FEM for problems with piezoelectric material

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Linear electro mechanics
  Field quantities and equations
  Material laws, restrictions
  Variational formulation

FEM-handling
  Linear system
  Solver
  Elements
  Error estimating

Numerical examples
  Test examples
  Crack example
  Adaptive mesh generation
  Iteration numbers

Open questions
Linear electro mechanics

- Coupling of deformation and electric field
- Interaction between mechanical and electrical quantities via piezo effects

Example:

\[ \Omega \]

\[ \Gamma_{N,u} \quad \Gamma_{N,\varphi} \]

\[ \Gamma_N \text{ homogenous} \] („free boundary“)

\[ \Gamma_D \]

(clamped, conductive)
Field quantities

\[ u_i \]
\[ \varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \]
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]
\[ \varphi \]
\[ E_k = -\partial_k \varphi \]
\[ D_i = \kappa_{ij} E_j \]
Field quantities

\[ \varepsilon_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \]

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} - e_{mij} E_m \]

\[ D_i = \kappa_{ij} E_j + e_{ikl} \varepsilon_{kl} \]

inverse piezo effect (actor)

piezo effect (sensor)
### Balance equations in $\Omega$, boundary conditions

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{div} \sigma + \vec{b} = 0$ in $\Omega$</td>
<td>Volume forces $b_i$</td>
</tr>
<tr>
<td>$u = u_0$ on $\Gamma_{D,u}$</td>
<td></td>
</tr>
<tr>
<td>$\sigma \cdot \vec{n} = \overline{T}$ on $\Gamma_{N,u}$</td>
<td>Boundary stresses $\overline{T}_i$</td>
</tr>
<tr>
<td>$\text{div} D - \omega_v = 0$ in $\Omega$</td>
<td>Volume charges $\omega_v$</td>
</tr>
<tr>
<td>$\varphi = \varphi_0$ on $\Gamma_{D,\varphi}$</td>
<td></td>
</tr>
<tr>
<td>$D \cdot \vec{n} = -\overline{\omega}<em>s$ on $\Gamma</em>{N,\varphi}$</td>
<td>Boundary charges $\overline{\omega}_s$</td>
</tr>
</tbody>
</table>
Material laws, restrictions

- No case of full isotropic material here,
- Restriction to transversal isotropic material behaviour,
- The poling direction be: $x_2$,
- Reducing the problem to 2 dimensions: all independent of $x_3$
Material laws, restrictions

Reduction to 2D as generalization of plain strain:

\begin{align*}
u_3 &= 0, \quad \varepsilon_{i3} = 0, \quad E_3 = 0 \\
\sigma_{13} &= \sigma_{23} = 0, \quad D_3 = 0
\end{align*}

(\(\sigma_{33}\) derivable from other quantities, but not part of other equations)

Use vector notation:

\[
\sigma = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}
\]
Material laws, restrictions

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_{12} \\
D_1 \\
D_2
\end{bmatrix} = 
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12} \\
D_1 \\
D_2
\end{bmatrix} = 
\begin{bmatrix}
C & B \\
B^T & -K
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12} \\
\varphi,1 \\
\varphi,2
\end{bmatrix}
\]

with \( x_2 \) poling direction and

\[
C = \begin{pmatrix}
c_{11} & c_{12} & 0 \\
c_{12} & c_{22} & 0 \\
0 & 0 & c_{33}
\end{pmatrix},
B = \begin{pmatrix}
0 & e_{12} \\
0 & e_{22} \\
e_{31} & 0
\end{pmatrix},
K = \begin{pmatrix}
k_{11} & 0 \\
0 & k_{22}
\end{pmatrix}.
\]
Variational formulation

\begin{align*}
c(u, v) + b(\varphi, v) &= F(v) \quad \forall v \in \mathcal{V} \quad (1) \\
b(\psi, u) - k(\psi, \varphi) &= -G(\psi) \quad \forall \psi \in \mathcal{Q} \quad (2)
\end{align*}

with bilinear forms
\begin{align*}c(u, v) &= \int_{\Omega} \varepsilon^T(u) C \varepsilon(v) \, d\Omega, \\
b(\psi, v) &= \int_{\Omega} \varepsilon^T(v) B \nabla \psi \, d\Omega, \\
k(\varphi, \psi) &= \int_{\Omega} \nabla^T \varphi K \nabla \psi \, d\Omega
\end{align*}

and linear forms
\begin{align*}F(v) &= \int_{\Omega} b \cdot v \, d\Omega + \int_{\Gamma_{N,u}} T \cdot v \, d\Gamma, \\
G(\psi) &= \int_{\Omega} \omega_v \psi \, d\Omega + \int_{\Gamma_{N,\varphi}} \omega_s \psi \, d\Gamma.
\end{align*}

