Chemnitz Scientific Computing Preprints – ISSN 1864-0087

TECHNISCHE UNIVERSITÄT CHEMNITZ

Michael Weise

A note on the second derivatives of rHCT basis functions – extended

CSC/15-02

Chemnitz Scientific Computing Preprints

Impressum:

Chemnitz Scientific Computing Preprints — ISSN 1864-0087
(1995–2005: Preprintreihe des Chemnitzer SFB393)

Herausgeber:
Professuren für
Numerische und Angewandte Mathematik
an der Fakultät für Mathematik
der Technischen Universität Chemnitz

Postanschrift:
TU Chemnitz, Fakultät für Mathematik
09107 Chemnitz
Sitz:
Reichenhainer Str. 41, 09126 Chemnitz

http://www.tu-chemnitz.de/mathematik/csc/

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TECHNISCHE UNIVERSITÄT CHEMNITZ

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A note on the second derivatives of rHCT basis functions – extended

CSC/15-02

Abstract

We consider reduced Hsieh-Clough-Tocher basis functions with respect to a splitting into subtriangles at an arbitrary interior point of the original triangular element. This article gives a proof that the second derivatives of those functions, which in general may jump at the subtriangle boundaries, do not jump at the splitting point.

CSC/15-02 ISSN 1864-0087 January 2014

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$$
\begin{aligned}
&\mu_+^2(x_{k+1,\mathrm{s}}y_{k,\mathrm{s}} + x_{k,\mathrm{s}}y_{k+1,\mathrm{s}}) + 2\mu_k\mu_{k+1}x_{k+1,k}y_{k+1,k} - \mu_k\mu_{k-1}(x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k}) \\
&-\mu_{k+1}\mu_{k-1}(x_{k+1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k+1,k}) \\
&\qquad = (\mu_k x_{k+1,k} + \mu_{k-1}x_{k+1,k-1})(\mu_{k+1}y_{k,k+1} + \mu_{k-1}y_{k,k-1}) \\
&\qquad\quad + (\mu_{k+1}x_{k,k+1} + \mu_{k-1}x_{k,k-1})(\mu_k y_{k+1,k} + \mu_{k-1}y_{k+1,k-1}) \\
&\qquad\quad + 2\mu_k\mu_{k+1}x_{k+1,k}y_{k+1,k} - \mu_k\mu_{k-1}(x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k}) \\
&\qquad\quad - \mu_{k+1}\mu_{k-1}(x_{k+1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k+1,k}) \\
&\qquad = \mu_k\mu_{k+1}x_{k+1,k}y_{k,k+1} + \mu_k\mu_{k-1}x_{k+1,k}y_{k,k-1} + \mu_{k+1}\mu_{k-1}x_{k+1,k-1}y_{k,k+1} \\
&\qquad\quad + \mu_{k-1}^2x_{k+1,k-1}y_{k,k-1} + \mu_k\mu_{k+1}x_{k,k+1}y_{k+1,k} + \mu_{k+1}\mu_{k-1}x_{k,k+1}y_{k+1,k-1} \\
&\qquad\quad + \mu_k\mu_{k-1}x_{k,k-1}y_{k+1,k} + \mu_{k-1}^2x_{k,k-1}y_{k+1,k-1} \\
&\qquad\quad + 2\mu_k\mu_{k+1}x_{k+1,k}y_{k+1,k} - \mu_k\mu_{k-1}(x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k}) \\
&\qquad\quad - \mu_{k+1}\mu_{k-1}(x_{k+1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k+1,k}) \\
&\qquad = \mu_{k-1}^2(x_{k-1,k+1}y_{k-1,k} + x_{k-1,k}y_{k-1,k+1}),
\end{aligned}
$$

which shows $C_{2:} = A_{2:}$.

In summary we have shown $\underline{A} = \underline{B} = \underline{C}$ for an arbitrary k, therefore

$$(D_2\Psi_k|_{T_1})(\hat{a}_\mathrm{s}) = (D_2\Psi_k|_{T_2})(\hat{a}_\mathrm{s}) = (D_2\Psi_k|_{T_3})(\hat{a}_\mathrm{s}) \quad \forall\, k = 1, 2, 3.$$

This proves the stated hypothesis

$$(D_2\Psi|_{T_1})(\hat{a}_\mathrm{s}) = (D_2\Psi|_{T_2})(\hat{a}_\mathrm{s}) = (D_2\Psi|_{T_3})(\hat{a}_\mathrm{s}).$$

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As above, a comparison of coefficients with (5) gives

$$
\begin{aligned}
-\mu_+^2 & x_{k-1,\mathrm{s}} y_{k-1,\mathrm{s}} + \mu_k^2 x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} x_{k,k-1} y_{k-1,k+1} - \mu_k \mu_{k+1} x_{k-1,k+1} y_{k,k-1} \\
&= -(\mu_k x_{k-1,k} + \mu_{k+1} x_{k-1,k+1})(\mu_k y_{k-1,k} + \mu_{k+1} y_{k-1,k+1}) \\
&\quad + \mu_k^2 x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} x_{k,k-1} y_{k-1,k+1} - \mu_k \mu_{k+1} x_{k-1,k+1} y_{k,k-1} \\
&= -\mu_k^2 x_{k-1,k} y_{k-1,k} - \mu_k \mu_{k+1} x_{k-1,k} y_{k-1,k+1} - \mu_k \mu_{k+1} x_{k-1,k+1} y_{k-1,k} \\
&\quad - \mu_{k+1}^2 x_{k-1,k+1} y_{k-1,k+1} + \mu_k^2 x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} x_{k,k-1} y_{k-1,k+1} \\
&\quad - \mu_k \mu_{k+1} x_{k-1,k+1} y_{k,k-1} \\
&= -\mu_{k+1}^2 x_{k-1,k+1} y_{k-1,k+1},
\end{aligned}
$$

