A-posteriori Error Control for Stationary Coupled Bulk-Surface Equations

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We consider a system of two coupled elliptic equations, one defined on a bulk domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ with piecewise smooth boundary and the second one on the boundary surface $\Gamma := \partial\Omega$ that is decomposed into a finite set of patches $\{\Gamma^i\}_{i=1}^N$. Such coupled reaction diffusion processes in the bulk and on the surface have recently attracted interest from an analytical point of view [2] and in different application areas such as biology and chemistry, see e.g. [3]. We seek the solution $u : \Omega \to \mathbb{R}$ and $v : \Gamma \to \mathbb{R}$ of the stationary coupled diffusion-reaction problem

$$-\Delta u + u = f \qquad \text{in } \Omega, \tag{1a}$$
$$(\alpha u - \beta v) + \partial_n u = 0 \qquad \text{on } \Gamma, \tag{1b}$$
$$-\underline{\Delta}_\Gamma v + v + \partial_n u = g \qquad \text{on } \Gamma, \tag{1c}$$
$$\underline{\nabla}_{\Gamma^i} v \cdot n^i + \underline{\nabla}_{\Gamma^j} v \cdot n^j = 0 \qquad \text{on } \partial\Gamma^i \cap \partial\Gamma^j\ . \tag{1d}$$

A-priori analysis for domains with smooth boundary has been established in [2] and a-posteriori analysis for a pure surface problem can be found in [1]. Here, a-posteriori error control is proved. We derive a fully computable residual estimator that takes into account the approximation errors due to discretization with lowest order continuous finite elements in space as well as errors due to polyhedral approximation of the surface. An adaptive refinement algorithm is described which controls the overall error. Numerical experiments illustrate the performance of the a-posteriori error estimator and the proposed adaptive algorithm.

References:

[1] A. Bonito, J. M. Cascón, P. Morin, and R. H. Nochetto. *AFEM for geometric PDE: the Laplace-Beltrami operator.* In Analysis and Numerics of Partial Differential Equations, pp. 257–306. Springer, 2013.
[2] C.M. Elliott and T. Ranner. *Finite element analysis for a coupled bulk–surface partial differential equation.* IMA J. Num. Anal., 33(2):377–402, 2013.
[3] I. L. Novak, F. Gao, Y.-S. Choi, D. Resasco, James C. Schaff, and B. M. Slepchenko. *Diffusion on a curved surface coupled to diffusion in the volume: Application to cell biology.* JCP, 226(2):1271–1290, 2007.

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