Here, \( \mathcal{V} = (H^1(\Omega))^d \), \( \mathcal{Q} = H^1(\Omega) \) with Dirichlet-cond.
FEM-handling: Linear system

FEM with $V_h = \text{span}\{\phi_i\} \subset \mathcal{V}$, $Q_h = \text{span}\{\rho_j\} \subset \mathcal{Q}$ of finite dimension leads to block structured system:

$$
\begin{bmatrix}
C & B \\
B^T & -K
\end{bmatrix}
\begin{bmatrix}
u \\
\varphi
\end{bmatrix}
= 
\begin{bmatrix}
f \\
g
\end{bmatrix}
$$

with

$$
\begin{cases}
C = (c(\phi_i, \phi_j)) \\
B = (b(\rho_i, \phi_j)) \\
K = (k(\rho_i, \rho_j))
\end{cases}
$$

(3)

Properties:

symmetric, but not positiv definite

$C$ and $K$ are s.p.d. (from ellipticity of $c(.,.)$, $k(.,.)$),

can use Bramble-Pasciak-CG
**Solver: Variant of Bramble-Pasciak-CG**

- Choose preconditioner $C_0$ for $C$, choose scalar $\gamma > 0$ s.t. $C - \gamma C_0$ positive definite, choose preconditioner $B_0$ for the Schur-complement $S = B^\top C_0^{-1} B + \gamma K$ and scalar $\delta > 0$

\[
\mathcal{A} = \begin{pmatrix}
    C_0^{-1} & 0 \\
    \delta B_0^{-1} B^\top C_0^{-1} & -\gamma \delta B_0^{-1}
\end{pmatrix}
\begin{pmatrix}
    C & B \\
    B^\top & -K
\end{pmatrix}
\]

and a matching scalar product

\[
\langle x, y \rangle := ((C - \gamma C_0)\overline{x}, \overline{y}) + (\delta^{-1} B_0 \overline{x}, y) \quad \text{with} \quad x = \begin{pmatrix}
    \overline{x} \\
    \overline{x}
\end{pmatrix}
\]

$\implies \mathcal{A}$ symmetric positive definite w.r.t. $\langle \cdot, \cdot \rangle$

(Bramble/Pasciak 88, generalized: Meyer/Steidten 01)
FEM-handling: Elements

System structure similar to Stokes or mixed elasticity formulations
⇒ inf-sup-condition for elements?
FEM-handling: Elements

System structure similar to Stokes or mixed elasticity formulations
⇒ inf-sup-condition for elements?
An equivalent notation:

\[
\begin{bmatrix}
C & B \\
-B^T & K
\end{bmatrix}
\begin{pmatrix}
u \\
\varphi
\end{pmatrix}
=
\begin{pmatrix}
f \\
g
\end{pmatrix}
\]

so,

\[
\begin{pmatrix}
v \\
\psi
\end{pmatrix}^T
A
\begin{pmatrix}
v \\
\psi
\end{pmatrix}
=
v^T C v + \psi^T K \psi = c(v, v) + k(\psi, \psi)
\]

uniform elliptic

Canonical ansatz: same elements for \(u_i\) and \(\varphi\)
FEM-handling: Error estimating

Idea: Generalisation of the residual-type error estimators for elasticity problems

- ∀ edges $E$ derive edge jumps

$$R_{E,\sigma} = \left\lfloor \sigma(u_h, \varphi_h) \cdot n \right\rfloor_E$$

and

$$R_{E,D} = \left\lfloor D(u_h, \varphi_h) \cdot n \right\rfloor_E$$

- Look on $r_{E,\sigma} = |E| \int_E |R_{E,\sigma}|^2$, $r_{E,D} = |E| \int_E R_{E,D}^2$

