I_1}) \times$$
$$\times \mathrm{e}^{-(s_2 - s_1) D^2} F_{\mathbb{T}}(\alpha_{I_2}) \cdots \mathrm{e}^{-(s_L - s_{L-1}) D^2} F_{\mathbb{T}}(\alpha_{I_L}) \mathrm{e}^{-(1 - s_L) D^2} \Big) \mathrm{d}s_1 \cdots \mathrm{d}s_L.$$

By definition we the $N = 0$ part of the Chern character is given explicitly by

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0) = \mathrm{Str}\left(c(\alpha_0') \mathrm{e}^{-D^2} \right), \tag{3.1}$$

and the $N = 1$ part is given explicitly by

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0 \otimes \alpha_1) = -\mathrm{Str}\left(\int_0^1 c(\alpha_0') \mathrm{e}^{-sD^2} F_{\mathbb{T}}(\alpha_1) \mathrm{e}^{-(1-s)D^2} \mathrm{d}s \right). \tag{3.2}$$

By the Lichnerowicz formula we have

$$D^2 = \nabla^{\dagger} \nabla + (1/4)\mathrm{scal}, \tag{3.3}$$

so that the $N = 0$ piece of $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$ is given by the probabilistic expression

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0) = \int_M \mathrm{e}^{-tH}(x,x) \mathrm{Str}_x \left(c(\alpha_0')(x) \mathbb{E}_t^{x,x} \left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x) \mathrm{d}s} //_{\nabla}^{x}(t)^{-1} \right] |_{t=2} \right) \mathrm{d}\mu(x),$$

with Str_x the $\mathbb{Z}_2$-graded trace on $\mathrm{End}(\mathscr{S}_x)$. We are going to use Theorem 2.9 to deduce a probabilistic representation for the $N = 1$ piece of $\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})$:

Theorem 3.1. *Let $\mathscr{M}$ be a compact even dimensional Riemannian spin manifold. Then for all $\alpha_0, \alpha_1 \in \Omega_{\mathbb{T}}(\mathscr{M})$ one has*

$$\mathrm{Ch}_{\mathbb{T}}(\mathscr{M})(\alpha_0 \otimes \alpha_1)$$
$$= \int_{\mathscr{M}} \mathrm{e}^{-tH}(x,x) \mathrm{Str}_x \left(c(\alpha_0')(x) \mathbb{E}_t^{x,x} \left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x)\mathrm{d}s} \int_0^t //_{\nabla}^{x}(s)^{-1} \Big(2c(*\mathrm{d}\mathsf{b}_s^x \lrcorner \alpha_1') - c(\alpha_1'')(\mathsf{b}_s^x)\mathrm{d}s \Big) //_{\nabla}^{x}(s) //_{\nabla}^{x}(t)^{-1} \right] |_{t=2} \right) \mathrm{d}\mu(x).$$

Proof: Applying Theorem 2.9 with $V := (1/8)\mathrm{scal}$, $\tilde{V} := \gamma$ and $P := F_{\mathbb{T}}(\alpha_1)$, and noting that by (3.3) one has $H_V^{\nabla} = D^2$, for all $x, y \in \mathscr{M}$, we immediately get

$$\mathrm{Str}\left(\int_0^1 \mathrm{e}^{-sD^2} F_{\mathbb{T}}(\alpha_1) \mathrm{e}^{-(1-s)D^2} \mathrm{d}s \right) =$$
$$\int_{\mathscr{M}} \mathrm{e}^{-tH}(x,x) \mathbb{E}_t^{x,y} \left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x)\mathrm{d}s} \int_0^t //_{\nabla}^{x}(s)^{-1} \big(\sigma_1(F(\alpha_1))^{\flat}(\mathrm{d}\mathsf{b}_s^x) + F(\alpha_1)_{\nabla}(\mathsf{b}_s^x)\mathrm{d}s \big) //_{\nabla}^{x}(s) //_{\nabla}^{x}(t)^{-1} \right] |_{t=2} \mathrm{d}\mu(x).$$

