Finite Element Techniques for Two-Phase Incompressible Flows

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We consider a domain $\Omega \subset \mathbb{R}^3$ which contains two different immiscible incompressible newtonian phases (fluid-fluid or fluid-gas). The time-dependent domains which contain the phases are denoted by $\Omega_1 = \Omega_1(t)$ and $\Omega_2 = \Omega_2(t)$. The interface between the two phases $(\partial\Omega_1 \cap \partial\Omega_2)$ is denoted by $\Gamma = \Gamma(t)$. To model the forces at the interface we make the standard assumption that the surface tension balances the jump of the normal stress on the interface, i.e., we have a free boundary condition

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = \tau\kappa\mathbf{n} \ ,$$

with $\mathbf{n} = \mathbf{n}_\Gamma$ the unit normal at the interface, τ the surface tension coefficient (material parameter), κ the curvature of Γ and $\boldsymbol{\sigma}$ the stress tensor. We assume continuity of the velocity across the interface. In combination with the conservation laws of mass and momentum this yields the following standard model:

$$\begin{cases} \rho_i\big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) = -\nabla p + \rho_i \mathbf{g} + \mathrm{div}(\mu_i \mathbf{D}(\mathbf{u})) & \text{in } \ \Omega_i \\ \mathrm{div}\,\mathbf{u} = 0 & \text{in } \ \Omega_i \end{cases} \quad \text{for } i = 1,2$$

$$[\boldsymbol{\sigma}\mathbf{n}]_\Gamma = \tau\kappa\mathbf{n}, \quad [\mathbf{u}\cdot\mathbf{n}]_\Gamma = 0 \ .$$

The vector $\mathbf{g}$ is a known external force (gravity). In addition we need initial conditions for $\mathbf{u}(x,0)$ and boundary conditions at $\partial\Omega$. For simplicity we assume homogeneous Dirichlet boundary conditions.

In this talk we present an overview of a solver that has been developed and implemented in our group. Important characteristics of the method are the following. For capturing the interface between the two phases the level set method is applied. The spatial discretization is based on a stable hierarchy of consistent tetrahedral grids. For discretization of velocity, pressure and the level set function we use conforming finite elements. For the pressure variable an extended linear finite element space (XFEM) is used which allows an accurate approximation of the pressure discontinuity across the interface. For the treatment of the surface tension force a special Laplace-Beltrami method has been developed. The time discretization is based on a variant of the fractional step θ-scheme. For solving the linearized discrete problems we use inexact Uzawa techniques and Krylov subspace methods combined with multigrid preconditioners. We apply a variant of the Fast Marching method for the reparametrization of the level set function.

We will discuss certain aspects of our solver in more detail. Results of numerical experiments for a three dimensional instationary two-phase fluid-fluid flow problem are presented.

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