Mixed finite elements for linear elasticity

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In this talk, we derive a mixed variational formulation for the equations of linear elasticity

$$\begin{aligned} -\mathrm{div}(\sigma) &= f \\ A\sigma &= \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T). \end{aligned}$$

Here u denotes the displacement field, and σ the stress tensor.

We choose $u \in H(\mathrm{curl})$, which implies the following continuity condition for the stresses

$$\mathrm{div}\sigma \in H(\mathrm{curl})^* = H^{-1}(\mathrm{div}).$$

We refer to this space as $H(\mathrm{div}\,\mathrm{div})$. We see that the variational problem is well posed.

We discretise the problem using Nédélec finite elements for the displacements. For σ, we propose a new family of symmetric, tensor-valued finite elements. The normal-normal component $\sigma_{nn} = n^T\sigma n$ is continuos across interfaces, therefore we see that they are conforming for the space $H(\mathrm{div}\,\mathrm{div})$. We present stable elements of arbitrary order.

These elements are suitable for the discretization of beams or shells, as they do not suffer from shear locking. We see that the optimal order error estimates for the solution still hold true in the case of thin structures. Also, there occurs no volume locking, when the material gets nearly incompressible, i.e. the Poisson ratio ν approaches 1/2.

The mixed formulation corresponds to a saddle point problem. In order to obtain a positive definite matrix, we hybridize the stress space. This means leave the stress-space noncontinuous, and introduce Lagrange parameters which guarantee the required normal-normal continuity. We use these additional variables to find a preconditioner that works for nearly incompressible materials.

We give numerical results, including the cases of nearly incompressible materials and thin structures.

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