With the product

$$\star : \Gamma_{C^\infty}(\mathscr{M}, T\mathscr{M} \otimes \mathscr{S}) \otimes \Omega_{C^\infty}(\mathscr{M}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathscr{S}), \quad (X \otimes \varphi) \star \alpha := c(X \lrcorner \alpha)\varphi,$$

where $X \lrcorner \alpha$ denotes the contraction of the form α by the vector field X, we are going to prove in a moment the formula

$$[D, c(\alpha)]\varphi = c((\mathrm{d} + \mathrm{d}^{\dagger})\alpha)\varphi - 2(\nabla\varphi)^{\sharp \otimes \mathrm{Id}} \star \alpha, \quad \alpha \in \Omega_{C^\infty}(\mathscr{M}), \quad \varphi \in \Gamma_{C^\infty}(\mathscr{M}, \mathscr{S}). \tag{3.4}$$

Given this identity, we find

$$\sigma_1(F_{\mathbb{T}}(\alpha_1))^\flat(X) = 2c(X\lrcorner\alpha_1') \quad \text{for all vector fields } X \text{ on } \mathscr{M},$$

and furthermore

$$F_{\mathbb{T}}(\alpha_1)_\nabla = -c(\mathrm{d}^\dagger\alpha_1') - c(\alpha_1''),$$

so that the above is

$$= \int_{\mathscr{M}} \mathrm{e}^{-tH}(x,x)\mathbb{E}_t^{x,x}\left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x)\mathrm{d}s}\int_0^t /\!/_{\nabla}^{x}(s)^{-1}\Big(2c(\mathrm{d}\mathsf{b}_s^x\lrcorner\alpha_1') - c(\mathrm{d}^\dagger\alpha_1')(\mathsf{b}_s^x) - c(\alpha_1'')(\mathsf{b}_s^x)\mathrm{d}s\Big)/\!/_{\nabla}^{x}(s)/\!/_{\nabla}^{x}(t)^{-1}\right]|_{t=2}\mathrm{d}\mu(x).$$

Using the Itô-to-Stratonovic rule

$$c(\mathrm{d}\mathsf{b}_s^x\lrcorner\alpha') = c(*\mathrm{d}\mathsf{b}_s^x\lrcorner\alpha') + \frac{1}{2}c(\mathrm{d}^\dagger\alpha')(\mathsf{b}_s^x)\mathrm{d}s,$$

we arrive at

$$\begin{aligned}&\mathrm{Str}\left(\int_0^1 \mathrm{e}^{-D^2}F_{\mathbb{T}}(\alpha_1)\mathrm{e}^{-(1-s)D^2}\mathrm{d}s\right)\\ &= \int_{\mathscr{M}} \mathrm{e}^{-tH}(x,x)\mathbb{E}_t^{x,y}\left[\mathrm{e}^{-(1/8)\int_0^t \mathrm{scal}(\mathsf{b}_s^x)\mathrm{d}s}\int_0^t /\!/_{\nabla}^{x}(s)^{-1}\Big(2c(*\mathrm{d}\mathsf{b}_s^x\lrcorner\alpha_1') - c(\alpha_1'')(\mathsf{b}_s^x)\mathrm{d}s\Big)/\!/_{\nabla}^{x}(s)/\!/_{\nabla}^{x}(t)^{-1}\right]|_{t=2}\mathrm{d}\mu(x),\end{aligned}$$

which is the claimed formula.

It remains to prove (3.4). To this end, denote with $\mathbb{C}\mathrm{l}(\mathscr{M}) \to \mathscr{M}$ the Clifford bundle and with

$$\tilde{}: \Omega_{C^\infty}(\mathscr{M}) \longrightarrow \Gamma_{C^\infty}(\mathscr{M}, \mathbb{C}\mathrm{l}(\mathscr{M}))$$

the natural isomorphism. Then we have

$$(\widetilde{\mathrm{d}+\mathrm{d}^\dagger})\alpha = D^{\mathbb{C}\mathrm{l}(M)}\tilde{\alpha}$$

(cf. [LM89], Chapter II, Thm. 5.12), with $D^{\mathbb{C}\mathrm{l}(M)}$ the natural Dirac operator on $\mathbb{C}\mathrm{l}(\mathscr{M}) \to \mathscr{M}$. Assume now $\alpha \in \Omega^p(\mathscr{M})$ and pick a local orthonormal frame $(e_1, \ldots, e_m)$. Write $\alpha = \sum_I \alpha_I \mathrm{e}_{i_1}^* \wedge \ldots \wedge e_{i_p}^*$ with some increasingly ordered multi-index $I = (i_1, \ldots, i_p)$. One has