- Mark $E$ for Coarsening when $r_{E,\sigma}$ and $r_{E,D}$ small

- Mark $E$ for Refining when $r_{E,\sigma}$ or $r_{E,D}$ relative large
Numerical examples

Experimental program SPC-PM-2Adpiez for linear piezoelectric problems in the 2D transversal isotropic case

- reduced biquadratic or bilinear quadrilateral elements, quadratic or linear triangle elements
- hierarchical preconditioner (Yserentant) – effective in 2D
- adaptive mesh refinement
- contact with obstacle is possible
Test examples: 1. Mini benchmark

\[ \sigma_{yy} = 10^3 \frac{N}{m^2}, \sigma_{xy} = 0, D_y = 1 \frac{As}{m^2} \]

slip b.c. for \( u \), Dirichlet b.c. \( \varphi = 0 \)

PZT4
Test examples

1. Mini benchmark

![Graph showing test_FG - (8 nodes) with U_x, U_y, and P values.](image)
Test examples: 2. Contact included

\[ \sigma_{yy} = 0, \sigma_{xy} = 0, D_y = -1 \frac{As}{m^2} \]

slip b.c. for \( u \), Dirichlet b.c. \( \phi = 0 \)

PZT4

slip b.c. for \( u \leftrightarrow \), Dirichlet b.c. \( \phi = 0 \)

Obstacle
Test examples

2. Contact example: displacement $u_x, u_y$

![Contact example](image_url)
Test examples

2. Contact example: electric potential and adapt. mesh

conttest - (3314 nodes)
2. Contact example: stress $\sigma_{xx}$ and $\sigma_{yy}$
Crack example: Griffith–crack in an infinite medium

\[ \sigma_{yy}^\infty = \text{const.}, \quad D_y^\infty = \text{const.}, \quad \sigma_{xy}^\infty = 0 \]

\[ \sigma_{yy} = \sigma_{xy} = D_y = 0 \quad \text{on } (-a, a) \times \{0\} \]
Crack example: opening/closing

- Exclude simple linear scaling
- So, all depends on the ratio $\lambda = \frac{D_y^\infty}{\sigma_y^\infty} \cdot 10^{-10} \frac{m}{V}$
- Possible crack opening or closing
- Non-physical overlapping not allowed

Known from analysis:
Crack closing for $\lambda < -7.9$, otherwise exact solution is given
Self-penetration effects for $\lambda > 102.5$
Crack example: opening/closing

Zoom in deformed net for crack opening ($\lambda = 102.5$)

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crac4 - (3379 nodes)
Crack example: opening/closing

Zoom in deformed net for closing without contact ($\lambda = -100$)

cracneg - (3021 nodes)
Crack example: opening/closing

Zoom in deformed net for closing with contact ($\lambda = -100$)

cracnegC - (3937 nodes)
Crack example – phenomena with strong $\lambda$

Isolines of $u_x$ and $u_y$ in domain ($\lambda = 1025$)
Crack example

Isolinines of $\sigma_{11}$ and $\sigma_{22}$ in domain ($\lambda = 1025$)
Crack example

Zoom of $\sigma_{22}$ and $D_y$ in $[0, 1.5] \times [0, 1]$ ($\lambda = 1025$)
Crack example

mech. and electric part of $\sigma_{22}$ zoomed in $[0, 1.5] \times [0, 1]$
Instability

Look on material law:

\[ \sigma = C : \varepsilon - B \cdot E \]

\[ \sigma_{\text{mech}}^{22} - B \cdot E \]

\[ \sigma_{\text{el}}^{22} \]

Approximated mechanical and electrical part in some points:

<table>
<thead>
<tr>
<th>(x, y)</th>
<th>(\sigma_{22}^{\text{mech}})</th>
<th>(\sigma_{22}^{\text{el}})</th>
<th>(\frac{\sigma_{22}^{\text{mech}}}{\sigma_{22}^{\text{el}}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.0703,0)</td>
<td>3.97589 E+8</td>
<td>-3.91575 E+8</td>
<td>0.015</td>
</tr>
<tr>
<td>(1.0703,0.25)</td>
<td>1.82728 E+8</td>
<td>-1.77676 E+8</td>
<td>0.027</td>
</tr>
<tr>
<td>(0.9375, 0.0625)</td>
<td>1.42623 E+8</td>
<td>-1.44335 E+8</td>
<td>0.016</td>
</tr>
</tbody>
</table>