$$
\begin{aligned}
\mu_+^2 & (x_{k-1,\mathrm{s}} y_{k,\mathrm{s}} + x_{k,\mathrm{s}} y_{k-1,\mathrm{s}}) + 2\mu_k \mu_{k-1} x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} (x_{k,k-1} y_{k+1,k} + x_{k+1,k} y_{k,k-1}) \\
-\mu_{k+1} & \mu_{k-1} (x_{k,k-1} y_{k-1,k+1} + x_{k-1,k+1} y_{k,k-1}) \\
&= (\mu_k x_{k-1,k} + \mu_{k+1} x_{k-1,k+1})(\mu_{k+1} y_{k,k+1} + \mu_{k-1} y_{k,k-1}) \\
&\quad + (\mu_{k+1} x_{k,k+1} + \mu_{k-1} x_{k,k-1})(\mu_k y_{k-1,k} + \mu_{k+1} y_{k-1,k+1}) \\
&\quad + 2\mu_k \mu_{k-1} x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} (x_{k,k-1} y_{k+1,k} + x_{k+1,k} y_{k,k-1}) \\
&\quad - \mu_{k+1} \mu_{k-1} (x_{k,k-1} y_{k-1,k+1} + x_{k-1,k+1} y_{k,k-1}) \\
&= \mu_k \mu_{k+1} x_{k-1,k} y_{k,k+1} + \mu_k \mu_{k-1} x_{k-1,k} y_{k,k-1} + \mu_{k+1}^2 x_{k-1,k+1} y_{k,k+1} \\
&\quad + \mu_{k+1} \mu_{k-1} x_{k-1,k+1} y_{k,k-1} + \mu_k \mu_{k+1} x_{k,k+1} y_{k-1,k} + \mu_{k+1}^2 x_{k,k+1} y_{k-1,k+1} \\
&\quad + \mu_{k-1} \mu_k x_{k,k-1} y_{k-1,k} + \mu_{k+1} \mu_{k-1} x_{k,k-1} y_{k-1,k+1} \\
&\quad + 2\mu_k \mu_{k-1} x_{k,k-1} y_{k,k-1} - \mu_k \mu_{k+1} (x_{k,k-1} y_{k+1,k} + x_{k+1,k} y_{k,k-1}) \\
&\quad - \mu_{k+1} \mu_{k-1} (x_{k,k-1} y_{k-1,k+1} + x_{k-1,k+1} y_{k,k-1}) \\
&= \mu_{k+1}^2 (x_{k-1,k+1} y_{k,k+1} + x_{k,k+1} y_{k-1,k+1}),
\end{aligned}
$$

which shows $B_{2:} = A_{2:}$. Similarly, one gets

$$
\begin{aligned}
-\mu_+^2 & x_{k+1,\mathrm{s}} y_{k+1,\mathrm{s}} + \mu_k^2 x_{k+1,k} y_{k+1,k} - \mu_k \mu_{k-1} x_{k+1,k} y_{k-1,k+1} - \mu_k \mu_{k-1} x_{k-1,k+1} y_{k+1,k} \\
&= -(\mu_k x_{k+1,k} + \mu_{k-1} x_{k+1,k-1})(\mu_k y_{k+1,k} + \mu_{k-1} y_{k+1,k-1}) \\
&\quad + \mu_k^2 x_{k+1,k} y_{k+1,k} - \mu_k \mu_{k-1} x_{k+1,k} y_{k-1,k+1} - \mu_k \mu_{k-1} x_{k-1,k+1} y_{k+1,k} \\
&= -\mu_k^2 x_{k+1,k} y_{k+1,k} - \mu_k \mu_{k-1} x_{k+1,k} y_{k+1,k-1} - \mu_k \mu_{k-1} x_{k+1,k-1} y_{k+1,k} \\
&\quad - \mu_{k-1}^2 x_{k+1,k-1} y_{k+1,k-1} + \mu_k^2 x_{k+1,k} y_{k+1,k} - \mu_k \mu_{k-1} x_{k+1,k} y_{k-1,k+1} \\
&\quad - \mu_k \mu_{k-1} x_{k-1,k+1} y_{k+1,k} \\
&= -\mu_{k-1}^2 x_{k-1,k+1} y_{k-1,k+1},
\end{aligned}
$$

1 Introduction

Some relevant problems such as the biharmonic problem or the plate problem can be described by a partial differential equation of fourth order. The weak formulation of any such problem features functions from the Sobolev space H^2. Thus, the functions themselves as well as their first and second generalised derivatives have to be square-integrable over the considered domain. The natural approach to solving such problems numerically by the finite element method is to use conforming finite elements. This means that the FE basis functions belong to a finite-dimensional subspace of the appropriate space H^2. This is fulfilled for FE basis functions which are globally C^1-continuous.

One example of C^1-continuous elements is the reduced Hsieh–Clough–Tocher (rHCT) element, which goes back to [1]. It is a triangular element with piecewise cubic shape functions defined on three subtriangles. The shape functions are constructed in such a way that the resulting global basis functions are C^1-continuous. The element uses the values of the function and both first derivatives at all three vertices as degrees of freedom, which sums up to 9 in total. Global C^1-continuity is achieved by inner C^1-continuity and the condition that the restriction of the normal derivative of any shape function to any element edge has to be linear with respect to the local line coordinate. The splitting into three subtriangles may be based on an arbitrary interior point, which is called splitting point.

The goal of this article is to show the following remarkable property. While the second derivatives of rHCT shape functions may jump across internal edges, they do not jump at the splitting point of the element. The current article is an extension of [4], where this property was shown for a splitting based on the barycenter.

Our practical motivation for this article comes from remarks 7.10 and 7.11 in [3]. The above property was used there to get rid of nodal jump terms in the construction of an a posteriori error estimator for rHCT elements for plate and laminate problems, but no proof was given.

2 Shape functions

There exist several approaches to the definition of rHCT shape functions. They all lead to the same functions eventually; only the formulations differ. We consider the method given in [5], which is a generalisation of [2] to an arbitrary interior point as splitting point. The construction of shape functions is shortly recapitulated in this section.

Consider a split of the original triangle T with the vertices

$$a_j = [x_j, y_j]^\mathsf{T}, \quad j = 1, 2, 3$$

based on an arbitrary interior point

$$a_\mathrm{s} = [x_\mathrm{s}, y_\mathrm{s}]^\mathsf{T} \in \operatorname{int} T.$$

Shape functions that belong to node a_j of the triangle T are written as a row vector

$$\Psi_j(a) = [\psi_j^{(0)}(a), \psi_j^{(1)}(a), \psi_j^{(2)}(a)]$$

and the full vector of all shape functions takes the form

$$\Psi(a) = [\Psi_1(a), \Psi_2(a), \Psi_3(a)]$$

at an arbitrary point $a = [x, y]^\mathsf{T}$. Shape functions with superscript (0) are related to the function value at the respective node and those with superscripts (1) and (2) are related to the function derivative with respect to x and y at the respective node.