(3.5)

$$\begin{aligned}&[D, c(\alpha)]\varphi = Dc(\alpha)\varphi - (-1)^p c(\alpha)\varphi\\ &= \sum_{j=1}^n \sum_I \left(e_j \cdot \nabla_{e_j}(\alpha_I e_{i_1} \cdots e_{i_p} \cdot \varphi) + (-1)^{p+1}\alpha_I e_{i_1} \cdots e_{i_p} \cdot e_j \cdot \nabla_{e_j}\varphi\right)\\ &= \sum_{j=1}^n \sum_I \Big(e_j \cdot \nabla_{e_j}^{\mathbb{C}\mathrm{l}(\mathscr{M})}(\alpha_I e_{i_1} \cdots e_{i_p}) \cdot \varphi + e_j \cdot \alpha_I e_{i_1} \cdots e_{i_p}\nabla_{e_j}\varphi + (-1)^{p+1}\alpha_I e_{i_1} \cdots e_{i_p} \cdot e_j \cdot \nabla_{e_j}\varphi\Big)\\ &= \sum_{j=1}^n \sum_I \Big(e_j \cdot \nabla_{e_j}^{\mathbb{C}\mathrm{l}(\mathscr{M})}(\alpha_I e_{i_1} \cdots e_{i_p}) \cdot \varphi + \alpha_I(e_j \cdot e_{i_1} \cdots e_{i_p} + (-1)^{p+1} e_{i_1} \cdots e_{i_p} \cdot e_j) \cdot \nabla_{e_j}\varphi\Big)\end{aligned}$$

(3.6)

$$= (D^{\mathbb{C}\mathrm{l}(\mathscr{M})}\tilde{\alpha}) \cdot \varphi + \sum_{j=1}^n \sum_I \left(\alpha_I(e_j \cdot e_{i_1} \cdots e_{i_p} + (-1)^{p+1} e_{i_1} \cdots e_{i_p} \cdot e_j) \cdot \nabla_{e_j}\varphi\right) .$$

Fix now I and j. In case $j \neq i_k$ for all $k = 1, \ldots, p$, one has

$$e_j \cdot e_{i_1} \cdots e_{i_p} = (-1)^p e_{i_1} \cdots e_{i_p} \cdot e_j \, .$$

In case $j = i_k$ for some $1 \leqslant k \leqslant p$, one has

$$e_j \cdot e_{i_1} \cdots e_{i_p} = e_{i_k} \cdot e_{i_1} \cdots e_{i_p} = (-1)^{k-1} e_{i_1} \cdots e_{i_k} \cdot e_{i_k} \cdots e_{i_p} = (-1)^k e_{i_1} \cdots \widehat{e_{i_k}} \cdots e_{i_p}$$

and

$$(-1)^{p+1}e_{i_1}\cdots e_{i_p}\cdot e_j = (-1)^{p+1}e_{i_1}\cdots e_{i_p}\cdot e_{i_k} = (-1)^{p+1+p-k}e_{i_1}\cdots e_{i_k}\cdot e_{i_k}\cdots e_{i_p}$$
$$= (-1)^{2p+2-k}e_{i_1}\cdots \widehat{e_{i_k}}\cdots e_{i_p} = (-1)^{k}e_{i_1}\cdots \widehat{e_{i_k}}\cdots e_{i_p}\,.$$

So the RHS of (3.6) equals

$$c((\mathrm{d}+\mathrm{d}^\dagger)\alpha)\varphi - 2\sum_I\sum_{k=1}^{p}(-1)^{k-1}\alpha_I e_{i_1}\cdots\widehat{e_{i_k}}\cdots e_{i_p}\cdot\nabla_{e_{i_k}}\varphi\,. \tag{3.7}$$

Assume again I and j are fixed and that $j = i_k$ for some k. Then by the definition of the product $\star$,

$$\begin{aligned}(e_j\otimes\nabla_{e_j}\varphi)\star\alpha_I e^*_{i_1}\wedge\ldots\wedge e^*_{i_p} &= c(e_j\lrcorner\alpha_I e^*_{i_1}\wedge\ldots\wedge e^*_{i_p})\nabla_{e_j}\varphi\\ &= c((-1)^{k-1}\alpha_I e^*_{i_1}\wedge\ldots\wedge\widehat{e_{i_k}}\wedge\ldots\wedge e^*_{i_p})\nabla_{e_{i_k}}\varphi\\ &= (-1)^{k-1}\alpha_I e_{i_1}\cdots\widehat{e_{i_k}}\cdots e_{i_p}\cdot\nabla_{e_{i_k}}.\end{aligned}$$

