Edge elements and coercivity

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Abstract

Many variational problems from computational electromagnetism are set in Sobolev spaces of differential 1-forms and feature non-elliptic sesqui-linear forms. In order to establish a rigorous convergence theory of Galerkin discretizations the inherent coercivity of the variational problems has to be exploited. This can be accomplished when using finite elements or boundary elements that can be viewed as discrete differential forms. Key is the commuting diagram property of nodal interpolation operators that underlies discrete counterparts of Hodge-type decompositions.

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1 Coercivity

We say that a sesqui-linear form $a : V \times V \mapsto \mathbb{C}$ satisfies a *generalized Gårding inequality*, if there is an isomorphism $X : V \mapsto V$ and a *compact* operator $K : V \mapsto V'$ such that

$$\exists c > 0 : \quad \left| a(u, Xu) + \langle Ku, \overline{u} \rangle_{V' \times V} \right| \geq c \, \|u\|_V^2 \quad \forall u \in V \; . \tag{1}$$

By a Fredholm alternative argument [10, Ch. 2], we see that, if $a(\cdot, \cdot)$ is *injective*, that is,

$$a(u, v) = 0 \quad \forall v \in V \quad \Rightarrow \quad u = 0 \, , \tag{2}$$

then

$$\forall f \in V' : \quad \exists_1 u \in V : \quad a(u, v) = \langle f, v \rangle_{V' \times V} \quad \forall v \in V \; .$$

A classical example is the bilinear form

$$a(u, v) := \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, \overline{v} \, \mathrm{d}\mathbf{x} - \omega^2 \int_\Omega u \overline{v} \, \mathrm{d}\mathbf{x} + i\eta \int_{\partial\Omega} u \overline{v} \, \mathrm{d}S = \int_\Omega f \overline{v} \, \mathrm{d}\mathbf{x} \; , \tag{3}$$

$u, v \in H^1(\Omega)$, $\omega, \eta > 0$, related to the source problem for the Helmholtz equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with simple first order absorbing boundary conditions. Thanks to the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ the sesqui-linear form from (3) satisfies (1) with $V = H^1(\Omega)$ and $X = Id$.

2 Problems from Electromagnetism

The Maxwell source problem models the propagation of time-harmonic electromagnetic waves in a bounded cavity. When simple first order absorbing boundary conditions are imposed on the the walls, the associated bilinear form reads, see [6, Sect. 5],

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \mu^{-1}\, \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x} - \omega^2 \int_\Omega \epsilon\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + i\eta\int_{\partial\Omega}(\mathbf{u}\times\mathbf{n})\cdot(\overline{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S\;, \tag{4}$$

$\mathbf{u},\mathbf{v} \in V := \{\mathbf{w}\in \boldsymbol{H}(\mathbf{curl};\Omega) : (\mathbf{w}\times\mathbf{n}) \in \boldsymbol{L}^2(\partial\Omega)\}$. Here, the infinite-dimensional kernel of **curl** thwarts the simple argument available for (3).

The crucial device for dealing with (4) is a stable *Helmholtz-type* direct decomposition of V:

$$V = Z \oplus N \quad , \quad N \subset \mathrm{Ker}(\mathbf{curl}) \quad , \quad Z \hookrightarrow L^2_{\mathbf{t}}(\Omega) \text{ compact}\;. \tag{5}$$

Simple examples are provided by the true $\boldsymbol{L}^2(\Omega)$-orthogonal Helmholtz decomposition and so-called regular splittings [6, Sect. 2.4]. For the latter, we even have $Z \subset (H^1(\Omega))^3$ and the compactness of the embedding into $L^2_{\mathbf{t}}(\Omega) := \{\mathbf{v}\in\boldsymbol{L}^2(\Omega) : (\mathbf{v}\times\mathbf{n}) \in L^2(\partial\Omega)$ is a consequence of Sobolev embedding theorems. Denoting by P_Z, P_N the projections associated with (5), a generalized Gårding inequality (1) for $a(\cdot,\cdot)$ from (4) is recovered with the “sign flipping isomorphism” $X := P_Z - P_N = 2P_Z - Id$.

A structurally similar bilinear form arises from the *electric field integral equations* that is used for the numerical simulation of scattering of time-harmonic electromagnetic wave at a bounded perfectly conducting object Ω [4, Sect. 7]:

$$a(\boldsymbol{\xi},\boldsymbol{\eta}) := \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\Big(\mathrm{div}_\Gamma\boldsymbol{\xi}(\mathbf{x})\mathrm{div}_\Gamma\overline{\boldsymbol{\eta}(\mathbf{y})} - \omega^2\boldsymbol{\xi}(\mathbf{x})\cdot\overline{\boldsymbol{\eta}(\mathbf{y})}\Big)\,\mathrm{d}S(\mathbf{x},\mathbf{y})\;, \tag{6}$$

defined for tangential surfaced vectorfields in the trace space $H^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$ of $\boldsymbol{H}(\mathbf{curl};\Omega)$, see [2]. For this bilinear form we can rely on induced regular splittings of $H^{-1/2}(\mathrm{div}_\Gamma,\partial\Omega)$ in order to obtain a generalized Gårding inequality [4, Sect. 6].