In order to shorten the following expressions, we introduce some abbreviations to be used throughout the article. We use $x_{i,j}$ and $y_{i,j}$ to denote $x_i - x_j$ and $y_i - y_j$, respectively. This implies $x_{i,j} = -x_{j,i}$ and $y_{i,j} = -y_{j,i}$. Furthermore, all indices $k, k-1, k+1$ run from 1 to 3 and $k \pm 1$ is always understood implicitly as

$$k \pm 1 \;\mapsto\; \big((k \pm 1 - 1) \mod 3\big) + 1$$

to stay in the admissible index set $\{1, 2, 3\}$. Formulas that use k as an index are valid for $k = 1, 2, 3$.

The outer edges of the element are denoted by E_k and the inner edges by f_k. Their orientation is as given in Figure 1, which leads to the formulas

$$E_k = \begin{bmatrix} x_{k-1,k+1} \\ y_{k-1,k+1} \end{bmatrix} \text{ and } f_k = \begin{bmatrix} x_{k,\mathrm{s}} \\ y_{k,\mathrm{s}} \end{bmatrix}.$$

We define normals of the outer edges with the same length by

$$N_k = \begin{bmatrix} -y_{k-1,k+1} \\ x_{k-1,k+1} \end{bmatrix}.$$

The subtriangle containing E_k is denoted T_k. The Jacobians of the mappings from the reference triangle to the three subtriangles, confer also section 3, are

$$\underline{J}_k = \begin{bmatrix} x_{k+1,\mathrm{s}} & x_{k-1,\mathrm{s}} \\ y_{k+1,\mathrm{s}} & y_{k-1,\mathrm{s}} \end{bmatrix}.$$

$$\begin{aligned}
B_{2:} &= \Big(\underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k+1})^\mathsf{T} \\
&\qquad + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k+1}\underline{M}_k\Big)_{2:} \\
&= -\tfrac{1}{\mu_{k+1}^2} x_{k-1,\mathrm{s}} y_{k-1,\mathrm{s}} (c^k)^\mathsf{T} + \tfrac{1}{\mu_{k+1}^2}(x_{k-1,\mathrm{s}} y_{k,\mathrm{s}} + x_{k,\mathrm{s}} y_{k-1,\mathrm{s}})(b^k_{k+1})^\mathsf{T} \\
&\quad + \tfrac{1}{6\mu_+^2\mu_{k+1}^2}\Big(6x_{k,k-1}y_{k,k-1}(\mu_k^2 c^k + 2\mu_k\mu_{k-1}b^k_{k+1} + 2\mu_k\mu_{k+1}b^k_{k-1}) \\
&\qquad + 6\mu_{k+1}x_{k,k-1}(-\mu_k y_{k-1,k+1}c^k - (\mu_k y_{k+1,k} + \mu_{k-1}y_{k-1,k+1})b^k_{k+1} \\
&\qquad\quad - (\mu_k y_{k,k-1} + \mu_{k+1}y_{k-1,k+1})b^k_{k-1}) \\
&\qquad - 6\mu_{k+1}y_{k,k-1}(\mu_k x_{k-1,k+1}c^k + (\mu_k x_{k+1,k} + \mu_{k-1}x_{k-1,k+1})b^k_{k+1} \\
&\qquad\quad + (\mu_k x_{k,k-1} + \mu_{k+1}x_{k-1,k+1})b^k_{k-1})\Big)^\mathsf{T} \\
&= \tfrac{1}{\mu_+^2\mu_{k+1}^2}(-\mu_+^2 x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}} + \mu_k^2 x_{k,k-1}y_{k,k-1} - \mu_k\mu_{k+1}x_{k,k-1}y_{k-1,k+1} \\
&\qquad - \mu_k\mu_{k+1}x_{k-1,k+1}y_{k,k-1})(c^k)^\mathsf{T} \\
&\quad + \tfrac{1}{\mu_+^2\mu_{k+1}^2}(\mu_+^2(x_{k-1,\mathrm{s}}y_{k,\mathrm{s}} + x_{k,\mathrm{s}}y_{k-1,\mathrm{s}}) + 2\mu_k\mu_{k-1}x_{k,k-1}y_{k,k-1} \\
&\qquad - \mu_k\mu_{k+1}(x_{k,k-1}y_{k+1,k} + x_{k+1,k}y_{k,k-1}) \\
&\qquad - \mu_{k+1}\mu_{k-1}(x_{k,k-1}y_{k-1,k+1} + x_{k-1,k+1}y_{k,k-1}))(b^k_{k+1})^\mathsf{T} \\
&\quad - \tfrac{1}{\mu_+^2}(x_{k,k-1}y_{k-1,k+1} + x_{k-1,k+1}y_{k,k-1})(b^k_{k-1})^\mathsf{T},
\end{aligned}$$

$$\begin{aligned}
C_{2:} &= \Big(\underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k-1})^\mathsf{T} \\
&\qquad + \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k-1}\underline{M}_k\Big)_{2:} \\
&= -\tfrac{1}{\mu_{k-1}^2} x_{k+1,\mathrm{s}} y_{k+1,\mathrm{s}} (c^k)^\mathsf{T} + \tfrac{1}{\mu_{k-1}^2}(x_{k+1,\mathrm{s}} y_{k,\mathrm{s}} + x_{k,\mathrm{s}} y_{k+1,\mathrm{s}})(b^k_{k-1})^\mathsf{T} \\
&\quad + \tfrac{1}{6\mu_+^2\mu_{k-1}^2}\Big(6x_{k+1,k}y_{k+1,k}(\mu_k^2 c^k + 2\mu_k\mu_{k-1}b^k_{k+1} + 2\mu_k\mu_{k+1}b^k_{k-1}) \\
&\qquad + 6\mu_{k-1}x_{k+1,k}(-\mu_k y_{k-1,k+1}c^k - (\mu_k y_{k+1,k} + \mu_{k-1}y_{k-1,k+1})b^k_{k+1} \\
&\qquad\quad - (\mu_k y_{k,k-1} + \mu_{k+1}y_{k-1,k+1})b^k_{k-1}) \\
&\qquad - 6\mu_{k-1}y_{k+1,k}(\mu_k x_{k-1,k+1}c^k + (\mu_k x_{k+1,k} + \mu_{k-1}x_{k-1,k+1})b^k_{k+1} \\
&\qquad\quad + (\mu_k x_{k,k-1} + \mu_{k+1}x_{k-1,k+1})b^k_{k-1})\Big)^\mathsf{T} \\
&= \tfrac{1}{\mu_+^2\mu_{k-1}^2}(-\mu_+^2 x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}} + \mu_k^2 x_{k+1,k}y_{k+1,k} - \mu_k\mu_{k-1}x_{k+1,k}y_{k-1,k+1} \\
&\qquad - \mu_k\mu_{k-1}x_{k-1,k+1}y_{k+1,k})(c^k)^\mathsf{T} \\
&\quad - \tfrac{1}{\mu_+^2}(x_{k+1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k+1,k})(b^k_{k+1})^\mathsf{T} \\
&\quad + \tfrac{1}{\mu_+^2\mu_{k-1}^2}(\mu_+^2(x_{k+1,\mathrm{s}}y_{k,\mathrm{s}} + x_{k,\mathrm{s}}y_{k+1,\mathrm{s}}) + 2\mu_k\mu_{k+1}x_{k+1,k}y_{k+1,k} \\
&\qquad - \mu_k\mu_{k-1}(x_{k+1,k}y_{k,k-1} + x_{k,k-1}y_{k+1,k}) \\
&\qquad - \mu_{k+1}\mu_{k-1}(x_{k+1,k}y_{k-1,k+1} + x_{k-1,k+1}y_{k+1,k}))(b^k_{k-1})^\mathsf{T}.
\end{aligned}$$

