# A differential topological invariant on spin manifolds from supersymmetric path integrals

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**Abstract**

We show that the $N = 1/2$ supersymmetric path integral on a closed even dimensional Riemannian spin manifold, realized via Chen forms and recent results from noncommutative geometry, induces a differential topological invariant (which does not depend on the Riemannian metric).

## 1 Motivation

Let $X$ be a compact even dimensional topological spin manifold[1]. The fixed topological spin structure induces an orientation (cf. Corollary E in [16]) on the Fréchet manifold $LX$ of smooth loops $\gamma : \mathbb{T} \to X$, whose tangent space $T_\gamma LX$ at a fixed loop $\gamma \in LX$ is given by the space of vector fields on $X$ along $\gamma$, that is, smooth maps $A : \mathbb{T} \to TX$ with $\dot\gamma(s) \in T_{\gamma(t)}X$ for all $s \in \mathbb{T}$. Given a Riemannian metric $g$ on $X$ let $E^g \in C^\infty(LX)$ and $\omega^g \in \Omega^2(LX)$ denote the energy functional and, respectively, the presymplectic form

$$E^g_\gamma := \int_{\mathbb{T}} g(\dot\gamma,\dot\gamma), \quad \omega^g_\gamma(A,B) := \int_{\mathbb{T}} g(\nabla_{\dot\gamma} A,B),$$

where we will occasionally identify $\mathbb{T} = [0,1]/\sim$. The following $N = 1/2$ supersymmetric path integral plays a crucial role in the context of Duistermaat-Heckman localization on $LX$: with

$$\widehat{\Omega}(LX) := \prod_{j=0}^{\infty} \Omega^j(LX)$$

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[1]We work exclusively in the category of smooth manifolds without boundary.

the space smooth differential forms o $LX$, one formally sets

$$\mathfrak{I}^g : \widehat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \mathfrak{I}^g[\alpha] := \int_{LX} e^{-E^g + \omega^g} \wedge \alpha. \tag{1.1}$$

Note that even though $LX$ is oriented, as it stands, the definition of $\mathfrak{I}^g$ does not make sense for (at least) the following reasons:

- there exists no infinite dimensional Lebesgue measure;
- the integral of an inhomogeneous differential form (which are the ones of interest) should by definition be the integral of its top degree part, however, $LX$ is infinite dimensional;
- $LX$ is noncompact, so even if one finds a natural way to integrate differential forms on $LX$, some care has to be taken concerning the question of finding a class of 'integrable' (smooth) differential forms.

As we are going to explain in a moment, the mathematical solution of these problems is tied together and manifests itself in a construction of $\mathfrak{I}^g$ via Chen integrals and the differential graded Chern character on $(M,g)$. However, in order to motivate our main results, let us continue with our heuristic observations for the moment.
With $\iota$ the contraction by the vector field $\mathbb{A}$ on $LX$ given by $\gamma \mapsto \dot{\gamma}$, which generates the natural $\mathbb{T}$-action on $LX$ given by rotating loops, and

$$\widehat{\Omega}_{\mathbb{T}}(LX) := \{\alpha \in \widehat{\Omega}(LX) : \mathcal{L}_{\mathbb{A}}\alpha = 0\}$$

the space of $\mathbb{T}$-invariant differential forms, there is a supercomplex

$$\cdots \xrightarrow{d+\iota} \widehat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widehat{\Omega}^-_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widehat{\Omega}^+_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \cdots, \tag{1.2}$$

and (with a slight abuse of notation) the dual supercomplex

$$\cdots \xrightarrow{d+\iota} \widehat{\Omega}_+^{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widehat{\Omega}_-^{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widehat{\Omega}_+^{\mathbb{T}}(LX) \xrightarrow{d+\iota} \cdots. \tag{1.3}$$

Note that these complexes are actually well-defined within the differential calculus of Fréchet manifolds. Now, supersymmetry takes the form

$$\mathfrak{I}^g[(d+\iota)\alpha] = 0 \quad \text{for all } \alpha \in \widehat{\Omega}(LX).$$

Moreover, $\mathfrak{I}^g$ is an even current, as $LX$ is formally even-dimensional, so that $\mathfrak{I}^g$ determines an even homology class in the homology of (1.3). Finally, one can derive the following infinite dimensional analogue of the Duistermaat-Heckman localization formula,

$$\mathfrak{I}^g[\alpha] = \int_X \widehat{A}(M,g) \wedge \alpha|_X \quad \text{for all } \alpha \in \widehat{\Omega}(LX) \text{ with } (d+\iota)\alpha = 0,$$

which leads to a simple and differential geometric 'proof' of the Atiyah-Singer index theorem [3, 2, 1], and which was in fact, the main motivation that lead to the discovery of $\mathfrak{J}^g$.

