Superconvergence analysis of Galerkin and Streamline Diffusion FEM for a singularly perturbed convection-diffusion problem with characteristic layers

Sebastian Franz[1] Torsten Linß[2]

For the model convection-diffusion equation

$$Lu := -\varepsilon \Delta u - bu_x + cu = f \text{ in } \Omega = (0,1)^2,$$

subject to Dirichlet boundary conditions

$$u = 0 \text{ on } \Gamma = \partial\Omega$$

with $b \geq \beta > 0$, we analyse the superconvergence properties of the Galerkin FEM and of the streamline-diffusion finite element method (SDFEM) using bilinear functions. The presence of the small perturbation parameter ε with $0 < \varepsilon \ll 1$ gives rise to an exponential layer of width $\mathcal{O}(\varepsilon)$ near the outflow boundary at $x = 0$ and to two parabolic layers of width $\mathcal{O}(\sqrt{\varepsilon})$ near the characteristic boundaries at $y = 0$ and $y = 1$. To resolve the layers we use appropriate Shishkin meshes. For the SDFEM we give an optimal choice for the stabilisation parameter δ, that ensures maximal stability in the induced streamline-diffusion norm without loosing the superconvergence property

$$|||u^N - u^I|||_{SD} \leq CN^{-2} \ln^2 N.$$

In the characteristic (or parabolic) boundary layer we are able to show that δ can be chosen of order $\delta = C\varepsilon^{-1/4}N^{-2}$ which is confirmed by numerical results. Using the superconvergence property we construct an enhanced numerical solution by postprocessing. The resulting numerical solution converges with almost second order accuracy.

References:

[1] http://www.math.tu-dresden.de/~sfranz/papers.html

[1] TU-Dresden, Institut für numerische Mathematik, 01062 Dresden, Germany, sebastian.franz@tu-dresden.de

[2] TU-Dresden, Institut für numerische Mathematik, 01062 Dresden, Germany, Torsten.Linss@tu-dresden.de