

## Nonlinear Optimization Exercises 4

1. Prove the missing part in Lemma III.4 (by following the dog-leg-path the step size is nondecreasing and the model value nonincreasing).
2. Denote by  $p_{k+1}$  the step direction that is generated in iteration  $k$  by the linear CG algorithm for  $f(x) = \frac{1}{2}x^T Ax - b^T x$  with  $A \succ 0$ . Prove that  $p_{k+1}$  is a descent direction.
3. Given  $A \in \mathbb{R}^{n \times n}$  positive definite and vectors  $p_0, \dots, p_{n-1} \in \mathbb{R}^n \setminus \{0\}$  that are conjugate with respect to  $A$  ( $v_i^T A v_j = 0$  for  $i \neq j$ ). Prove:

(a) (1)  $p_0, \dots, p_{n-1}$  are linearly independent.

(b) (3) Given the the minimization problem  $\min_{x \in \mathbb{R}^n} \frac{1}{2}x^T Ax - b^T x$  and some starting point  $x_0 \in \mathbb{R}^n$ , the sequence  $x_k$  given by the iteration

$$x_{k+1} = x_k + \alpha_k p_k \quad \text{with} \quad r_k = Ax_k - b \quad \text{and} \quad \alpha_k = -\frac{r_k^T p_k}{p_k^T A p_k}$$

converges in at most  $n$  steps to the optimum  $x^*$  and  $x^* = \sum_{i=0}^{n-1} \left(\frac{p_i^T b}{p_i^T A p_i}\right) p_i$ .

(c) (2) The points  $x_k$  defined in (b) satisfy

$$x_{k+1} = \arg \min_{x \in \{x_0\} + \text{span}\{p_0, \dots, p_k\}} \frac{1}{2}x^T Ax - b^T x.$$

4. Rosen's projected gradient method: Given the optimization problem

$$\min f(x) \quad \text{s.t.} \quad Ax = b$$

with  $A \in \mathbb{R}^{m \times n}$  having full row rank. Show that the step direction  $p = -(I - A^T(AA^T)^{-1}A)\nabla f(x)$  ( $x$  feasible, i.e.  $Ax = b$ ) yields a feasible descent direction if there is one (hint:  $I - A^T(AA^T)^{-1}A$  is the projection matrix onto the null space of  $A$ ). Using this step direction, develop a globally convergent optimization method for this problem (under suitable conditions on  $f$ ) [a practical implementation is encouraged but not required here].