TU Chemnitz
C. Helmberg

Chemnitz, January 17, 2011
due on January 28, 2011

Nonlinear Optimization
Exercises 7

1. Given $c \in \mathbb{R}^n$, determine analytically the optimal solution of the problem

$$\begin{array}{ll} \min & \sum_{i=1}^n x_i \ln x_i - c^T x, \\ \text{s.t.} & \sum_{i=1}^n x_i = 1, \\ & x > 0. \end{array}$$

2. Prove Theorem IX.8: There is no direction $d \in F_1(\bar{x})$ for which $d^T \nabla f(\bar{x}) < 0$ if and only if there exists a vector $\lambda \in \mathbb{R}^n$ so that

$$\nabla f(\bar{x}) = \sum_{i \in \mathcal{A}(\bar{x})} \lambda_i \nabla c_i(\bar{x}) \quad \text{with } \lambda_i \geq 0 \text{ for } i \in \mathcal{A}(\bar{x}) \cap \mathcal{I}.$$

   Hint: Use Farkas' Lemma.

3. Given the setting of the proof of Theorem IX.13, show that the following matrix is regular,

$$\left[\begin{array}{cc} \nabla^2 f(x^*) - \sum \lambda_i^* \nabla^2 c_i(x^*) & -[\nabla c_1(x^*), \ldots, \nabla c_m(x^*)] \\ [\nabla c_1(x^*), \ldots, \nabla c_m(x^*)]^T & 0 \end{array} \right].$$

4. Given a smooth concave function $c : \mathbb{R}^n \to \mathbb{R}$, show that for all $x, y \in \mathbb{R}^n$

$$c(y) \leq c(x) + \nabla c(x)^T (y - x).$$

5. Given a smooth concave function $c : \mathbb{R}^n \to \mathbb{R}$, show that $-\log(c(x))$ is convex on $\{x \in \mathbb{R}^n : c(x) > 0\}$.

6. Consider the following quadratic program in $\mathbb{R}^2$

$$\begin{array}{ll} \min & \frac{1}{2} x^T Q x - bx \\ \text{s.t.} & x_1 + 1 \geq 0 \\ & -x_1 + 1 \geq 0 \\ & x_2 + 1 \geq 0 \\ & -x_2 + 1 \geq 0 \; . \end{array}$$

   Determine optimal solutions, Lagrange multipiers and (strongly/weakly) active index sets for the cases

   $(a) \quad Q = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 14 \\ 7 \end{bmatrix} \qquad (b) \quad Q = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

   $(c) \quad Q = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

   What is the infinitesimal change of the value of the optimal solution in each case, if the right hand side is shifted by some small $d \in \mathbb{R}^4$ (use Th.IX.13).