Optimal control subject to singularly perturbed convection-diffusion-equations

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We consider the optimal control problem

$$\min_{y,u} J(y,u) := \min_{y,u} \left(\frac{1}{2} \|y - y_0\|_0^2 + \frac{\lambda}{2} \|u\|_0^2 \right)$$

subject to a convection dominated differential state equation

$$Ly := -\varepsilon y'' + ay' + by = f + u \text{ in } (0,1),$$
$$y(0) = y(1) = 0.$$

The solutions of such singularly perturbed differential equations typically exhibit boundary layers. The optimality condition leads to the enhanced system of the state equation and its adjoint form. The change of sign of the convection term in the adjoint equation induces a boundary layer of the adjoint state at the opposite side of the domain compared to the primal state. By the coupling of the two differential equations via the optimality system those layers lead to additional boundary layers of a weaker form in the other part of the solution. Our analysis shows the layer structure of the solution of such an optimality system.

Using linear finite elements on adapted grids of Shishkin type we treat the effects of the arising layers at the boundaries of the domain. Furthermore we proof uniform error estimates with respect to the perturbation parameter ϵ. We show that the weak boundary layers also have an impact on the quality of numerical algorithms for solving the optimality system. Numerical results supporting our analysis are presented.

Moreover, we discuss extensions to the case of box constraints for the control u.

References:

[1] T. Linss and M. Stynes, *Numerical Solution of Systems of Singularly Perturbed Differential Equations,* Computational Methods in Applied Mathematics, 9 (2009), pp. 165–191

[2] H.-G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations, 2nd Edition,* Springer-Verlag, 2008

[3] F. Tröltzsch, *Optimal Control of Partial Differential Equations,* American Mathematical Society, 2010

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