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Nonlinear Optimization

Exercises 1

1. (1) Compute the gradient $\nabla f_i(x)$ and Hessian matrix $\nabla^2 f_i(x)$, for $i = 1, 2$
   - $f_1(x) = x^T Qx + b^T x$, with $Q^T = Q \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.
   - $f_2(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, $x \in \mathbb{R}^2$ (Rosenbrock function.)

2. (5) Let A be a symmetric matrix of order n. Show that the following statements are equivalent:

   (i) A is positive semidefinite ($A \succeq 0$), i.e., $v^T Av \geq 0 \ \ \forall v \in \mathbb{R}^n$.

   (ii) $\lambda_i \geq 0$ where λ_i is an eigenvalue of A, $i = 1, 2, \ldots, n$.

   (iii) $\exists C \in \mathbb{R}^{m\times n}$ such that $A = C^T C$ (and $\operatorname{rank}(C) = \operatorname{rank}(A)$).

   What changes, if A is positive definite ($A \succ 0$ or $v^T Av > 0 \ \forall v \in \mathbb{R}^n \setminus \{0\}$)?

3. (1) Generate various symmetric 2×2 matrices via specifying their eigenvalues (not necessarily all positive) and eigenvectors and, using Matlab, produce mesh and contour plots of the respective quadratic functions. What do you know about the gradient by looking at contour plots?

4. (3) Using Matlab, have a look at the contour plot of the Rosenbrock function and write a routine that generates for an arbitrary point in $[-1, 2] \times [-\frac{1}{2}, 3]$ a mesh plot of the Rosenbrock function with its linear and quadratic model.

5. (5) Landau Symbols (see Wikipedia): proof that for $f \in C^2(\mathbb{R}^n)$ and $h \to 0$ it holds that
   $$f(x + h) = f(x) + \nabla f(x)^T h + o(\|h\|),$$
   $$f(x + h) = f(x) + \nabla f(x)^T h + O(\|h\|^2).$$
   What can be said for $f \in C^3(\mathbb{R}^n)$ and the quadratic model?

6. (5) Suppose Newton's method is applied to $f(x) = \cos x$.

   a) Show that there is a starting point so that the sequence of iterates diverges.

   b) Show that there is a starting point so that the iterates alternate between two non stationary points.

   c) If the procedure starts from a point near 0, then to what point does the method converge? Why?

   d) How can one determine the maximal interval containing π so that for all starting points in this interval the iterates remain in this interval and converge to π?

   Determine the starting points and values numerically and try Newton's method on them.