Numerical analysis of a quasilinear Neumann equation under minimal regularity of the data

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We consider the finite element approximation of the following quasilinear Neumann problem

$$\begin{cases} -\operatorname{div}\left[a(x,u(x))\nabla u(x)\right] + f(x,u(x)) = 0 & \text{in } \Omega\,, \\ a(x,u(x))\partial_\nu u(x) = g(x) & \text{on } \Gamma\,. \end{cases} \tag{1}$$

The consideration is restricted to polygonal domains of dimension two and polyhedral domains of dimension three. For numerical purposes we discretize Ω by a regular triangulation and the mesh on Γ is induced by that on Ω. We approximate the PDE (1) by linear and continuous finite elements.

In spite that f is considered monotone nondecreasing with respect to u, the above equation is not monotone because of the dependence of the coefficients of the operator with respect to the function u. This lack of monotonicity causes the main difficulty in the numerical analysis of the above equation: the uniqueness of a solution of the discretized version of (1) is an open question, even though (1) has a unique solution. This non-monotonicity also complicates the discussion of any linearization of (1) which is needed, in particular, to apply the well-known Aubin-Nitsche trick for optimal L^2 error estimates for the approximation of (1).

In contrast to well-known results in the literature, the C^2 regularity and the boundedness of the coefficient a are not necessary to derive error estimates. In most of the cases it suffices to require only a local Lipschitz property of a.

The boundary datum g is supposed to be in $L^s(\Gamma)$ $(s \geq 2)$, therefore the regularity of the solution u of (1) cannot be $H^2(\Omega)$ but only $H^{3/2}(\Omega)$. Because of this lower regularity of u, the analysis becomes more difficult than in the regular case.

The assumption on Ω introduces a new difficulty: the regularity of elliptic equations in corner domains needs special care. For this reason, we distinguish two cases: whether Ω is convex or not. These two different situations yield different order of convergence for the discretization of equation (1) in the $L^2(\Omega)$ norm.

The main aims of this talk are twofold. First, error estimates in different function spaces are presented and second the issue of uniqueness of a solution to the approximate discrete equation is addressed.

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