which are the same coefficients as in $A_{1:}$ with $1/(\mu_+^2\mu_{k+1}^2)$ factored out. The coefficients of $(b_{k-1}^k)^\mathsf{T}$ in $A_{1:}$ and $B_{1:}$ are evidently also equal. Therefore, it holds $B_{1:} = A_{1:}$. Similarly, one gets $C_{1:} = A_{1:}$. The coefficients of $(b_{k+1}^k)^\mathsf{T}$ in $A_{1:}$ and $C_{1:}$ are evidently equal and it holds

$$\begin{aligned}
&\mu_+^2 y_{k+1,\mathrm{s}}^2 - \mu_k^2 y_{k+1,k}^2 + 2\mu_{k-1}\mu_k y_{k+1,k} y_{k-1,k+1} \\
&\quad= (\mu_k y_{k+1,k} + \mu_{k-1} y_{k+1,k-1})^2 - \mu_k^2 y_{k+1,k}^2 + 2\mu_{k-1}\mu_k y_{k+1,k} y_{k-1,k+1} \\
&\quad= \mu_k^2 y_{k+1,k}^2 + 2\mu_k\mu_{k-1} y_{k+1,k} y_{k+1,k-1} + \mu_{k-1}^2 y_{k+1,k-1}^2 \\
&\qquad - \mu_k^2 y_{k+1,k}^2 + 2\mu_{k-1}\mu_k y_{k+1,k} y_{k-1,k+1} \\
&\quad= \mu_{k-1}^2 y_{k-1,k+1}^2, \\
&-\mu_+^2 y_{k,\mathrm{s}} y_{k+1,\mathrm{s}} - \mu_k\mu_{k+1} y_{k+1,k}^2 + \mu_{k-1}\mu_k y_{k+1,k} y_{k,k-1} + \mu_{k-1}\mu_{k+1} y_{k+1,k} y_{k-1,k+1} \\
&\quad= -(\mu_{k+1} y_{k,k+1} + \mu_{k-1} y_{k,k-1})(\mu_k y_{k+1,k} + \mu_{k-1} y_{k+1,k-1}) \\
&\qquad - \mu_k\mu_{k+1} y_{k+1,k}^2 + \mu_{k-1}\mu_k y_{k+1,k} y_{k,k-1} + \mu_{k-1}\mu_{k+1} y_{k+1,k} y_{k-1,k+1} \\
&\quad= \mu_{k+1}\mu_k y_{k,k+1}^2 - \mu_{k-1}\mu_{k+1} y_{k-1,k+1} y_{k+1,k} - \mu_{k-1}\mu_k y_{k,k-1} y_{k+1,k} \\
&\qquad + \mu_{k-1}^2 y_{k,k-1} y_{k+1,k-1} \\
&\qquad - \mu_k\mu_{k+1} y_{k+1,k}^2 + \mu_{k-1}\mu_k y_{k+1,k} y_{k,k-1} + \mu_{k-1}\mu_{k+1} y_{k+1,k} y_{k-1,k+1} \\
&\quad= \mu_{k-1}^2 y_{k+1,k-1} y_{k,k-1},
\end{aligned}$$

which are the coefficients of $(c^k)^\mathsf{T}$ and $2(b_{k-1}^k)^\mathsf{T}$ with $1/(\mu_+^2\mu_{k-1}^2)$ factored out.

$A_{3:} = B_{3:} = C_{3:}$ follows analogously with all y_* replaced by the corresponding x_*; a double '$-$' cancels out.

Finally, we consider $A_{2:}$, $B_{2:}$, and $C_{2:}$ and get

$$\begin{aligned}
A_{2:} &= \left(\underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_k\,\underline{M}_k\right)_{2:} \\
&= \tfrac{1}{6\mu_+^2\mu_k^2}\Big(6x_{k-1,k+1}y_{k-1,k+1}(\mu_k^2 c^k + 2\mu_k\mu_{k-1}b_{k+1}^k + 2\mu_k\mu_{k+1}b_{k-1}^k) \\
&\qquad + 6\mu_k x_{k-1,k+1}(-y_{k-1,k+1}\mu_k c^k - (\mu_k y_{k+1,k} + \mu_{k-1}y_{k-1,k+1})b_{k+1}^k \\
&\qquad\quad - (\mu_k y_{k,k-1} + \mu_{k+1}y_{k-1,k+1})b_{k-1}^k) \\
&\qquad - 6\mu_k y_{k-1,k+1}(x_{k-1,k+1}\mu_k c^k + (\mu_k x_{k+1,k} + \mu_{k-1}x_{k-1,k+1})b_{k+1}^k \\
&\qquad\quad + (\mu_k x_{k,k-1} + \mu_{k+1}x_{k-1,k+1})b_{k-1}^k)\Big)^\mathsf{T} \\
&= \tfrac{1}{\mu_+^2}\Big(-x_{k-1,k+1}y_{k-1,k+1}c^k + (x_{k-1,k+1}y_{k,k+1} + x_{k,k+1}y_{k-1,k+1})b_{k+1}^k \\
&\qquad + (x_{k-1,k+1}y_{k-1,k} + x_{k-1,k}y_{k-1,k+1})b_{k-1}^k\Big)^\mathsf{T},
\end{aligned}$$

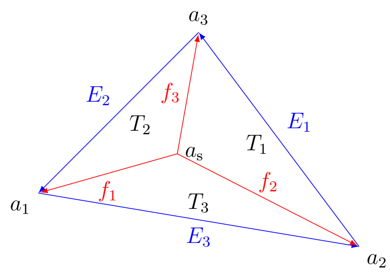

Figure 1: Triangle T with splitting

Their determinants are abbreviated as

$$\mu_k = \det \underline{J}_k = x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}} - x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}}.$$

The final shape functions are constructed to fulfil three propositions.