*The aim of this paper is to examine the dependence of* $\mathfrak{J}^g$ *on* $g$. To this end, let $g_\bullet = (g_t)_{t\in[0,1]}$ be a smooth family of Riemannian metrics on $X$ and define for every fixed $t \in [0,1]$ a differential form

$$\beta_t^{g_\bullet} \in \Omega^1(LX), \quad \beta_{t,\gamma}^{g_\bullet}(A) := -\frac{1}{2}\int_{\mathbb{T}} g_t(\dot{\gamma},A),$$

and the induced odd current

$$\mathfrak{C}_t^{g_\bullet} : \widehat{\Omega}(LX) \longrightarrow \mathbb{C}, \quad \mathfrak{C}_t^{g_\bullet}(\alpha) := \mathfrak{J}^{g_t}(\beta_t^{g_\bullet} \wedge \alpha).$$

In the appendix, we are going to derive the formula

$$(d/dt)\mathfrak{J}^{g_t} = (d+\iota)\mathfrak{C}_t^{g_\bullet} \quad \text{for all } t \in [0,1]. \tag{1.4}$$

This equality has an important consequence: defining the (odd) Chern-Simons current $\mathfrak{C}^{g_\bullet}$ by

$$\mathfrak{C}^{g_\bullet} := \int_0^1 \mathfrak{C}_t^{g_\bullet} dt : \widehat{\Omega}(LX) \longrightarrow \mathbb{C},$$

one gets the transgression formula

$$\mathfrak{J}^{g_1} - \mathfrak{J}^{g_0} = (d+\iota)\mathfrak{C}^{g_\bullet},$$

so that the homology class induced by $\mathfrak{J}^g$ in the homology of (1.3) does not depend on a particular choice of a Riemannian metric $g$ on $X$. These heuristic considerations show that any mathematically rigorous definition of the supersymmetric current $\mathfrak{J}^g$ should lead to a differential topologic invariant of $X$.

# 2 Main results

Let us explain now how these heuristic considerations can be verified in a mathematically rigorous way. To this end, we first explain the natural class of (smooth) integrable differential forms on $LX$: we turn $\widehat{\Omega}(LX)$ into a complete locally convex Hausdorff space by equipping $\Omega^j(LX)$ with the family of seminorms $\nu_f(\alpha) := \nu(f^*\alpha)$, where $f$ is a smooth map from a finite dimensional manifold $Y$ to $LX$, and $\nu$ is a continuous seminorm on the Fréchet space $\Omega^j(Y)$, and by equipping $\widehat{\Omega}(LX)$ with the product topology. Given $\alpha \in \Omega(X)$ and $t \in \mathbb{T}$ one defines $\alpha(t) \in \Omega(LX)$ to be the pullback of $\alpha$ with respect to the evaluation $\gamma \mapsto \gamma(t)$.

Consider the Fréchet space of $\mathbb{T}$-invariant differential forms $\Omega_{\mathbb{T}}(X \times \mathbb{T})$ on $X \times \mathbb{T}$, with $\mathbb{T}$ acting on the second slot. With $\vartheta_{\mathbb{T}} \in \Omega(\mathbb{T})$ the volume form, any $\theta \in \Omega_{\mathbb{T}}(X \times \mathbb{T})$ can be uniquely written in the form $\theta = \theta' + \vartheta_{\mathbb{T}} \wedge \theta''$ with $\theta',\theta'' \in \Omega(X)$.

Associated to this construction, there is the space of *entire chains* $\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$ which is defined as the completion of

$$\mathsf{C}_{\mathbb{T}}(X) := \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N},$$

with

$$\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N} := \Omega_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N}/(\mathbb{C} \cdot 1)$$

and where $\mathsf{C}_{\mathbb{T}}(X)$ is equipped with the following family of seminorms: given any continuous seminorm $\nu$ on $\Omega_{\mathbb{T}}(X \times \mathbb{T})$, one gets the induced projective tensor norm

$$\pi_{\nu,N} \quad \text{on} \quad \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N},$$

and then a seminorm $\epsilon_\nu$ on $\mathsf{C}_{\mathbb{T}}(X)$ by setting

$$\epsilon_\nu(c) := \sum_{N=0}^{\infty} \frac{\pi_{\nu,N}(c_N)}{\lfloor N/2 \rfloor !}, \tag{2.1}$$

if

$$c = \sum_{N=0}^{\infty} c_N \in \mathsf{C}_{\mathbb{T}}(X), \quad \text{with } c_N \in \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{\otimes N} \text{ for all } N.$$

The required family of seminorms is now given by $\epsilon_\nu$, where $\nu$ is a continuous seminorm on $\Omega_{\mathbb{T}}(X \times \mathbb{T})$.
There exists a uniquely determined continuous map [6], the equivariant *Chen iterated integral map*,

$$\Psi : \mathsf{C}^{\epsilon}_{\mathbb{T}}(X) \longrightarrow \widehat{\Omega}(LX).$$

such that for all $N \in \mathbb{N}_{\geq 0}$, $\theta_0, \ldots, \theta_N \in \theta \in \Omega_{\mathbb{T}}(X \times \mathbb{T})$, one has

$$\Psi(\theta_0 \otimes \cdots \otimes \theta_N) \tag{2.2}$$
$$= \int_{\{0 \leq t_1 \leq \cdots \leq t_N \leq 1\}} \theta_0(0) \wedge (\iota\theta_1'(t_1) - \theta_1''(t_1)) \wedge \cdots \wedge (\iota\theta_N'(t_N) - \theta_N''(t_N))\, dt_1 \cdots dt_N. \tag{2.3}$$

**Definition 2.1.** The space of *integrable Chen forms* $\widetilde{\Omega}(LX) \subset \widehat{\Omega}(LX)$ is defined as the image of $\Psi$.