So, about 6 binary digits are erased in computing \(\sigma\)
### Instability–2

Approximated ratio \(\frac{\sigma_{22}}{\sigma_{22}^{\text{mech}}}\) in some points \((x, y)\) for different \(\lambda\):

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(P_1) ((1.0703, 0))</th>
<th>(P_2) ((1.0703, 0.25))</th>
<th>(P_3) ((0.9375, 0.0625))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1025</td>
<td>0.015</td>
<td>0.027</td>
<td>0.016</td>
</tr>
<tr>
<td>102.5</td>
<td>0.07</td>
<td>0.11</td>
<td>0.025</td>
</tr>
<tr>
<td>10.25</td>
<td>0.44</td>
<td>0.57</td>
<td>0.37</td>
</tr>
<tr>
<td>4.25</td>
<td>0.69</td>
<td>0.88</td>
<td>0.66</td>
</tr>
</tbody>
</table>

\(\Downarrow\)

**Decreasing instability of \(\sigma\) for smaller \(\lambda\)**
Crack example—stress at some lines

\( \sigma_{22} \) with \( \lambda = 1025 \), along \( y = 0 \) (on ligament)
Crack example—stress at some lines

\[ \sigma_{22} \text{ with } \lambda = 102.5, \text{ along } y = 0 \]
Crack example— stress at some lines

\[ \sigma_{22} \quad \text{with } \lambda = 10.25, \text{ along } y = 0 \]
Crack example– stress at some lines

\[ \sigma_{22} \quad \text{with} \quad \lambda = 1025, \text{along} \quad x = 1 \quad (\text{orthogonal to the crack tip}) \]
Crack example—stress at some lines

\[ \sigma_{22} \] with \( \lambda = 102.5 \), along \( x = 1 \)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)

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uriss - (224 nodes)
Adaptive mesh generation (Crack example)

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uriss - (248 nodes)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)

uriss - (633 nodes)

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Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)

![Adaptive mesh generation diagram](image_url)

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uriss - (938 nodes)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)

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uriss - (1458 nodes)
Adaptive mesh generation (Crack example)
Adaptive mesh generation (Crack example)

uriss - (2017 nodes)
Iteration numbers– Contact example
Iteration numbers– Crack example ($\lambda = 102.5$)
### Iteration numbers – Crack example ($\lambda = 1025$)

<table>
<thead>
<tr>
<th>nodes</th>
<th>Iter (quadrat.)</th>
<th>nodes</th>
<th>Iter (linear)</th>
</tr>
</thead>
<tbody>
<tr>
<td>118</td>
<td>211</td>
<td>118</td>
<td>188</td>
</tr>
<tr>
<td>200</td>
<td>181</td>
<td>200</td>
<td>153</td>
</tr>
<tr>
<td>248</td>
<td>172</td>
<td>269</td>
<td>187</td>
</tr>
<tr>
<td>332</td>
<td>177</td>
<td>315</td>
<td>150</td>
</tr>
<tr>
<td>436</td>
<td>195</td>
<td>584</td>
<td>198</td>
</tr>
<tr>
<td>633</td>
<td>176</td>
<td>830</td>
<td>193</td>
</tr>
<tr>
<td>1210</td>
<td>182</td>
<td>1126</td>
<td>184</td>
</tr>
<tr>
<td>2017</td>
<td>195</td>
<td>2117</td>
<td>204</td>
</tr>
<tr>
<td>3569</td>
<td>215</td>
<td>3903</td>
<td>206</td>
</tr>
<tr>
<td>5258</td>
<td>228</td>
<td>5339</td>
<td>229</td>
</tr>
<tr>
<td>6460</td>
<td>204</td>
<td>7132</td>
<td>225</td>
</tr>
<tr>
<td>7454</td>
<td>201</td>
<td>9410</td>
<td>218</td>
</tr>
</tbody>
</table>
estimated error ($\sigma$-part) – crack example
Open questions

Future work:

- Stable calculation of Jacobians by using edge-vectors
- Sufficiently stable calculation of stresses near the crack tip
- Combination with crack growth
- Comparison with other solver techniques
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Thanks for attention!
Danke für Ihre Aufmerksamkeit!