As one has $(\nabla\varphi)^{\sharp\otimes\mathrm{d}} = \sum_{j=1}^n e_j\otimes\nabla_{e_j}\varphi$, (3.7) equals

$$c((\mathrm{d}+\mathrm{d}^\dagger)\alpha)\varphi - 2\sum_I\sum_{j=1}^{n}(e_j\otimes\nabla_{e_j}\varphi)\star\alpha_I \mathrm{e}^*_{i_1}\wedge\cdots\wedge \mathrm{e}^*_{i_p} = c((\mathrm{d}+\mathrm{d}^\dagger)\alpha)\varphi - 2(\nabla\varphi)^{\sharp\otimes\mathrm{d}}\star\alpha\,,$$

completing the proof. ■

Appendix A. Facts on sectorial forms and operators

In this appendix, we have collected some definitions and facts on sectorial forms and operators, following the presentation from section VI in [K].
A densely defined sesqui-linear form h in a Hilbert space $\mathscr{H}$ is called *sectorial*, if there exist numbers $\beta\in[0,\pi/2), \gamma\in\mathbb{R}$ such that

$$\{h(\Psi,\Psi) : \Psi\in\mathrm{Dom}(h), \|\Psi\| = 1\}\subset\{z\in\mathbb{C} : |\arg(z-\gamma)|\leqslant\beta\}.$$

Above, γ is called a *vertex* of h and β an *angle* of h.

A sectorial form h in $\mathscr{H}$ is called h is called *closed* if for all $\Psi\in\mathscr{H}$ which admit a sequence $(\Psi_n)\subset\mathrm{Dom}(h)$ with

$$\|\Psi_n-\Psi\|\to 0,\quad h(\psi_n-\psi_l,\psi_n-\psi_l)\to 0\quad\text{as } n,l\to\infty,$$

and h is called *closable* if it has a closed extension; in this case h has a smallest closed extension $\overline{h}$, called the *closure* of h. Sums of sectorial forms are sectorial, and sums of closed forms are closed (on their natural domain of definition; Theorem 131 p. 319 in [K]).

An densely defined operator S in $\mathscr{H}$ is called *sectorial*, if the form h_S given by $\mathrm{Dom}(h_S) = \mathrm{Dom}(S)$ and $h_S(\Psi_1,\Psi_2) = \langle S\Psi_1,\Psi_2\rangle$ is sectorial. If a form h in $\mathscr{H}$ is induced by a sectorial operator S in $\mathscr{H}$, in the sense that $h = h_S$, then h is closable (Theorem 1.27 p. 318 in [K]).

Theorem A.1. *If h is sectorial and the form h' satisfies* $\mathrm{Dom}(h)\subset\mathrm{Dom}(h')$ *and admits constants $a\in[0,\infty)$, $b\in[0,1)$ such that*

$$|h'(\Psi,\Psi)|\leqslant a\,\|\Psi\|^2 + b|h(\Psi,\Psi)|\quad \textit{for all } \Psi\in\mathrm{Dom}(h),$$

then the form $h+h'$ is

- *sectorial,*
- *closed if and only if h is closed,*
- *closable if and only if h is closable; and then* $\mathrm{Dom}(\overline{h+h'}) = \mathrm{Dom}(\overline{h})$.

Proof: This is Theorem 1.33 p. 320 in [K]. ■

Given $\beta \in (0, \pi/2]$ set

$$\Sigma_\beta = \{r\mathrm{e}^{\sqrt{-1}\alpha} : r > 0, \alpha \in (-\beta, \beta)\}$$

and

$$\Sigma_{0,\beta} := \Sigma_\beta \cup \{0\} = \{r\mathrm{e}^{\sqrt{-1}\alpha} : r \geqslant 0, \alpha \in (-\beta, \beta)\}.$$

A family of bounded operators $(T_z)_{z\in\Sigma_{0,\beta}}$ in $\mathscr{H}$, with some $\beta \in (0, \pi/2]$, is called a *holomorphic semigroup*, if

- $z \mapsto T_z$ is holomorphic[3] in $z \in \Sigma_\beta$,
- $T_{z+z'} = T_z T_{z'}$ for all $z, z' \in \Sigma_{0,\beta}$,
- $z \mapsto T_z$ is strongly continuous in $z = 0$ and $T(0) = 1$.