Set

$$\widetilde{\Omega}_{\mathbb{T}}(LX) := \widetilde{\Omega}(LX) \cap \widehat{\Omega}_{\mathbb{T}}(LX).$$

The following result follows essentially from calculations made in [6]. A detailed proof will be given in Section 3

**Proposition 2.2.** *There is a well-defined supercomplex*

$$\cdots \xrightarrow{d+\iota} \widetilde{\Omega}^{+}_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widetilde{\Omega}^{-}_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \widetilde{\Omega}^{+}_{\mathbb{T}}(LX) \xrightarrow{d+\iota} \cdots . \tag{2.4}$$

The associated dual supercomplex will be denoted with

$$\cdots \xrightarrow{d+\iota} \widetilde{\Omega}^{\mathbb{T}}_{+}(LX) \xrightarrow{d+\iota} \widetilde{\Omega}^{\mathbb{T}}_{-}(LX) \xrightarrow{d+\iota} \widetilde{\Omega}^{\mathbb{T}}_{+}(LX) \xrightarrow{d+\iota} \cdots . \tag{2.5}$$

Let us now give the formula for $\mathfrak{I}^g$. Recall that we have fixed a topologic spin structure on $X$. Consider the spinor bundle $\Sigma_g \to X$ induced by $g$, with its (essentially self-adjoint) Dirac operator $D_g$ on the super-Hilbert-space of $L^2$-spinors $\Gamma_{L^2}(X,\Sigma_g)$, and the (natural extension to differential forms of all degrees of the) Clifford multiplication

$$c_g : \Omega(X) \longrightarrow \Gamma_{C^\infty}(X,\mathrm{End}(\Sigma_g)).$$

Given any $N \in \mathbb{N}_{\geq 1}$ and any tupel $(\theta_1,\dots,\theta_N)$ of elements of $\Omega_{\mathbb{T}}(X\times\mathbb{T})$, define a differential operator $F_g(\theta_1,\dots,\theta_N)$ in $\Sigma_g \to X$ as follows,

$$\begin{aligned} &F_g(\theta) = c_g(d\theta') - [D_g, c_g(\theta')] - c_g(\theta'') \\ &F_g(\theta_1,\theta_2) = (-1)^{|\theta_1'|}\big(c_g(\theta_1'\theta_2') - c_g(\theta_1')c_g(\theta_2')\big), \\ &F_g(\theta_1,\dots,\theta_N) = 0, \quad \text{if } N \geq 3, \end{aligned}$$

where here and in the sequel all commutators are super-commutators. For $M \leq N$ denote with $P_{M,N}$ all tuples $I = (I_1,\dots,I_M)$ of subsets of $\{1\dots,N\}$ with $I_1\cup\cdots\cup I_M = \{1\dots,N\}$ and with each element of $I_a$ smaller than each element of $I_b$ whenever $a<b$. Given

$$\theta_1,\dots,\theta_N \in \Omega_{\mathbb{T}}(X\times\mathbb{T}), \quad I = (I_1,\dots,I_M) \in P_{M,N}, \quad 1 \leq a \leq M,$$

set

$$\theta_{I_a} := (\theta_{i+1},\dots,\theta_{i+m}), \quad \text{if } I_a = \{j \mid i < j \leq i+m\} \text{ for some } i,m.$$

With these preparations, the following is the main result of [7]:

**Theorem 2.3.** *There exists a uniquely determined current $\mathfrak{I}^g : \widetilde{\Omega}(LX) \to \mathbb{C}$ such that for all $N \in \mathbb{N}_{\geq 0}$, $\theta_0,\dots,\theta_N \in \Omega_{\mathbb{T}}(X\times\mathbb{T})$ one has*

$$\mathfrak{I}^g\left[\int_{\{0\leq t_1\leq\cdots\leq t_N\leq 1\}} \theta_0(0)\wedge(\iota\theta_1'(t_1)-\theta_1''(t_1))\wedge\cdots\wedge(\iota\theta_N'(t_N)-\theta_N''(t_N))\,dt_1\cdots dt_N\right] \tag{2.6}$$

$$\begin{aligned} = \sum_{M=1}^{N}(-1)^M \sum_{I\in P_{M,N}} \int_{\{0\leq t_1\leq\cdots\leq t_M\leq 1\}} &\mathrm{Str}_g\Big(c_g(\theta_0)e^{-t_1D_g^2}F_g(\theta_{I_1})\times \\ &\times e^{-(t_2-t_1)D_g^2}F_g(\theta_{I_2})\cdots e^{-(t_M-t_{M-1})D_g^2}F_g(\theta_{I_M})e^{-(1-t_M)D_g^2}\Big)\;dt_1\cdots dt_M, \end{aligned}$$

