On the Stability and Conditioning of Anisotropic Finite-Element-Runge-Kutta Methods

Jens Lang[1] Weizhang Huang[2] Lennard Kamenski[3]

In [HKL2010,HKL2013] anisotropic mesh adaptation methods for elliptic problems are studied. In a next step, we have investigated the influence of anisotropic meshes upon the time stepping and the conditioning of the linear systems arising from linear finite element approximations of linear parabolic equations. Here, we present stability results and estimates for the condition number. Both explicit and implicit time integration schemes are considered. For stabilized explicit Runge-Kutta methods, it is shown that the allowed maximal step size depends only on the number of the elements in the mesh and a measure of the non-uniformity of the mesh viewed in the metric specified by the inverse of the diffusion matrix. Particularly, it is independent of the mesh non-uniformity in volume measured in the Euclidean metric [HKL2016]. For the implicit time stepping situation, bounds are established for the condition number of the resulting linear system with and without diagonal preconditioning for the implicit Euler (the simplest implicit RK method) and general implicit RK methods. It is shown that the conditioning of an implicit RK method behaves like that of the implicit Euler method. The obtained bounds for the condition number have explicit geometric interpretations and take the interplay between the diffusion matrix and the mesh geometry into full consideration. They show that there are three mesh-dependent factors that can affect the conditioning: the number of elements, the mesh non-uniformity measured in the Euclidean metric, and the mesh non-uniformity with respect to the inverse of the diffusion matrix. They also reveal that the preconditioning using the diagonal of the system matrix, the mass matrix, or the lumped mass matrix can effectively eliminate the effects of the mesh non-uniformity measured in the Euclidean metric [HKL2017]. Illustrative numerical examples are given.

References:

[1] W. Huang, L. Kamenski, J. Lang (HKL2010), A new anisotropic mesh adaptation method based upon hierarchical a posteriori error estimates, J. Comp. Phys. 229 (2010), pp. 2179 2198.
[2] W. Huang, L. Kamenski, J. Lang (HKL2013), Adaptive finite elements with anisotropic meshes, Numerical Mathematics and Advanced Applications 2011: Proceedings of ENUMATH 2011, the 9th European Conference on Numerical Mathematics and Advanced Applications, Leicester, September 2011, A. Cangiani et al. (eds.), pp. 33-42, Springer 2013.
[3] W. Huang, L. Kamenski, J. Lang (HKL2016), Stability of explicit one-step methods for P1-finite element approximation of linear diffusion equations on anisotropic meshes, SIAM J. Numer. Anal. 54 (2016), pp. 1612-1634.
[4] W. Huang, L. Kamenski, J. Lang (HKL2017), Conditioning of implicit Runge-Kutta integration for finite element approximation of linear diffusion equations on anisotropic meshes, arXiv:1703.06463.

[1] Technische Universitaet Darmstadt, Mathematics, Darmstadt, Germany,
`lang@mathematik.tu-darmstadt.de`

[2] Kansas University,
`whuang@ku.edu`

[3] WIAS Berlin,
`kamenski@wias-berlin.de`