It follows that the restriction of T to $[0, \infty)$ is a strongly continuous semigroup, and if S is the generator of this semigroup, then for every $\Psi_0 \in \mathscr{H}$, the function

$$[0, \infty) \ni t \longmapsto T(t)\Psi_0 \in \mathscr{H}$$

is the uniquely determined strongly continuous function $\Psi : [0, \infty) \to \Gamma_{L^2}(\mathscr{M}, \mathscr{E})$ which is strongly differentiable on $(0, \infty)$ taking values in $\mathrm{Dom}(S)$ thereon, such that

$$(\mathrm{d}/\mathrm{d}t)\Psi(t) = S\Psi(t), \quad t > 0, \quad \Psi(0) = \Psi_0.$$

Thus, one essential property of holomorphic semigroups is that the above initial value problem has a unique solution for every initial value in $\mathscr{H}$, rather than just for initial values in the domain of the generator (cf. Remark 1.22 on p. 492 in [K]).

Finally, there is the following representation theorem:

Theorem A.2. *For every closed sectorial form h in $\mathscr{H}$ there exists a unique densely defined, closed, and sectorial operator S in $\mathscr{H}$ such that* $\mathrm{Dom}(S) \subset \mathrm{Dom}(h)$ *and*

$$h(\Psi_1, \Psi_2) = \langle S\Psi_1, \Psi_2\rangle \quad \textit{for all } \Psi_1 \in \mathrm{Dom}(S), \Psi_2 \in \mathrm{Dom}(h). \tag{A.1}$$

Moreover, $-S$ generates a holomorphic semigroup in $\mathscr{H}$, to be denoted with $z \mapsto \mathrm{e}^{-zS}$.

Proof: The existence of a densely defined, closed, and sectorial S satisfying (A.1) is the statement of Theorem 2.1 on p. 322 in [K]. In fact, it is stated there that S is actually m-sectorial, which by Theorem 1.24 on p. 492 in [K] implies that $-S$ generates a holomorphic semigroup, as for some $r \in \mathbb{R}$, the form induced by $S + r$ has a vertex 0 (see also Theorem 1.14 in ([A86])). ■

References

[AS] Aizenman, M. & Simon, B.: *Brownian motion and Harnack inequality for Schrödinger operators.* Comm. Pure Appl. Math. 35 (1982), no. 2, 209–273.

[A86] Arendt, W.; Grabosch, A.; Greiner, G.; Groh, U.; Lotz, H. P.; Moustakas, U.; Nagel, R.; Neubrander, F.; Schlotterbeck, U. *One-parameter semigroups of positive operators.* Lecture Notes in Mathematics, 1184. Springer-Verlag, Berlin, 1986.

[A83] Atiyah, M. F.: *Circular symmetry and stationary-phase approximation.* Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). Astérisque No. 131 (1985), 43–59.

[B86] Bérard, P. H.: *Spectral geometry: direct and inverse problems. With appendixes by Gérard Besson, and by Bérard and Marcel Berger.* Lecture Notes in Mathematics, 1207. Springer-Verlag, Berlin, 1986.

[BGV92] Berline, N. & Getzler, E. & Vergne, M.: *Heat kernels and Dirac operators.* Grundlehren der Mathematischen Wissenschaften, 298. Springer-Verlag, Berlin, 1992.

[B84] Bismut, J.-M.: *The Atiyah-Singer theorems: a probabilistic approach. I. The index theorem.* J. Funct. Anal. 57 (1984), no. 1, 56–99.

[3]Here, weak/strong/norm holomorphy are equivalent by the uniform boundedness principle.

[B85] Bismut, J.-M.: *Index theorem and equivariant cohomology on the loop space.* Comm. Math. Phys. 98 (1985), no. 2, 213–237.

[BG20] Boldt, S. & Güneysu, B.: *Scattering Theory and Spectral Stability under a Ricci Flow for Dirac Operators.* arXiv:2003.10204, 2020.

[BrG] Braun, M. & Güneysu, B.: *Heat flow regularity, Bismut–Elworthy–Li's derivative formula,and pathwise couplings on Riemannian manifoldswith Kato bounded Ricci curvature.* Preprint, 2020.