*where* $\mathrm{Str}_g$ *denotes the supertrace in $\Gamma_{L^2}(X,\Sigma_g)$. Moreover, $\mathfrak{I}^g$ is even and $(d+\iota)\mathfrak{I}^g = 0$, so that $\mathfrak{I}^g$ defines an even homology class in the homology of (2.5), and one has the localization formula*

$$\mathfrak{I}^g[\alpha] = \int_X \widehat{A}(M,g)\wedge\alpha|_X \quad \textit{for all } \alpha\in\widetilde{\Omega}(LX) \textit{ with } (d+\iota)\alpha = 0.$$

That this definition of $\mathfrak{J}^g$ is natural, in the sense that it really serves as an *implementation* of the right hand side of (1.1), has been indicated in [9] using the Pfaffian line bundle. A probabilistic representation of $\mathfrak{J}^g$ has been derived in [10], generalizing the earlier result from [5] for $N = 1$ to all orders.
Here comes the main result of this note:

**Theorem 2.4.** *Assume $g_\bullet = (g_t)_{t\in[0,1]}$ is a smooth family of Riemannian metrics on $X$. Then there exists a canonically given odd current $\mathfrak{C}^{g_\bullet} : \widetilde{\Omega}(LX) \to \mathbb{C}$ with $\mathfrak{J}^{g_1} - \mathfrak{J}^{g_0} = (d+\iota)\mathfrak{C}^{g_\bullet}$; in particular, the homology class induced by $\mathfrak{J}^g$ in the homology of (2.5) does not depend on a particular choice of a Riemannian metric $g$ on $X$.*

Our main result yields a new differential topological invariant:

**Corollary 2.5.** *Let $M$ and $N$ be compact even-dimensional, oriented spin manifolds with fixed topological spin-structures. Assume there exists a diffeomorphism $f : M \to N$ preserving orientations and topological spin-structures. Then, for any choice of Riemannian metrics $g$ and $h$ on $M$ resp. on $N$, the homology class induced by $\mathfrak{J}^g_M$ in the homology of (2.5) equals the homology class of $f^*\mathfrak{J}^h_N$.*

*Proof.* Setting $g_1 := f^*h$, the diffeomorphism $f$ becomes an orientation and metric spin-structure preserving isometry $f : (M,g_1) \to (N,h)$ furnishing unitary equivalences between Clifford multiplications and Dirac operators on $(M,g_1)$ and $(N,h)$. Formula (2.6) shows that $\mathfrak{J}^{g_1}_M$ and $f^*\mathfrak{J}^h_N$ are equal, and Theorem 2.4 establishes the claim. □

# 3 Proof of Proposition 2.2

We have to show that $d + \iota$ maps

$$\widetilde{\Omega}_{\mathbb{T}}(LX) = \widetilde{\Omega}(LX) \cap \widehat{\Omega}_{\mathbb{T}}(LX)$$

to itself. We give $\Omega_{\mathbb{T}}(X \times \mathbb{T})$ the $\mathbb{Z}$-grading

$$\theta' + \vartheta_{\mathbb{T}} \wedge \theta'' \in \Omega_{\mathbb{T}}(X \times \mathbb{T})^j \Leftrightarrow \theta' \in \Omega^j(X), \theta'' \in \Omega^{j+1}(X)$$

and turn it into a locally convex DGA using the differential $d + \iota_{\partial_{\mathbb{T}}}$ with $\partial_{\mathbb{T}}$ the canonic vector field on $\mathbb{T}$. Then $\mathsf{C}_{\mathbb{T}}(X)$ inherits the $\mathbb{Z}$-grading induced by

$$\mathsf{C}_{\mathbb{T}}(X) = \bigoplus_{N=0}^{\infty} \Omega_{\mathbb{T}}(X \times \mathbb{T}) \otimes \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]^{\otimes N},$$

where $\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]$ denotes $\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})$ as a set with the shifted grading

$$\underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})[1]^j := \underline{\Omega}_{\mathbb{T}}(X \times \mathbb{T})^{j+1}.$$

With $b$ the Hochschild differential and $B$ the Connes differential in the $\mathbb{Z}$-graded category, the space $\mathsf{C}_{\mathbb{T}}(X)$ becomes a supercomplex with the differential $d+\iota_{\partial_{\mathbb{T}}}+b+B$. By continuity,

the same holds true for $\mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$.
Let

$$\mathbf{A} : \widehat{\Omega}(LX) \longrightarrow \widehat{\Omega}(LX)$$

be the idempotent linear operator obtained by averaging the $\mathbb{T}$-action on $LX$. Then as shown in [6] one has the formulae

$$\Psi(d + \iota_{\partial_{\mathbb{T}}} + b) = d\Psi, \quad \Psi B + \mathbf{A}\Psi\iota_{\partial_{\mathbb{T}}} = \mathbf{A}\iota\Psi,$$

noting that in the notation of [6] one has $\rho = \mathbf{A}\Psi$. Note also that $\mathbf{A}$ commutes with $d$ and $\iota$, so that

$$\mathbf{A}\Psi(d + \iota_{\partial_{\mathbb{T}}} + b + B) = (d + \iota)\mathbf{A}\Psi.$$