1. The functions Ψ are cubic polynomials in each subtriangle, are continuous within T, and fulfil
$$\begin{aligned}\Psi_j(a_i) &= [1,0,0]\,\delta_{ij} \\ \nabla\Psi_j(a_i) &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\delta_{ij} \quad \forall\, i,j = 1,2,3\end{aligned}$$
with the Kronecker delta
$$\delta_{ij} = \begin{cases}1 & i = j,\\ 0 & i \neq j.\end{cases}$$
2. The normal derivatives of all functions are linear along outer element edges with respect to the local line coordinate.
3. The functions are C^1-continuous inside T.

The final shape functions are defined with the help of basic functions and some transformations in order to assure the above propositions. We shortly repeat the results here, the whole derivation can be found in [5] together with [2].

The formulas for all shape functions on subtriangle T_k read

$$\begin{aligned}
\Psi_k|_{T_k} &= \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_k, \\
\Psi_{k+1}|_{T_k} &= \hat{\Phi}_1\,\underline{H}_k + \hat{\beta}\,(b_k^{k+1})^\mathsf{T} + \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_{k+1}, \\
\Psi_{k-1}|_{T_k} &= \hat{\Phi}_2\,\underline{H}_k + \hat{\beta}\,(b_k^{k-1})^\mathsf{T} + \hat{\Phi}_0\,\underline{H}_k\,\underline{M}_{k-1}
\end{aligned} \tag{1}$$

with the basic functions

$$
\begin{aligned}
\hat{\Phi}_0(\hat{a}) &= (1-\hat{x}-\hat{y})^2\,[1+2\hat{x}+2\hat{y},\ \hat{x},\ \hat{y}],\\
\hat{\Phi}_1(\hat{a}) &= \hat{x}^2\,[3-2\hat{x},\ \hat{x}-1,\ \hat{y}],\\
\hat{\Phi}_2(\hat{a}) &= \hat{y}^2\,[3-2\hat{y},\ \hat{x},\ \hat{y}-1],\\
\hat{\beta}(\hat{a}) &= \hat{x}\hat{y}(1-\hat{x}-\hat{y})
\end{aligned}
\tag{2}
$$

given on the reference triangle

$$
\hat{T} = \{[\hat{x},\hat{y}]^\mathsf{T} \in \mathbb{R}^2 : \hat{x} \ge 0, \hat{y} \ge 0, \hat{x}+\hat{y} \le 1\}
\tag{3}
$$

and the auxiliary terms

$$
\begin{aligned}
\underline{H}_k &= \begin{bmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & \underline{J}_k^\mathsf{T} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_{k+1,\mathrm{s}} & y_{k+1,\mathrm{s}} \\ 0 & x_{k-1,\mathrm{s}} & y_{k-1,\mathrm{s}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \quad 0 \\ 0 & f_{k+1}^\mathsf{T} \\ 0 & f_{k-1}^\mathsf{T} \end{bmatrix},\\
b_k^{k+1} &= \frac{1}{|E_k|^2}\begin{bmatrix} 6E_k^\mathsf{T} f_{k-1} \\ 3\mu_k N_k + 2|E_k|^2 f_{k-1} \end{bmatrix},\\
b_k^{k-1} &= \frac{1}{|E_k|^2}\begin{bmatrix} -6E_k^\mathsf{T} f_{k+1} \\ 3\mu_k N_k + 2|E_k|^2 f_{k+1} \end{bmatrix},\\
c^k &= \begin{bmatrix} 6 \\ -2f_k \end{bmatrix},\\
\underline{S} &= -2\mu_+ \begin{bmatrix} 3 & f_1^\mathsf{T} \\ 3 & f_2^\mathsf{T} \\ 3 & f_3^\mathsf{T} \end{bmatrix} = -2\mu_+ \begin{bmatrix} 3 & x_{1,\mathrm{s}} & y_{1,\mathrm{s}} \\ 3 & x_{2,\mathrm{s}} & y_{2,\mathrm{s}} \\ 3 & x_{3,\mathrm{s}} & y_{3,\mathrm{s}} \end{bmatrix},\\
\underline{S}^{-1} &= -\frac{1}{6\mu_+^2}\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ 3N_1 & 3N_2 & 3N_3 \end{bmatrix},\\
\underline{T}_k &= e_{k-1}(\mu_k b_{k+1}^k)^\mathsf{T} + e_{k+1}(\mu_k b_{k-1}^k)^\mathsf{T} + e_k(\mu_{k-1} b_{k+1}^k + \mu_{k+1} b_{k-1}^k + \mu_k c^k)^\mathsf{T},\\
\underline{M}_k &= -\underline{S}^{-1}\underline{T}_k.
\end{aligned}
$$

The e_j in the formula for $\underline{T}_k$ denote the j-th unit vectors with $(e_j)_i = \delta_{ij}$.

3 Transformation of second derivatives

The shape functions (1) are formulated with the help of the basic functions (2), which are given on the reference triangle (3). Each of the three subtriangles is mapped to the reference triangle by an affine linear mapping like illustrated in