3 Galerkin discretization

We assume that we are given a sequence V_n, $n\in\mathbb{N}$, of finite-dimensional subspaces of V that are approximating in the sense that

$$\forall u \in V: \quad \lim_{n\to\infty}\inf_{v_n\in V_N}\|u-v_n\|_V = 0\;. \tag{7}$$

In order to get quasi-optimal convergence of the Galerkin discretization of $a(u,v) = \langle f,v\rangle_{V'\times V}$, $v\in V$, based on $\{V_n\}$, we have to establish a uniform discrete inf-sup condition

$$\exists C>0: \quad \sup_{v_n\in V_n}\frac{|a(u_n,v_n)|}{\|v_n\|_V} \geq C\,\|u_n\|_V \quad \forall u_n\in V_n,\ \forall n\in\mathbb{N}\;. \tag{8}$$

If $a(\cdot,\cdot)$ satisfies a strengthened Gårding inequality (1) with $X = Id$ and (2), we can directly infer (8) from (7) [14]: define a compact operator $S: V\mapsto V$ by

$$a(v,Su) = \langle Kv,\overline{u}\rangle_{V'\times V} \quad \forall u,v\in V\;.$$

Further, denote by P_n the V-orthogonal projection $P_n : V \mapsto V_n$, which strongly converges to Id. Then,

$$|a(u_n,P_n(Id+S)u_n)| \geq |a(u_n,u_n) + \langle Ku_n,\overline{u}_n\rangle_{V'\times V}| - \|a\|\,\|(Id-P_n)Su_n\|_V\,\|u_n\|_V\;,$$

and, for sufficiently large n, (8) follows from (1) and the fact that $(Id - P_n)S \to 0$ uniformly for $n \to \infty$ [9, Thm. 10.6].

If a sign-flipping isomorphism X based on a splitting (5) is involved, this strategy fails, because, in general, $X\,V_n \not\subset V_n$. Therefore, we need another projector $P_n^Z : Z \mapsto V_n$, that satisfies

$$\exists\{\epsilon_n\} \subset \mathbb{R}_+^{\mathbb{N}},\ \lim_{n\to\infty} e_n = 0 : \quad \|(Id - P_n^Z)P_Z u_n\|_V \leq e_n \|u_n\|_V \quad \forall u_n \in V_n,\ \forall n \in \mathbb{N}\,. \tag{9}$$

Then, see [6, Sect. 5], for suffciently large n, we conclide (8) from

$$\begin{aligned}&|a(u_n,(P_n^Z X + P_n S)u_n)| = \\ &|a(u_n, Xu_n) + \langle Ku_n, \overline{u}_n\rangle_{V'\times V} - a((Id - P_n^Z)2P_Z u_n, u_n) - a((Id - P_n)Su_n, u_n)| \\ &\qquad\qquad \geq (c - \|a\|(2\epsilon_n + \|(Id - P_n)S\|))\,\|u_n\|_V^2\ .\end{aligned}$$

The approximation assumption (9) has the geometric meaning that angle enlosed by the subspaces $P_Z V_n \subset Z$ and $V_n \subset V$ tends to zero as $n \to \infty$.

4 Finite Elements

Edge elements are the lowest order representatives of families of $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming finite element spaces $\mathcal{W}^1(\mathcal{T}_h) \subset \boldsymbol{H}(\mathbf{curl};\Omega)$ required for the standard Galerkin discretization of (4). They are built on (heaxehedral or tetrahedral) triangulations $\mathcal{T}_h$ of Ω. They are perfect finite elements in the sense of [5]: they are locally polynomial, can be endowed with local degrees of freedom that ensure global tangential continuity, and possess bases comprised of locally supported functions, see [12, 13] and [6, Sect. 3]. Asymptotic density of $\mathcal{W}^1(\mathcal{T}_h)$ in $\boldsymbol{H}(\mathbf{curl};\Omega)$ follows from standard arguments.

In the usual fashion, the (global) degrees of freedom give rise to a nodal interpolation operator $\mathsf{I}_h : (C^\infty(\Omega))^3 \mapsto \mathcal{W}^1(\mathcal{T}_h)$. It enjoys the exceptional *commuting diagram property*

$$\mathbf{curl} \circ \mathsf{I}_h = \mathsf{J}_h \circ \mathbf{curl}\ , \tag{10}$$

where J_h is a projector onto $\mathbf{curl}\,\mathcal{W}^1(\mathcal{T}_h)$. Morever, it can be shown [1, Sect. 4.1] that I_h is bounded on

$$H_h := \{\mathbf{v} \in (H^1(\Omega))^3 :\ \mathbf{curl}\,\mathbf{v} \in \mathbf{curl}\,\mathcal{W}^1(\mathcal{T}_h)\}\ ,$$

and satisfies, with $C > 0$ only depending on the shape-regularity of $\mathcal{T}_h$ and $h > 0$ standing for the mesh-width of $\mathcal{T}_h$,

$$\|\mathbf{u}_h - \mathsf{I}_h\mathbf{u}\|_{L^2(\Omega)} \leq Ch\,\|\mathbf{u}\|_{H^1(\Omega)} \quad \forall \mathbf{u} \in H_h\ . \tag{11}$$

From the commuting diagram property (10) we immediately get that

$$\mathbf{curl}(\mathbf{u}_h - \mathsf{I}_h\mathbf{u}) = 0\ .$$

Hence, (11) and the stability of the regular decomposition imply (9) for the choice $P_n^Z = \mathsf{I}_h$.

We can now apply the abstract theory of Sect. 3 in the case of a sign flipping isomorphism derived from a regular splitting of $\boldsymbol{H}(\mathbf{curl};\Omega)$. This confirms the asymptotically quasi-optimal convergence of the Galerkin solutions for the h-version of $\boldsymbol{H}(\mathbf{curl};\Omega)$-conforming finite elements.

In the case of the electric field integral equation, a suitable boundary element space is immediately available by taking the rotated tangential trace of functions in $\mathcal{W}^1(\mathcal{T}_h)$. This procedure will yield div_Γ-compliant boundary element spaces for tangential vector fields that also possess projectors satisfying a commuting diagram property. Thus, convergence of boundary element Galerkin schemes can be established as above [8], see also [4, Sect. 9] and [7].

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