Assume that $\alpha \in \widetilde{\Omega}(LX)$ is $\mathbb{T}$-invariant. This means that $\alpha = \Psi(\theta)$ for some $\theta \in \mathsf{C}^{\epsilon}_{\mathbb{T}}(X)$ and that $\mathbf{A}\Psi(\theta) = \Psi(\theta)$. Clearly, $(d + \iota)\alpha = (d + \iota)\Psi(\theta)$, so that $(d + \iota)\alpha$ is in $\widetilde{\Omega}(LX)$, furthermore,

$$(d + \iota)\alpha = (d + \iota)\mathbf{A}\Psi(\theta) = \mathbf{A}\Psi((d + \iota_{\partial_{\mathbb{T}}} + b + B)\theta),$$

which shows that $(d + \iota)\alpha$ is $\mathbb{T}$-invariant. This completes the proof.

# 4 Proof of Theorem 2.4

We briefly recall the Bourguignion-Gauduchon machinery for metric changes of the Dirac operator [4]. For any $t \in [0,1]$, define a section $\mathcal{A}_t$ of $\mathrm{End}(TX)$ by

$$g_0(u,v) = g_t(\mathcal{A}_t u,v) \qquad \text{for all} \qquad x \in X, u,v \in T_xX\,.$$

Then $\mathcal{A}_t$ is strictly positive w.r.t. $g_t$ and $g_0$ and $\mathcal{A}_t^{-1/2}$ is a pointwise isometry $(TX,g_t) \to (TX,g_0)$. It therefore lifts canonically to an $\mathrm{SO}(n)$-equivariant bundle map

$$b_t : \mathrm{SO}(X,g_t) \longrightarrow \mathrm{SO}(X,g_0)\,,$$

where $\mathrm{SO}(X,g_t)$ denotes the bundle of oriented orthonormal frames of $X$ w.r.t. the Riemannian metric $g_t$.
Now recall that we have fixed a topological spin structure. This implies that every Riemannian metric $g_t$ canonically induces a Riemannian spin structure on $X$, i.e., a $\mathrm{Spin}(n)$-principal fibre bundle $P_t$ over $X$ together with a $\xi$-equivariant map $\pi_t : P_t \to \mathrm{SO}(X,g_t)$ such that $(P_t,\pi_t)$ is a $\xi$-reduction of $\mathrm{SO}(X,g_t)$. Here, $\xi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the canonically given double cover. Furthermore, $(P_t, \pi_t)$ being associated with a fixed topological spin structure, the map $b_t$ lifts to an equivariant bundle map $\widetilde{b}_t : P_t \to P_0$ and through the associated vector bundle construction, we obtain a fibrewise isometric vector bundle isomorphism

$$\beta_t : \Sigma_{g_t} \longrightarrow \Sigma_{g_0}\,,$$

which moreover satisfies

$$\beta_t(c_{g_t}(\theta)(\varphi)) = c_{g_0}({\mathcal{A}'}_t^{1/2}(\theta))(\beta_t(\varphi)) \quad \text{for all} \quad x \in X, \theta \in T^*_xX, \varphi \in (\Sigma_{g_t})_x\,,$$

where $\mathcal{A}' \in \mathrm{End}(T^*X)$ is the transpose of $\mathcal{A}$. With

$$0 < \rho_t = d\mu_{g_0}/d\mu_{g_t} \in C^\infty(X)$$

the Radon-Nikodym density of $\mu_{g_0}$ w.r.t. $\mu_{g_t}$, we obtain the canonically given unitary operator

$$U_t : \Gamma_{L^2}(X,\Sigma_{g_t}) \longrightarrow \Gamma_{L^2}(X,\Sigma_{g_0})$$
$$U_t\varphi(x) = \rho_t^{-1/2}\beta_t(\varphi(x)),$$

which we use to define a family of $\vartheta$-summable Fredholm modules $\mathcal{M}^{g_\bullet}$ over $\Omega(X)$ in the sense of Definition 2.1 in [7], by

$$\mathcal{M}^{g_\bullet}_t := (\Gamma_{L^2}(X,\Sigma_{g_0}),c^t,Q_t) := (\Gamma_{L^2}(X,\Sigma_{g_0}), U_t c_{g_t} U_t^*, U_t D_{g_t} U_t^*),$$

where $D_{g_t}$ is the Dirac operator acting on $L^2$-sections of $\Sigma_{g_t}$. Consider the Chern character

$$\mathrm{Ch}_{g_t} : \mathsf{C}^\epsilon_{\mathbb{T}}(X) \longrightarrow \mathbb{C},$$

whose value at

$$\theta_0 \otimes \cdots \otimes \theta_N \in \mathsf{C}^\epsilon_{\mathbb{T}}(X)$$

is given by the RHS of (2.6) for $g = g_t$. Then $\mathrm{Ch}_{g_t}$ vanishes on the kernel of $\Psi$ and this defines $\mathfrak{J}^{g_t}$. If we can show that $\mathcal{M}^{g_\bullet}$ satisfies the axioms of Definition 6.1 in [7], then (using that Chern characters are invariant under unitary transformations) it follows that the (odd) Chern-Simons form

