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Graph Theory

Exercise 6

Terms

- Ramsey numbers $R(H_1, ..., H_c)$ and $R(c, k, n)$,
- extremal edge number $\mathrm{ex}(n, H)$,
- Turán graphs, Turán's theorem, Mantel's theorem.

Tasks

1. Prove the upper bound $R(K_k, K_k) \leq 4^k$.

2. Given $p \in [0, 1]$ and $n \in \mathbb{N}$, let $G(n, p)$ be a randomly generated graph, in which each of the $\binom{n}{2}$ edges is present independently with probability p.

   (a) Determine an upper bound on the probability that $G(n, p)$ has a clique/independent set of size at least k.

   (b) Deduce the lower bound $R(K_k, K_k) \geq \sqrt{2}^k$.

3. Determine $\mathrm{ex}(n, K_{r,1})$ for all $n, r \in \mathbb{N}$.

4. *Mantel's theorem.* Let G be a triangle free graph with n vertices and m edges.

   (a) Prove every step in the following chain of inequalities:

$$\frac{4m^2}{n} = \frac{1}{n}\Big(\sum_{v\in V} d_G(v)\Big)^2 \leq \sum_{v\in V} d_G(v)^2 = \sum_{vw\in E} (d_G(v) + d_G(w)) \leq nm.$$

   (b) From this, prove Mantel's theorem: $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.

5. Let $G(n, p)$ be a random graph as described in task 2.

   (a) What is the expected number of edges of $G(n, p)$?

   (b) Prove the following upper bound on the expected number of C_k in $G(n, p)$?

$$\mathbb{E}(\#C_k) \leq \frac{(n)_k}{2k} p^k, \qquad \text{where } (n)_k = n(n-1)\cdots(n-k+1).$$

   (c) Deduce a lower bound on $\mathrm{ex}(n, C_k)$.

   (d) How can this be generalized to arbitrary forbidden subgraphs H?