

## Nonlinear Optimization Exercises 7

1. (2) Given  $c \in \mathbb{R}^n$ , determine analytically the optimal solution of the problem

$$\begin{aligned} \min \quad & \sum_{i=1}^n x_i \ln x_i - c^T x, \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = 1, \\ & x > 0. \end{aligned}$$

2. (4) Prove Theorem IX.8: There is no direction  $d \in F_1(\bar{x})$  for which  $d^T \nabla f(\bar{x}) < 0$  if and only if there exists a vector  $\lambda \in \mathbb{R}^n$  so that

$$\nabla f(\bar{x}) = \sum_{i \in \mathcal{A}(\bar{x})} \lambda_i \nabla c_i(\bar{x}) \quad \text{with } \lambda_i \geq 0 \text{ for } i \in \mathcal{A}(\bar{x}) \cap \mathcal{I}.$$

Hint: Use Farkas' Lemma.

3. (4) Given the setting of the proof of Theorem IX.13, show that the following matrix is regular,

$$\begin{bmatrix} \nabla^2 f(x^*) - \sum \lambda_i^* \nabla^2 c_i(x^*) & -[\nabla c_1(x^*), \dots, \nabla c_m(x^*)] \\ [\nabla c_1(x^*), \dots, \nabla c_m(x^*)]^T & 0 \end{bmatrix}.$$

4. (2) Given a smooth concave function  $c: \mathbb{R}^n \rightarrow \mathbb{R}$ , show that for all  $x, y \in \mathbb{R}^n$

$$c(y) \leq c(x) + \nabla c(x)^T (y - x).$$

5. (2) Given a smooth concave function  $c: \mathbb{R}^n \rightarrow \mathbb{R}$ , show that  $-\log(c(x))$  is convex on  $\{x \in \mathbb{R}^n : c(x) > 0\}$ .

6. (6) Consider the following quadratic program in  $\mathbb{R}^2$

$$\begin{aligned} \min \quad & \frac{1}{2} x^T Q x - b x \\ \text{s.t.} \quad & x_1 + 1 \geq 0 \\ & -x_1 + 1 \geq 0 \\ & x_2 + 1 \geq 0 \\ & -x_2 + 1 \geq 0. \end{aligned}$$

Determine optimal solutions, Lagrange multipliers and (strongly/weakly) active index sets for the cases

$$(a) \quad Q = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 7 \end{bmatrix} \quad (b) \quad Q = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$(c) \quad Q = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

What is the infinitesimal change of the value of the optimal solution in each case, if the right hand side is shifted by some small  $d \in \mathbb{R}^4$  (use Th.IX.13).