Technische Universität Chemnitz
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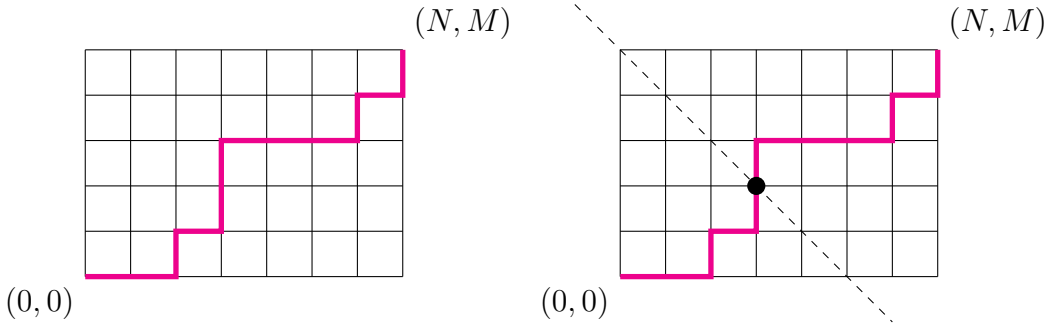
Chemnitz, 22. Oktober 2014
due: 29. Oktober 2014

Introduction to Discrete Mathematics
Exercise 2

1. Prove the following recursion for the number of unordered number partitions $P_{n,k}$: $P_{n,1} = P_{n,n} = 1$ and $P_{n,k} = P_{n-k,1} + P_{n-k,2} + \ldots + P_{n-k,k}$. (4 points)

2. Many identities for binomial coefficients can be obtained by counting paths in lattices. Consider the $N \times M$ lattice and count all possible paths from $(0,0)$ to (N,M) that only move rightwards or upwards.



   (a) How many such paths exist? (2 points)

   (b) Give an alternative proof of *Vandermonde's identity*

$$\binom{M+N}{M} = \sum_{i \geq 0} \binom{M}{M-i} \binom{N}{i},$$

   by classifying all paths from $(0,0)$ to (N,M) by their respective intersection point with the dashed line $x + y = M$. (4 points)

3. Let $a_n = \sum_{k=0}^{n} k\binom{n}{k}$ for natural n. Show that $a_{n+1} = 2a_n + 2^n$ holds for positive integral n. (3 points)

4. Prove $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ and use it to show that $\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$. (3 points)

5. How many possibilities are there to position k rooks on a $n \times n$ chessboard such that they cannot capture each other? How many rooks can be placed in this way? (The rook can move any number of squares along any rank or file) (4 points)