$$
\begin{aligned}
C_{1:} = \Big(&\underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b_{k-1}^k)^\mathsf{T}\\
&+ \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k-1}\underline{M}_k\Big)_{1:}\\
= {}& \tfrac{1}{\mu_{k-1}^2} y_{k+1,\mathrm{s}}^2 (c^k)^\mathsf{T} - \tfrac{2}{\mu_{k-1}^2} y_{k,\mathrm{s}} y_{k+1,\mathrm{s}} (b_{k-1}^k)^\mathsf{T}\\
&+ \tfrac{1}{6\mu_+^2\mu_{k-1}^2}\Big(-6y_{k+1,k}^2(\mu_k^2 c^k + 2\mu_k\mu_{k-1} b_{k+1}^k + 2\mu_k\mu_{k+1} b_{k-1}^k)\\
&\quad - 12\mu_{k-1} y_{k+1,k}(-\mu_k y_{k-1,k+1} c^k - (\mu_k y_{k+1,k} + \mu_{k-1} y_{k-1,k+1}) b_{k+1}^k\\
&\quad\quad - (\mu_k y_{k,k-1} + \mu_{k+1} y_{k-1,k+1}) b_{k-1}^k)\Big)^\mathsf{T}\\
= {}& \tfrac{1}{\mu_+^2\mu_{k-1}^2}(\mu_+^2 y_{k+1,\mathrm{s}}^2 - \mu_k^2 y_{k+1,k}^2 + 2\mu_{k-1}\mu_k y_{k+1,k} y_{k-1,k+1})(c^k)^\mathsf{T}\\
&+ \tfrac{2}{\mu_+^2} y_{k+1,k} y_{k-1,k+1} (b_{k+1}^k)^\mathsf{T}\\
&+ \tfrac{2}{\mu_+^2\mu_{k-1}^2}(-\mu_+^2 y_{k,\mathrm{s}} y_{k+1,\mathrm{s}} - \mu_k\mu_{k+1} y_{k+1,k}^2 + \mu_{k-1}\mu_k y_{k+1,k} y_{k,k-1}\\
&\quad + \mu_{k-1}\mu_{k+1} y_{k+1,k} y_{k-1,k+1})(b_{k-1}^k)^\mathsf{T}.
\end{aligned}
$$

We recall

$$
\mu_k y_{k,\mathrm{s}} + \mu_{k+1} y_{k+1,\mathrm{s}} + \mu_{k-1} y_{k-1,\mathrm{s}} = 0
$$

from [5] to show the auxiliary formula

$$
\begin{aligned}
\mu_+ y_{k,\mathrm{s}} &= \mu_k y_{k,\mathrm{s}} + \mu_{k+1} y_{k,\mathrm{s}} + \mu_{k-1} y_{k,\mathrm{s}}\\
&= -\mu_{k+1} y_{k+1,\mathrm{s}} - \mu_{k-1} y_{k-1,\mathrm{s}} + \mu_{k+1} y_{k,\mathrm{s}} + \mu_{k-1} y_{k,\mathrm{s}}\\
&= \mu_{k+1} y_{k,k+1} + \mu_{k-1} y_{k,k-1}.
\end{aligned}
\tag{5}
$$

Analogous results hold for $\mu_+ y_{k+1,\mathrm{s}}$ and $\mu_+ y_{k-1,\mathrm{s}}$ and also for all variants with y replaced by x. We consider the coefficients of $(c^k)^\mathsf{T}$ and $2(b_{k+1}^k)^\mathsf{T}$ in $B_{1:}$ with $1/(\mu_+^2\mu_{k+1}^2)$ factored out to shorten the equations. With the help of the above formula, we get

$$
\begin{aligned}
\mu_+^2 y_{k-1,\mathrm{s}}^2 &- \mu_k^2 y_{k,k-1}^2 + 2\mu_k\mu_{k+1} y_{k,k-1} y_{k-1,k+1}\\
&= (\mu_k y_{k-1,k} + \mu_{k+1} y_{k-1,k+1})^2 - \mu_k^2 y_{k,k-1}^2 + 2\mu_k\mu_{k+1} y_{k,k-1} y_{k-1,k+1}\\
&= \mu_k^2 y_{k-1,k}^2 + 2\mu_k\mu_{k+1} y_{k-1,k} y_{k-1,k+1} + \mu_{k+1}^2 y_{k-1,k+1}^2\\
&\quad - \mu_k^2 y_{k,k-1}^2 + 2\mu_k\mu_{k+1} y_{k,k-1} y_{k-1,k+1}\\
&= \mu_{k+1}^2 y_{k-1,k+1}^2,\\
-\mu_+^2 y_{k-1,\mathrm{s}} y_{k,\mathrm{s}} &- \mu_k\mu_{k-1} y_{k,k-1}^2 + \mu_k\mu_{k+1} y_{k,k-1} y_{k+1,k} + \mu_{k+1}\mu_{k-1} y_{k,k-1} y_{k-1,k+1}\\
&= -(\mu_k y_{k-1,k} + \mu_{k+1} y_{k-1,k+1})(\mu_{k-1} y_{k,k-1} + \mu_{k+1} y_{k,k+1})\\
&\quad - \mu_k\mu_{k-1} y_{k,k-1}^2 + \mu_k\mu_{k+1} y_{k,k-1} y_{k+1,k} + \mu_{k+1}\mu_{k-1} y_{k,k-1} y_{k-1,k+1}\\
&= \mu_k\mu_{k-1} y_{k,k-1}^2 - \mu_k\mu_{k+1} y_{k,k-1} y_{k+1,k} - \mu_{k+1}\mu_{k-1} y_{k,k-1} y_{k-1,k+1}\\
&\quad - \mu_{k+1}^2 y_{k-1,k+1} y_{k,k+1}\\
&\quad - \mu_k\mu_{k-1} y_{k,k-1}^2 + \mu_k\mu_{k+1} y_{k,k-1} y_{k+1,k} + \mu_{k+1}\mu_{k-1} y_{k,k-1} y_{k-1,k+1}\\
&= \mu_{k+1}^2 y_{k-1,k+1} y_{k+1,k},
\end{aligned}
$$

Next we use $N_k + N_{k+1} + N_{k-1} = 0$ to reformulate $\underline{M}_k$ as

$$
\begin{aligned}
\underline{M}_k &= -\underline{S}^{-1}\underline{T}_k \\
&= \tfrac{1}{6\mu_+^2}\begin{bmatrix}\mu_1 & \mu_2 & \mu_3 \\ 3N_1 & 3N_2 & 3N_3\end{bmatrix}\Big(e_{k-1}(\mu_k b_{k+1}^k)^\mathsf{T} + e_{k+1}(\mu_k b_{k-1}^k)^\mathsf{T} \\
&\qquad\qquad + e_k(\mu_{k-1}b_{k+1}^k + \mu_{k+1}b_{k-1}^k + \mu_k c^k)^\mathsf{T}\Big) \\
&= \tfrac{1}{6\mu_+^2}\begin{bmatrix} (\mu_k^2 c^k + 2\mu_k\mu_{k-1}b_{k+1}^k + 2\mu_k\mu_{k+1}b_{k-1}^k)^\mathsf{T} \\ 3\Big(N_k(\mu_k c^k + \mu_{k-1}b_{k+1}^k + \mu_{k+1}b_{k-1}^k)^\mathsf{T} + N_{k-1}(\mu_k b_{k+1}^k)^\mathsf{T} + N_{k+1}(\mu_k b_{k-1}^k)^\mathsf{T}\Big)\end{bmatrix} \\
&= \tfrac{1}{6\mu_+^2}\begin{bmatrix} (\mu_k^2 c^k + 2\mu_k\mu_{k-1}b_{k+1}^k + 2\mu_k\mu_{k+1}b_{k-1}^k)^\mathsf{T} \\ 3\Big(\mu_k N_k (c^k)^\mathsf{T} + (\mu_k N_{k-1} + \mu_{k-1}N_k)(b_{k+1}^k)^\mathsf{T} + (\mu_k N_{k+1} + \mu_{k+1}N_k)(b_{k-1}^k)^\mathsf{T}\Big)\end{bmatrix}.
\end{aligned}
$$