$$\mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}}) : \mathsf{C}^\epsilon_{\mathbb{T}}(X) \longrightarrow \mathbb{C}$$

constructed on page 31 in [7] satisfies

$$\mathrm{Ch}_{g_1} - \mathrm{Ch}_{g_0} = (d + \iota_{\partial_{\mathbb{T}}} + b + B)\mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}})$$

and vanishes on the kernel of $\Psi$, too. It follows that

$$\mathfrak{C}^{g_\bullet}(\Psi(\theta)) := \mathrm{CS}(\mathcal{M}^{g_\bullet}_{\mathbb{T}})(\theta), \quad \theta \in \mathsf{C}^\epsilon_{\mathbb{T}}(X),$$

is well-defined and, being invariant under $\mathbf{A}$ (which follows from its very construction), has the desired properties, in view of

$$\mathbf{A}\Psi(d + \iota_{\partial_{\mathbb{T}}} + b + B) = (d + \iota)\mathbf{A}\Psi.$$

It remains to show (H1) and (H2) from Definition 6.1 in [7], where (H1) is the condition

$$\sup_{t\in[0,1]} \mathrm{tr}\left(e^{-Q_t^2}\right) < \infty,$$

and (H2) is the condition

$$\sup_{t\in[0,1]}\left\|\dot{Q}_t(Q_t^2+1)^{-1/2}\right\| + \sup_{t\in[0,1]}\left\|(Q_t^2+1)^{-1/2}\dot{Q}_t\right\| < \infty.$$

Here, (H1) can be seen as follows: one can appeal to the Lichnerowicz formula for $D_t^2$ and semigroup domination (cf. Theorem 3.1 in [11]) to get

$$\mathrm{tr}\left(e^{-Q_t^2}\right) \leq \mathrm{rank}(\Sigma_0)e^{-\min_{x\in X}(1/4)\mathrm{scal}_{g_t}(x)}\mathrm{tr}\left(e^{-\Delta_{g_t}}\right),$$

which entails (H1), as $t\mapsto \min_{x\in X}(1/4)\mathrm{scal}_{g_t}(x)$ is clearly continuous, and $t\mapsto \mathrm{tr}\left(e^{-\Delta_{g_t}}\right)$ is smooth by Proposition 6.1 from [13].
To see (H2) note that from elliptic regularity, each $Q_t := U_{g_t}D_{g_t}U_{g_t}^*$ has the same domain of definition $W^{1,2}(X)$. Furthermore, $\dot{Q}_t := (d/dt)Q_t$ is a first order differential operator, which we consider as acting on smooth spinors. The proof of (H2) is based on the following lemma, which is a modification of Lemma 4.17 in [8]:

**Lemma 4.1.** *Let $S$ be a densely defined, closed linear operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, and let $T$ be a self-adjoint bounded linear operator in $\mathcal{H}_1$ with $T\geq -\lambda$ for some $\lambda\geq 0$. Assume that $S^*S+T\geq 0$. Then one has*

$$\|S(S^*S+T+1)^{-1/2}\| \leq \sqrt{\lambda+1}\,.$$

*Proof.* By assumption we have

$$S^*S+1 \leq S^*S+T+\lambda+1\,,$$

which means

$$\|(S^*S+1)^{1/2}f\| \leq \|(S^*S+T+\lambda+1)^{1/2}f\| \quad \text{for all} \quad f\in \mathrm{dom}(S^*S)^{1/2}\,.$$

From this we obtain

$$\|(S^*S+1)^{1/2}(S^*S+T+1)^{-1/2}h\| \leq \|(S^*S+T+\lambda+1)^{1/2}(S^*S+T+1)^{-1/2}h\|$$

for all $h\in\mathcal{H}_1$. Using the functional calculus associated with the operator $S^*S+T$, we calculate the norm of the operator appearing on the right hand side to be

$$\|(S^*S+T+\lambda+1)^{1/2}(S^*S+T+1)^{-1/2}\| \leq \sup_{t\geq 0}\sqrt{\frac{t+\lambda+1}{t+1}} = \sqrt{\lambda+1}\,,$$

which implies

$$\|(S^*S+1)^{1/2}(S^*S+T+1)^{-1/2}\| \leq \sqrt{\lambda+1}\,.$$

Now we can estimate

$$\begin{aligned}\|S(S^*S+T+1)^{-1/2}\| &= \|S(S^*S+1)^{-1/2}(S^*S+1)^{1/2}(S^*S+T+1)^{-1/2}\| \\ &\le \sqrt{\lambda+1}\|S(S^*S+1)^{-1/2}\| \\ &\le \sqrt{\lambda+1}\|(S^*S)^{1/2}(S^*S+1)^{-1/2}\| \\ &\le \sqrt{\lambda+1}\sup_{t\ge 0}\sqrt{\frac{t}{t+1}} \\ &\le \sqrt{\lambda+1}\,,\end{aligned}$$

where we have used the polar decomposition $S = U(S^*S)^{1/2}$ with a partial isometry $U$ on the third line and the functional calculus associated with the operator $S^*S$ on the fourth line. □

