Fractional Cahn-Hilliard Equation(s): Analysis, Properties and Approximation

Mark Ainsworth[1] Zhiping Mao[2]

The classical Cahn-Hilliard equation [1] is a non-linear, fourth order in space, parabolic par- tial differential equation which is often used as a diffuse interface model for the phase separation of a binary alloy. Despite the widespread adoption of the model, there are good reasons for preferring models in which fractional spatial derivatives appear [2,3]. We consider two such Fractional Cahn-Hilliard equations (FCHE). The first [4] corresponds to considering a gradient flow of the free energy functional in a negative order Sobolev space $H^{-\alpha}$, $\alpha \in [0,1]$ where the choice $\alpha = 1$ corresponds to the classical Cahn-Hilliard equation whilst the choice $\alpha = 0$ recovers the Allen-Cahn equation. It is shown that the equation preserves mass for all positive values of fractional order and that it indeed reduces the free energy. The well-posedness of the problem is established in the sense that the H^1-norm of the solution remains uniformly bounded. We then turn to the delicate question of the L_∞ boundedness of the solution and establish an L_∞ bound for the FCHE in the case where the non-linearity is a quartic polynomial. As a consequence of the estimates, we are able to show that the Fourier-Galerkin method delivers a spectral rate of convergence for the FCHE in the case of a semi-discrete approximation scheme. Finally, we present results obtained using computational simulation of the FCHE for a variety of choices of fractional order α. We then consider an alternative FCHE [3,5] in which the free energy functional involves a fractional order derivative.

References:

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[1] Division of Applied Mathematics, Brown University, Providence RI 02912, USA,
Mark_Ainsworth@brown.edu

[2] Division of Applied Mathematics, Brown University, Providence RI 02912, USA,
Zhiping_Mao@brown.edu