[BHL00] Broderix, K.& Hundertmark, D. & Leschke, H.: *Continuity properties of Schrödinger semigroups with magnetic fields.* Rev. Math. Phys. 12 (2000), no. 2, 181–225.

[1] Cheng, L.J. & Thalmaier, A.: & Thompson, J.: *Quantitative C^1-estimates by Bismut formulae.* J. Math. Anal. Appl. 465 (2018), no. 2, 803–813.

[CtE18] Chill, R. & ter Elst, A. F. M.: *Weak and strong approximation of semigroups on Hilbert spaces.* Integral Equations Operator Theory 90 (2018), no. 1, Paper No. 9, 22 pp.

[D83] Dodziuk J.: *Maximum principle for parabolic inequalities and the heat flow on open manifolds.* Indiana Univ. Math. J.,32(1983) no.5, 703–716.

[BD01] Driver, B. K. & Thalmaier, A.: *Heat equation derivative formulas for vector bundles.* J. Funct. Anal. 183 (2001), no. 1, 42–108.

[GP15] Güneysu, B. & Pallara, D.: *Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below.* Math. Ann. 363 (2015), no. 3-4, 1307–1331.

[G12] Güneysu, B. *On generalized Schrödinger semigroups.* J. Funct. Anal. 262 (2012), no. 11, 4639–4674.

[G17] Güneysu, B.: *Covariant Schrödinger semigroups on Riemannian manifolds.* Operator Theory: Advances and Applications, 264. Birkhäuser/Springer, Cham, 2017.

[GL19] Güneysu, B. & Ludewig, M.: *The Chern Character of ϑ-summable Fredholm Modules over dg Algebras and Localization on Loop Space.* Preprint, 2019.

[HT94] Hackenbroch, W. & Thalmaier, A.: *Stochastische Analysis.* B.G. Teubner, Stuttgart, 1994.

[H02] Hsu, E.P.: *Stochastic analysis on manifolds.* Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

[IW89] Ikeda, N. & Watanabe, S.: *Stochastic differential equations and diffusion processes.* Second edition. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[JLO88] Jaffe, A. & Lesniewski, A. & Osterwalder, K.: *Quantum K-theory. I. The Chern character.* Comm. Math. Phys. 118 (1988), no. 1, 1–14.

[K] Kato, T.: *Perturbation theory for linear operators.* Reprint of the 1980 edition. Springer, classics in mathematics.

[LM89] Lawson, Jr., H. B. & Michelsohn, M.-L.: *Spin geometry.* Princeton Math. Ser., vol. 38, Princeton Univ. Press, Princeton, NJ, 1989.

[M65] Milnor, J.W.: *Remarks concerning spin manifolds.* Differential and Combinatorial Topology, Princeton Univ. Press, 1965, pp. 55–62.

[N92] Norris, J.R.: *A complete differential formalism for stochastic calculus in manifolds. Séminaire de Probabilités, XXVI*, 189–209, Lecture Notes in Math., 1526, Springer, Berlin, 1992.

[O99] Ouhabaz, E.M.: *L^p contraction semigroups for vector valued functions.* Positivity 3 (1999), no. 1, 83–93.

[P03] Plank, H.: *Stochastic representation of the gradient and Hessian of diffusion semigroups on Riemannian manifolds.* Dissertation, Universität Regensburg, 2003.

[R08] Ren, Y.-F.: *On the Burkholder–Davis–Gundy inequalities for continuous martingales.* Statist. Probab. Lett. **78** (2008), no. 17, 3034–3039.

[IS97] Shigekawa, I.: *L^p contraction semigroups for vector valued functions.* J. Funct. Anal. 147 (1997), no. 1, 69–108.

[S82] Simon, B.: *Schrödinger semigroups.* Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 3, 447–526.

[S05] Simon, B.: *Functional integration and quantum physics.* Second edition. AMS Chelsea Publishing, Providence, RI, 2005.

[S77] Simon, B.: *An abstract Kato's inequality for generators of positivity preserving semigroups.* Indiana Univ. Math. J. 26 (1977), no. 6, 1067–1073.

[SV96] Stollmann, P. & Voigt, J.: *Perturbation of Dirichlet forms by measures.* Potential Anal. 5 (1996), no. 2, 109–138.

[S93] Sturm, K.-T.: *Schrödinger semigroups on manifolds.* J. Funct. Anal. 118 (1993), no. 2, 309–350.