A Mixed Discretization of Elliptic Problems on Anisotropic Hybrid Meshes

Ariel Lombardi[1] Alexis B. Jawtuschenko[2]

In this talk we consider the approximation in mixed form of elliptic problems in polyhedra. It is known that when the polyhedral domain is concave along edges or vertices, singularities may appear in the solution which degrade the numerical approximations. For the finite element method (FEM), strategies have been proposed in order to recover the optimal order of convergence, one of them being the use of meshes which are a priori adapted to the singularities. These meshes contain, in general, arbitrarily narrow elements, and as a consequence, the FEM tools to prove convergence have to manage this kind of elements. When mixed finite elements are considered, it is common the use of the $H(\text{div})$-conforming Raviart-Thomas spaces on tetrahedral meshes to approximate the vectorial variable, and then, interpolation error estimates for the Raviart-Thomas interpolation operator are one of the main tools to analyse the approximation error. In order to prove optimal convergence of the mixed method, anisotropic interpolation error estimates like

$$\|\mathbf{u}-\Pi_0\mathbf{u}\|_{L^2(T)} \leq C(\bar{c}) \left(\sum_{i=1}^{3} h_i \|\partial_{x_i}\mathbf{u}\|_{L^2(T)} + h_T \|\text{div}\,\mathbf{u}\|_{L^2(T)} \right), \tag{1}$$

are needed. Here, Π_0 is the Raviart-Thomas interpolation operator of lowest order [Nedelec, Raviart-Thomas], h_T is the diameter of T and h_i is the diameter of T in the x_i-direction. This estimate was proved in [Acosta et al.] for a tetrahedron T with the constant C depending on the regular vertex property of T. When meshes contain arbitrarily anisotropic tetrahedra (this happens when the solution exhibits edge singularities) the constant C becomes arbitrarily large for some elements (known as slivers), as can be deduced from the results in [Acosta et al.]. As a consequence, optimal error estimates can not be obtained for this kind of approximation.

For the simplest elliptic problem for the Laplace operator in polyhedra, with the aim to avoid the presence of slivers in anisotropic meshes, we propose a generalization of the standard mixed method mentioned before, which, in particular, allows for the use of hybrid meshes made up of triangularly right prisms, tetrahedra and pyramids. The meshes can contain arbitrarily anisotropic right prisms in order to deal with edge singularities, and isotropic tetrahedra to be able to consider general polyhedral domains. And (isotropic) pyramids are needed in order to glue right prisms and tetrahedra.

For such a kind of meshes we introduce and analyse a mixed Finite/Virtual Element Method [Brezzi et al.]. The local discrete spaces coincide with the lowest order Raviart-Thomas spaces (and its extensions [Nedelec]) on tetrahedral and triangularly right prismatic elements, and extend it to pyramidal elements. The local vectorial space on one

[1] Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Departamento de Matemática, Rosario, Argentina,
ariel@fceia.unr.edu.ar
[2] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires,
ajawtu@dm.uba.ar

element E of any shape is defined as

$$V_h(E) = \Big\{\mathbf{v} \in H(\text{div}\,, E) \cap H(\textbf{curl}\,, E) : \mathbf{v}\cdot\mathbf{n} \in \mathcal{P}_0(f)\ \forall f \text{ face of } E, \text{div}\,\mathbf{v} \in \mathcal{P}_0(E), \textbf{curl}\,\mathbf{v} = 0\Big\}$$

where $\mathbf{n}$ denotes the exterior normal to E, and $\mathcal{P}_0(S)$ the space of constant functions on S. As in the virtual element technology, the stiffness matrix can be computed from the known degrees of freedom of the shape functions. The discrete scheme is well posed and optimal error estimates are proved on meshes which allow for anisotropic elements. In particular, local interpolation error estimates for the virtual space are optimal and anisotropic on anisotropic right prisms, which can be used to obtain optimal approximation error estimates when the solution has edge or vertex singularities when suitably adapted meshes are used.

References:

[1] G. Acosta, Th. Apel, R.G. Durán, A.L. Lombardi. Error estimates for Raviart-Thomas interpolation of any order on anisotropic tetrahedra. Math. Comp. 80#273 (2011) 141–163.

[2] F. Brezzi, R.S. Falk, L.D. Marini. Basic principles of virtual element methods. Math. Model. Numer. Anal. 48 (2014), 1227–1240.

[3] J.C. Nédéléc. Mixed finite elements in R^3. Numer. Math. 35 (1980) 315–341.

[4] P.A. Raviart, J.-M. Thomas. A mixed finite element method for second order elliptic problems, in *Mathematical Aspects of the Finite Element Method*, I. Galligani, E. Magenes, eds. Lectures Notes in Math. 606. Springer–Verlag 1977.