An Optimal Order DG Time Discretization Scheme for Parabolic Problems with Non-homogeneous Constraints

Igor Voulis[1] Arnold Reusken[2]

We consider parabolic problems with non-homogeneous constraints. Standard problems of this kind include the heat equation with a non-homogeneous Dirchlet boundary condition and the following Stokes problem with an non-homogeneous divergence condition and a non-homogeneous boundary condition (in $\Omega \times [0,T]$, $\Omega \subset \mathbb{R}^d$):

$$\begin{aligned} u' - \Delta u + \nabla p &= f \\ \mathrm{div} u &= g \\ u|_{\partial\Omega} &= h \\ u(0) &= u_0. \end{aligned}$$

This problem can be seen as a parabolic problem in (an affine coset of) the space of divergence free functions with a Lagrange multiplier p and two non-homogeneous conditions: $\mathrm{div} u = g$ and $u|_{\partial\Omega} = h$.
If one applies standard DG in time sub-optimal results are obtained (cf. Table below). We present an analysis which explains the cause of this sub-optimal behavior. Based on this analysis we introduce a modification which leads to an optimal convergence order, non only for the energy norm of u, but also for the L^2 norm of the Lagrange multiplier p. Furthermore, an optimal nodal superconvergence result for u is obtained.
Our theoretical results are confirmed by numerical results. In the table below one can see that the temporal convergence order for the Lagrange multiplier is 1 for the standard method (SM) and 2 for our modified method (MM). In this experiment we used a $\mathcal{P}_2 - \mathcal{P}_1$ Taylor-Hood pair in space and linear functions in time. For the modified method we see that the spatial error dominates after a few temporal refinements (N_T).

N_T	4	8	16	32	64	128
SM	1.36231	0.73455	0.37695	0.19011	0.09513	0.04755
EOC_T		0.89112	0.96248	0.98752	0.99894	1.00042
MM	0.30828	0.07813	0.01984	0.00589	0.00286	0.00261
EOC_T		1.98035	1.97707	1.75344	1.04000	0.13472

Error in L^2-norm between exact p and the solution of the discrete problem.

References:

[1] V. Thomee, Galerkin Finite Element Methods for Parabolic Problems (Springer Series in Computational Mathematics). Springer-Verlag New York, Inc., 2006.
[2] S. Hussain, F. Schieweck, and S. Turek, “A note on accurate and efficient higher order Galerkin time stepping schemes for the nonstationary Stokes equations,” The Open Numerical Methods Journal, vol. 4, pp. 35–45, 2012.
[3] D. Schötzau and C. Schwab, “Time Discretization of Parabolic Problems by the hp-Version of the Discontinuous Galerkin Finite Element Method,” SIAM Journal on Numerical Analysis, vol. 38, no. 3, pp. 837–875, 2000.

[1] RWTH-Aachen University, Institut fuer Geometrie und Praktische Mathematik, Aachen, Germany, `voulis@igpm.rwth-aachen.de`
[2] Institut fuer Geometrie und Praktische Mathematik, RWTH Aachen, `reusken@igpm.rwth-aachen.de`