Discrete curvatures and optimal quasi-isometric manifold parameterizations

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Optimal grid generation and adaptation can be considered in the framework of manifold parameterization.

Consider the problem of untangling and optimization of 2D unstructured grids via node movement. The basic idea is to have unified treatment for any spatial dimensions and for different types of elements. The concept of cell shape/size optimization is quite clear. For each grid cell the mapping of canonical target cell onto real cell is considered. Certain distortion measure of this mapping is minimized using variational methods.

One can construct polyhedral manifold by gluing all target elements together (generally it cannot be embedded as a surface in 3D space). Grid optimization in 2D is just a flattening of this manifold: one-to-one mapping of this manifold onto plane with minimal distortion. For general non-simplicial grids the result of gluing is not polyhedral manifold, but it is still Alexandrov manifold (manifold of bounded curvature) [4]. The "optimal" parameterization is the one which provides minimal distortion of the flattening.

The problem of adaptation of hybrid/unstructured grids is equivalent to the problem of finding bilipschitz mapping of above polyhedral manifold onto another manifold where (intrinsic) adaptation metric is defined.

Problem of existence of bilipschitz mappings between Alexandrov manifolds is considered in [4], [7]. The distortion of parameterization is estimated via positive and negative instrinsic curvatures of manifold [4], [7]. The case of arbitrary genus is considered in recent paper by Yu.D. Burago [6].

In [3] it was conjectured that instead of positive curvature parameterization distortion should be estimated via certain constant from isoparametric inequalities - called "depth of pockets" in [3]. Numerical experiments and grid refinement studies results are in good agreement with suggested estimates.

In 3D the numerical treatment is similar to 2D, but theoretical understanding is still lacking. The polyhedral manifolds are defined similarly by gluing simplices, but theoretical results about existence of global bilipschitz parameterizations are not available. The concept of discrete curvatures in multidimensional case is still not very well defined. In practice manifolds are constructed by gluing together target hexahedra (as a rule chosen as cubes), prisms, pyramids and simplices.

Within above concept there is no difference between structured (block structured) mapped grids and unstructured/hybrid grids. As a result of this unified treatment the same functional [2], [3] satisfying polyconvexity conditions [1] is used for any dimensions and any types of elements [5]. The minimization problem for this functional is well posed. Recently the rigorous convergence results for the iterative minimization method were obtained [8].

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References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Rational Mech. Anal. 63, 1977, 337-403.

[2] Garanzha VA. Barrier variational generation of quasi-isometric grids. *Computational Mathematics and Mathematical Physics* 2000; **40**(11):1617–1637.

[3] V.A. Garanzha. Variational principles in grid generation and geometric modeling: theoretical justifications and open problems. *Numerical Linear Algebra with Applications* 2004; **9**(6-7).

[4] Yu.G. Reshetnyak: Two-Dimensional Manifolds of Bounded Curvature. In: Reshetnyak Yu.(ed.), Geometry IV (Non-regular Riemannian Geometry), pp. 3-165. Springer-Verlag, Berlin (1991).

[5] L.V. Branets and V.A. Garanzha Distortion measure for trilinear mapping. Application to 3-D grid generation. *Numerical Linear Algebra with Applications* 2002; **9**(6-7).

[6] A. Belenkiy, Yu. Burago. Bi-Lipschitz equivalent Alexandrov surfaces, I, arXive:math. DG/0409340. 2003.

[7] M. Bonk and U. Lang. Bi-Lipschitz parameterization of surfaces. Mathematische Annalen, 2003, DOI 10.1007/s00208-003-0443-8.

[8] V.A. Garanzha and I.E. Kaporin. On the convergence of a gradient method for the minimization of functionals in finite deformation elasticity and for the minimization of barrier grid functionals. *Comp. Math. and Math. Phys.* 2005, **45**(8):1400–1415.