Using this lemma, we are going to prove that one has (H2): first of all, note that $Q_t$ acting on $\Gamma_{C^\infty}(X,\Sigma_{g_0})$ is a first order differential operator whose coefficients depend smoothly on $t \in [0,1]$. Since $X$ is compact, it follows that

$$\left\langle \dot{Q}_t\varphi,\psi \right\rangle = (d/dt)\left\langle Q_t\varphi,\psi\right\rangle = (d/dt)\left\langle \varphi,Q_t\psi\right\rangle = \left\langle \varphi,\dot{Q}_t\psi \right\rangle$$

for all $\varphi,\psi \in \Gamma_{C^\infty}(X,\Sigma_{g_0})$, i.e., $\dot{Q}_t$ is symmetric.
Secondly, the operator $Q_t^2+1$ being elliptic, it follows from a classical result of Seeley [14] that $(Q_t^2+1)^{-1/2}$ is a pseudo-differential operator. In particular, it maps $\Gamma_{C^\infty}(X,\Sigma_{g_0})$ to itself.
Turning to operator norms, note that $\dot{Q}_t(Q_t^2+1)^{-1/2}$ is bounded if and only if

$$\sup\left\{\left|\left\langle \dot{Q}_t(Q_t^2+1)^{-1/2}\varphi,\varphi\right\rangle\right| \,:\, \varphi \in \Gamma_{C^\infty}(X,\Sigma_{g_0})\right\} < \infty\,.$$

The operators $\dot{Q}_t$ and $(Q_t^2+1)^{-1/2}$ being symmetric this, in turn, is equivalent to $(Q_t^2+1)^{-1/2}\dot{Q}_t$ being bounded. Hence, it suffices to show that

$$\sup_{t\in[0,1]}\left\|\dot{Q}_t(Q_t^2+1)^{-1/2}\right\| < \infty\,. \tag{4.1}$$

To this end, we first use the unitary invariance of the functional calculus to compute

$$\begin{aligned}\left\|\dot{Q}_t(Q_t^2+1)^{-1/2}\right\| &= \left\|\dot{Q}_t((U_tD_{g_t}U_t^*)^2+1)^{-1/2}\right\| = \left\|\dot{Q}_tU_t(D_{g_t}^2+1)^{-1/2}U_t^*\right\| \\ &= \left\|U_t^*\dot{Q}_tU_t(D_{g_t}^2+1)^{-1/2}\right\|\,.\end{aligned}$$

Next, we decompose

$$U_t^*\dot{Q}_tU_t = \sigma_t\circ\nabla_t + \tau_t,$$

with $\nabla_t$ the spinor connection of $\Sigma_{g_t}$, and

$$\sigma_t \in \Gamma_{C^\infty}(X,\mathrm{Hom}(T^*X\otimes\Sigma_{g_t},\Sigma_{g_t})),\quad \tau_t \in \Gamma_{C^\infty}(X,\mathrm{End}(\Sigma_{g_t})),$$

so that by the Lichnerowicz formula we have

$$U_t^* \dot{Q}_t U_t (D_{g_t}^2 + 1)^{-1/2} = \sigma_t \nabla \left(\nabla^*\nabla + \tfrac{1}{4}\mathrm{scal}_{g_t} + 1\right)^{-1/2} + \tau_t \left(D_{g_t}^2 + 1\right)^{-1/2} . \tag{4.2}$$

Because $\|(D_{g_t}^2 + 1)^{-1/2}\| \leq 1$, the operator norm of the second term on the right hand side is bounded by $\|\tau_t\|$, which is continuous in $t$. Hence,

$$\sup_{t\in[0,1]} \|\tau_t \left(D_{g_t}^2 + 1\right)^{-1/2} \| < \infty .$$

Regarding the first term on the right hand side of (4.2), we appeal to the above lemma with

$$S = \nabla, \quad T = (1/4)\mathrm{scal}_{g_t}, \quad \lambda_t := (1/4) \max_{x\in X} |\mathrm{scal}_{g_t}(x)|,$$

to see that

$$\|\sigma_t \nabla \left(\nabla^*\nabla + \tfrac{1}{4}\mathrm{scal}_{g_t} + 1\right)^{-1/2} \| \leq \|\sigma_t\| \sqrt{\lambda_t + 1} ,$$

which is also continuous in $t$, thereby completing the proof of (4.1) and hence also of Theorem 2.4.

# Appendix: formal proof of formula (1.4)

We start by calculating the derivative of $\mathfrak{I}^{g_t}$ w.r.t. $t$,

$$(d/dt)\mathfrak{I}^{g_t}[\alpha] = \int_{LX} (d/dt) e^{-E^{g_t}+\omega^{g_t}} \wedge \alpha = \int_{LX} e^{-E^{g_t}+\omega^{g_t}} \wedge (d/dt)\left(-E^{g_t} + \omega^{g_t}\right) \wedge \alpha .$$

