Technische Universität Chemnitz
M. Winter

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Graph Theory

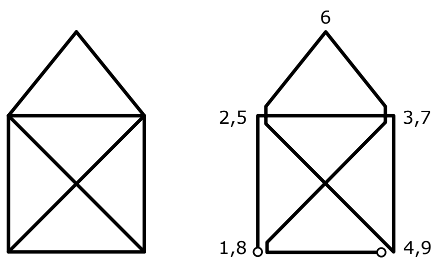
Exercise 4

Terms

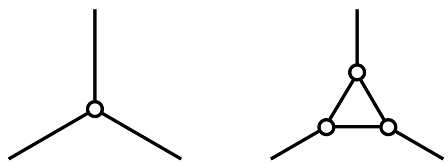
- Euler path, Euler cycle, Eulerian graph,
- Hamiltonian path, Hamiltonian cycle, Hamiltonian graph,
- degree sequence, toughness, Hamiltonian hull, theorem of Dirac, theorem of Ore,
- edge space $\mathcal{E}(G)$, cycle space $\mathcal{C}(G)$, cut space $\mathcal{C}^\perp(G)$, (simple) cycle base,
- plane graph, embedding, face, planar graph, algebraic planarity condition,
- Euler's polyhedral formula, discharging method, dual graph.

Tasks

1. How many ways are there to draw the "Haus vom Nikolaus" without lifting the pen?



2. Let G be some graph and G' another graph obtained from G by "blowing up" every vertex to a triangle (see figure). Show that G is Hamiltonian if and only if G' is.



3. Show that the edge-graph of the n-dimensional hypercube $Q_n := [-1,1]^n$ is Hamiltonian. Now, cut off the vertices of Q_n by a small plane cut. Is the edge-graph of the resulting polytope (the *truncated hypercube*) still Hamiltonian?

4. The cycle space $\mathcal{C}(G)$ and the cut space $\mathcal{C}^\perp(G)$ together do not necessarily span the whole edge space. For example, show that a bipartite Eulerian graph (with at least one edge) always satisfies

$$\mathcal{C}(G) \oplus \mathcal{C}^\perp(G) \subset \mathcal{E}(G).$$

5. Let G be some finite plane graph. Let V be its number of vertices, E its number of edges, F its number of faces and C its number of components. Prove that

$$V - E + F = C + 1.$$

6. Let G be bipartite and Eulerian. Show that G is not polyhedral.

7. Show that the Petersen graph is *not* planar by

   (a) using the algebraic planarity condition,

   (b) using the discharging method,

   (c) showing that it contains $K_{3,3}$ as a topological minor,

   (d) showing that it contains K_5 as a minor.