With these intermediate results, we are now ready to formulate the rows of $\underline{A}$, $\underline{B}$, and $\underline{C}$ as linear combinations of the row vectors $(c^k)^\mathsf{T}$, $(b_{k+1}^k)^\mathsf{T}$, and $(b_{k-1}^k)^\mathsf{T}$. Denote the i-th row of $\underline{A}$ by $A_{i:}$, for $\underline{B}$ and $\underline{C}$ respectively.

We first consider $A_{1:}$, $B_{1:}$, and $C_{1:}$ and get

$$
\begin{aligned}
A_{1:} &= \Big(\underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_k\,\underline{M}_k\Big)_{1:} \\
&= \tfrac{1}{6\mu_+^2\mu_k^2}\Big(-6y_{k-1,k+1}^2(\mu_k^2 c^k + 2\mu_k\mu_{k-1}b_{k+1}^k + 2\mu_k\mu_{k+1}b_{k-1}^k) \\
&\qquad - 12\mu_k y_{k-1,k+1}(-\mu_k y_{k-1,k+1}c^k - (\mu_k y_{k+1,k} + \mu_{k-1}y_{k-1,k+1})b_{k+1}^k \\
&\qquad\qquad - (\mu_k y_{k,k-1} + \mu_{k+1}y_{k-1,k+1})b_{k-1}^k)\Big)^\mathsf{T} \\
&= \tfrac{1}{\mu_+^2}(y_{k-1,k+1}^2 c^k + 2y_{k-1,k+1}y_{k+1,k}b_{k+1}^k + 2y_{k-1,k+1}y_{k,k-1}b_{k-1}^k)^\mathsf{T}, \\
B_{1:} &= \Big(\underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b_{k+1}^k)^\mathsf{T} \\
&\qquad + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k+1}\underline{M}_k\Big)_{1:} \\
&= \tfrac{1}{\mu_{k+1}^2}y_{k-1,\mathrm{s}}^2(c^k)^\mathsf{T} - \tfrac{2}{\mu_{k+1}^2}y_{k-1,\mathrm{s}}y_{k,\mathrm{s}}(b_{k+1}^k)^\mathsf{T} \\
&\quad + \tfrac{1}{6\mu_+^2\mu_{k+1}^2}\Big(-6y_{k,k-1}^2(\mu_k^2 c^k + 2\mu_k\mu_{k-1}b_{k+1}^k + 2\mu_k\mu_{k+1}b_{k-1}^k) \\
&\qquad - 12\mu_{k+1}y_{k,k-1}(-\mu_k y_{k-1,k+1}c^k - (\mu_k y_{k+1,k} + \mu_{k-1}y_{k-1,k+1})b_{k+1}^k \\
&\qquad\qquad - (\mu_k y_{k,k-1} + \mu_{k+1}y_{k-1,k+1})b_{k-1}^k)\Big)^\mathsf{T} \\
&= \tfrac{1}{\mu_+^2\mu_{k+1}^2}(\mu_+^2 y_{k-1,\mathrm{s}}^2 - \mu_k^2 y_{k,k-1}^2 + 2\mu_k\mu_{k+1}y_{k,k-1}y_{k-1,k+1})(c^k)^\mathsf{T} \\
&\quad + \tfrac{2}{\mu_+^2\mu_{k+1}^2}(-\mu_+^2 y_{k-1,\mathrm{s}}y_{k,\mathrm{s}} - \mu_k\mu_{k-1}y_{k,k-1}^2 + \mu_k\mu_{k+1}y_{k,k-1}y_{k+1,k} \\
&\qquad + \mu_{k+1}\mu_{k-1}y_{k,k-1}y_{k-1,k+1})(b_{k+1}^k)^\mathsf{T} \\
&\quad + \tfrac{2}{\mu_+^2}y_{k,k-1}y_{k-1,k+1}(b_{k-1}^k)^\mathsf{T},
\end{aligned}
$$

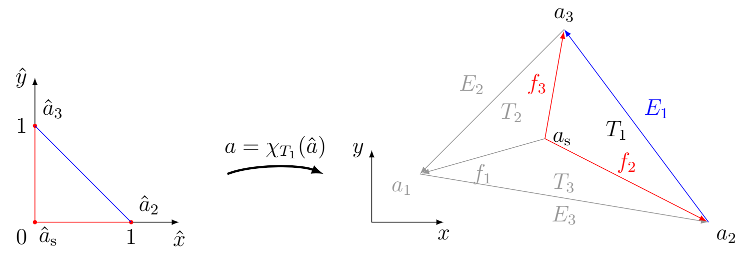

Figure 2: Mapping between the reference triangle and T_1

Figure 2. Inner edges are mapped to the axes of the reference triangle. This can be formulated as

$$
a = \chi_{T_k}(\hat{a}) = \underline{J}_k\hat{a} + a_\mathrm{s}, \quad \hat{a} = \hat{\chi}_{T_k}(a) = \chi_{T_k}^{-1}(a) = \underline{J}_k^{-1}(a - a_\mathrm{s}) \quad \text{for } a \in T_k
$$

with the Jacobian

$$
\underline{J}_k = [f_{k+1} \vdots f_{k-1}] = \begin{bmatrix} x_{k+1,\mathrm{s}} & x_{k-1,\mathrm{s}} \\ y_{k+1,\mathrm{s}} & y_{k-1,\mathrm{s}} \end{bmatrix}
$$

associated with the subtriangle T_k.