Let $\nabla(t)$ denote the Levi-Civita connection for $g_t$, and let $\gamma \in LX$, $X,Y \in T_\gamma LX$. The $t$-derivative appearing in the integrand on the right-hand side is

$$(d/dt)\left(-E_\gamma^{g_t} + \omega_\gamma^{g_t}\right)(Y,Z) = -\frac{1}{2}\int_{\mathbb{T}} g_t'(\dot\gamma,\dot\gamma) + \int_{\mathbb{T}} g_t'(Y,\nabla(t)_{\dot\gamma} Z) + \int_{\mathbb{T}} g_t(Y,\nabla(t)'_{\dot\gamma} Z) , \tag{4.3}$$

where we have used primes to denote derivatives w.r.t. $t$ and dots to denote derivatives w.r.t. the loop parameter.
Using that the covariant derivative commutes with every contraction, the second integral in (4.3) is equal to

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}} g_t'(Y,\nabla(t)_{\dot\gamma} Z) + \frac{1}{2}\int_{\mathbb{T}} \{\dot\gamma g_t'(Y,Z) - \nabla(t)_{\dot\gamma}(g_t'(Y,\cdot\,))(Z)\} \\
&\qquad = \frac{1}{2}\int_{\mathbb{T}} g_t'(Y,\nabla(t)_{\dot\gamma} Z) - \frac{1}{2}\int_{\mathbb{T}} \nabla(t)_{\dot\gamma}(g_t'(Y,\cdot\,))(Z) \\
&\qquad = \frac{1}{2}\int_{\mathbb{T}} \{g_t'(Y,\nabla(t)_{\dot\gamma} Z) - g_t'(Z,\nabla(t)_{\dot\gamma} Y)\} - \frac{1}{2}\int_{\mathbb{T}} (\nabla(t)_{\dot\gamma} g_t')(Y,Z) .
\end{aligned}$$

For the third term on the right-hand side of (4.3), we use the well-known formula (see, e.g., [15, Proposition 2.3.1]) for the time derivative of the Levi-Civita connection,

$$\int_{\mathbb{T}} g_t(Y,\nabla(t)'_{\dot\gamma}Z) = \frac{1}{2}\int_{\mathbb{T}} \{(\nabla(t)_Z g'(t))(Y,\dot\gamma) + (\nabla(t)_{\dot\gamma} g'_t)(Y,Z) - (\nabla(t)_Y g'_t)(Z,\dot\gamma)\}\ .$$

Putting the above together, we obtain

$$\begin{aligned}(d/dt)\left(-E^{g_t}_\gamma + \omega^{g_t}_\gamma\right)(Y,Z) = -\frac{1}{2}\int_{\mathbb{T}} g'_t(\dot\gamma,\dot\gamma) + \frac{1}{2}\int_{\mathbb{T}} \{g'_t(Y,\nabla(t)_{\dot\gamma}Z) - g'_t(Z,\nabla(t)_{\dot\gamma}Y)\} \\ -\frac{1}{2}\int_{\mathbb{T}} \{(\nabla(t)_Y g'_t)(\dot\gamma,Z) - (\nabla(t)_Z g'_t)(\dot\gamma,Y)\}\ . \quad (4.4)\end{aligned}$$

On the other hand, defining the 1-form $\sigma_t$ on $LX$ by

$$(\sigma_t)_\gamma(Y) = -\frac{1}{2}\int_{\mathbb{T}} g'_t(\dot\gamma, Y)\,,$$

its exterior derivative $d\sigma_t$ is defined by the Cartan formula,

$$d(\sigma_t)_\gamma(Y,Z) = Y\sigma^t(\widetilde{Z}) - Z\sigma^t(\widetilde{Y}) - \sigma_t([\widetilde{Y},\widetilde{Z}])\,,$$

where $\widetilde{Y}$ and $\widetilde{Z}$ are local extensions of $Y,Z$, i.e., vector fields defined on a neighborhood of $\gamma \in LX$ with $\widetilde{Y}_\gamma = Y$ and $\widetilde{Z}_\gamma = Z$ (this definition is independent of the extensions $\widetilde{Y}, \widetilde{Z}$). Using 1- and 2-parameter variations of $\gamma$ with variation vector fields $X$ and $Y$ respectively and formula (4.4), one easily computes

$$d(\sigma_t)_\gamma(Y,Z) = (d/dt)\left(-E^{g_t}_\gamma + \omega^{g_t}_\gamma\right)(Y,Z) - \iota\sigma_t\,.$$

Hence, for any differential form $\alpha$ on $LX$ we have

$$(d/dt)\mathfrak{I}^{g_t}[\alpha] = \int_{LX} e^{-E^{g_t}+\omega^{g_t}} \wedge (d+\iota)\sigma_t \wedge \alpha = \int_{LX} e^{-E^{g_t}+\omega^{g_t}} \wedge \sigma_t \wedge (d+\iota)\alpha\,,$$

where the last equality follows from

$$(d+\iota)\mathfrak{I}^{g_t}[\alpha] = \mathfrak{I}^{g_t}[(d+\iota)\alpha] = 0.$$

Defining

$$\mathfrak{C}^{g_\bullet}_t(\alpha) := \int_{LX} e^{-E^{g_t}+\omega^{g_t}} \wedge \sigma_t \wedge \alpha\,,$$

we end up with

$$(d/dt)\mathfrak{I}^{g_t} = (d+\iota)\mathfrak{C}^{g_\bullet}_t,$$

formally proving (1.4).

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