TU Chemnitz
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Nonlinear Optimization
Exercises 4

1. (2) Prove $(1 - \frac{(g^\top g)^2}{g^\top Bg\, g^\top B^{-1}g}) \geq 0$ in the dog-leg Lemma III.4.

2. (2) Denote by p_{k+1} the step direction that is generated in iteration k by the linear CG algorithm for $f(x) = \frac{1}{2}x^T Ax - b^T x$ with $A \succ 0$. Prove that p_{k+1} is a descent direction.

3. (6) Given $A \in \mathbb{R}^{n\times n}$ positive definite and vectors $p_0, \ldots, p_{n-1} \in \mathbb{R}^n \setminus \{0\}$ that are conjugate with respect to A ($v_i^T Av_j = 0$ for $i \neq j$). Prove:

   (a) (1) $p_0, \ldots, p_{n-1}$ are linearly independent.

   (b) (3) Given the minimization problem $\min_{x\in\mathbb{R}^n} \frac{1}{2}x^T Ax - b^T x$ and some starting point $x_0 \in \mathbb{R}^n$, the sequence x_k given by the iteration

   $$x_{k+1} = x_k + \alpha_k p_k \quad \text{with} \quad r_k = Ax_k - b \quad \text{and} \quad \alpha_k = -\frac{r_k^T p_k}{p_k^T Ap_k}$$

   converges in at most n steps to the optimum x^* and $x^* = \sum_{i=0}^{n-1}(\frac{p_i^T b}{p_i^T Ap_i})p_i$.

   (c) (2) The points x_k defined in (b) satisfy

   $$x_{k+1} = \arg \min_{x\in\{x_0\}+\text{span}\{p_0,\ldots,p_k\}} \frac{1}{2}x^T Ax - b^T x.$$

4. (6) Rosen's projected gradient method: Given the optimization problem

   $$\min f(x) \quad \text{s.t.} \quad Ax = b$$

   with $A \in \mathbb{R}^{m\times n}$ having full row rank. Show that the step direction $p = -(I - A^T(AA^T)^{-1}A)\nabla f(x)$ (x feasible, i.e. $Ax = b$) yields a feasible descent direction if there is one (hint: $I - A^T(AA^T)^{-1}A$ is the projection matrix onto the null space of A). Using this step direction, develop a globally convergent optimization method for this problem (under suitable conditions on f)[a practical implementation is encouraged but not required here].