A FEM Approach for a Surface Navier-Stokes Equation on Manifolds with Arbitrary Genus

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We consider a compact smooth Riemannian surface S without boundary and an incompressible surface Navier-Stokes equation

$$\begin{aligned} \partial_t v + \nabla_v v &= -\nabla_S p + \frac{1}{\mathrm{Re}} \left(-\Delta_S^{dR} v + 2\kappa v \right) \\ \nabla_S \cdot p &= 0 \end{aligned}$$

in $S \times (0, \infty)$ with initial condition $v\,(x, t = 0) = v_0(x) \in T_x S$. Thereby $v(x,t) = (v_1, v_2) \in TS$ denotes the tangential surface velocity, $p(x,t) \in \mathbb{R}$ the surface pressure, Re the surface Reynolds number, κ the Gaussian curvature, $T_x\S$ the tangent space on $x \in S$, $TS = \cup_{s \in S} T_x S$ the tangent bundle and $\nabla_v, \nabla_S \cdot$ and Δ_S^{dR} the covariant directional derivative, surface divergence and surface Laplace-DeRham operator, respectively.

As in flat space the equation results from conservation of mass and (tangential) linear momentum. However, differences are found in the appearing operators and the additional term including the Gaussian curvature. The unusual sign results from the definition of the Laplave-DeRham operator. While a huge literature exists for the two-dimensional Navier-Stokes equation in flat space, results for its surface counterpart are rare. We introduce a surface finite element approach which is also esirable for surfaces with genus $g(S \neq 0$, as it allows to deal with harmonic vector fields and demonstrate the dependency of the solution on the topology of the surface on various examples.

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