The derivatives with respect to the coordinates x and y can be obtained from the derivatives with respect to the master coordinates $\hat{x}$ and $\hat{y}$ via a simple transformation. It can be written for the second derivatives as

$$
(D_2\Psi|_{T_k})(\hat{a}) = \underline{F}_k\,(\hat{D}_2\Psi|_{T_k})(\hat{a}) \tag{4}
$$

with the matrix differential operators

$$
\hat{D}_2 = \left[\frac{\partial^2}{\partial\hat{x}^2}, \frac{\partial^2}{\partial\hat{x}\partial\hat{y}}, \frac{\partial^2}{\partial\hat{y}^2}\right]^\mathsf{T}, \quad D_2 = \left[\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial y^2}\right]^\mathsf{T}
$$

and an appropriate transformation matrix $\underline{F}_k$. The transformation matrix takes the form

$$
\begin{aligned}
\underline{F}_k &= \frac{1}{\mu_k^2}\begin{bmatrix} (\underline{J}_k)_{22}^2 & -2(\underline{J}_k)_{21}(\underline{J}_k)_{22} & (\underline{J}_k)_{21}^2 \\ -(\underline{J}_k)_{12}(\underline{J}_k)_{22} & (\underline{J}_k)_{12}(\underline{J}_k)_{21} + (\underline{J}_k)_{11}(\underline{J}_k)_{22} & -(\underline{J}_k)_{11}(\underline{J}_k)_{21} \\ (\underline{J}_k)_{12}^2 & -2(\underline{J}_k)_{11}(\underline{J}_k)_{12} & (\underline{J}_k)_{11}^2 \end{bmatrix} \\
&= \frac{1}{\mu_k^2}\begin{bmatrix} y_{k-1,\mathrm{s}}^2 & -2y_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}} & y_{k+1,\mathrm{s}}^2 \\ -x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}} & x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}} + x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}} & -x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}} \\ x_{k-1,\mathrm{s}}^2 & -2x_{k-1,\mathrm{s}}x_{k+1,\mathrm{s}} & x_{k+1,\mathrm{s}}^2 \end{bmatrix}
\end{aligned}
$$

as shown in section 8.3 of [3].

4 Second derivatives at the splitting point

The splitting point of the complete triangle has the master coordinates $\hat{a}_\mathrm{s} = [0,0]^\mathsf{T}$ for all three subtriangles. Our hypothesis that the second derivatives of the shape functions do not jump at the splitting point therefore reads

$$(D_2\Psi|_{T_1})(\hat{a}_\mathrm{s}) = (D_2\Psi|_{T_2})(\hat{a}_\mathrm{s}) = (D_2\Psi|_{T_3})(\hat{a}_\mathrm{s}).$$

This is a comparison of 3×9 values evaluated on 3 subelements, which gives 81 values which are to be shown as being 3 same sets of 27 values per set. After splitting the vector Ψ into the vertex related parts Ψ_1, Ψ_2, Ψ_3, one can use (1) (reformulated such that now the index of Ψ_* is constant and the index of T_* varies) and (4) to write

$$\begin{aligned}
\underline{A} &:= (D_2\Psi_k|_{T_k})(\hat{a}_\mathrm{s}) = \underline{F}_k\,(\hat{D}_2\Psi_k|_{T_k})(\hat{a}_\mathrm{s}) = \underline{F}_k\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_k\,\underline{M}_k,\\
\underline{B} &:= (D_2\Psi_k|_{T_{k+1}})(\hat{a}_\mathrm{s}) = \underline{F}_{k+1}\,(\hat{D}_2\Psi_k|_{T_{k+1}})(\hat{a}_\mathrm{s})\\
&= \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s})\underline{H}_{k+1} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k+1})^\mathsf{T} + \underline{F}_{k+1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k+1}\underline{M}_k,\\
\underline{C} &:= (D_2\Psi_k|_{T_{k-1}})(\hat{a}_\mathrm{s}) = \underline{F}_{k-1}\,(\hat{D}_2\Psi_k|_{T_{k-1}})(\hat{a}_\mathrm{s})\\
&= \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s})\underline{H}_{k-1} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k-1})^\mathsf{T} + \underline{F}_{k-1}\,(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_{k-1}\underline{M}_k
\end{aligned}$$

for any fixed k from 1 to 3. It remains to show $\underline{A} = \underline{B} = \underline{C}$; this is done in the following by evaluating all necessary terms.

The second master derivatives of all basic functions at the splitting point are

$$(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s}) = \begin{bmatrix} -6 & -4 & 0\\ -6 & -2 & -2\\ -6 & 0 & -4 \end{bmatrix}, \qquad (\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s}) = \begin{bmatrix} 6 & -2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix},$$

$$(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s}) = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 6 & 0 & -2 \end{bmatrix}, \qquad (\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s}) = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}.$$

This yields

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s}) = \frac{1}{\mu_k^2}\begin{bmatrix} 6y^2_{k-1,\mathrm{s}} & -2y^2_{k-1,\mathrm{s}} & 0\\ -6x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}} & 2x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}} & 0\\ 6x^2_{k-1,\mathrm{s}} & -2x^2_{k-1,\mathrm{s}} & 0 \end{bmatrix},$$

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s}) = \frac{1}{\mu_k^2}\begin{bmatrix} 6y^2_{k+1,\mathrm{s}} & 0 & -2y^2_{k+1,\mathrm{s}}\\ -6x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}} & 0 & 2x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}}\\ 6x^2_{k+1,\mathrm{s}} & 0 & -2x^2_{k+1,\mathrm{s}} \end{bmatrix},$$

$$\underline{F}_k(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s}) = \frac{1}{\mu_k^2}\begin{bmatrix} -2y_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}}\\ x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}} + x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}}\\ -2x_{k-1,\mathrm{s}}x_{k+1,\mathrm{s}} \end{bmatrix},$$

and

$$\begin{aligned}
&\underline{F}_k(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\\
&= \tfrac{1}{\mu_k^2}\left[\begin{matrix} -6(y_{k-1,\mathrm{s}}-y_{k+1,\mathrm{s}})^2 & -4y^2_{k-1,\mathrm{s}}+4y_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}} & \cdots\\ 6(x_{k-1,\mathrm{s}}-x_{k+1,\mathrm{s}})(y_{k-1,\mathrm{s}}-y_{k+1,\mathrm{s}}) & 4x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}}-2(x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}}+x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}}) & \cdots\\ -6(x_{k-1,\mathrm{s}}-x_{k+1,\mathrm{s}})^2 & -4x^2_{k-1,\mathrm{s}}+4x_{k+1,\mathrm{s}}x_{k-1,\mathrm{s}} & \cdots \end{matrix}\right.\\
&\qquad\qquad \left.\begin{matrix} 4y_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}}-4y^2_{k+1,\mathrm{s}}\\ 4x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}}-2(x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}}+x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}})\\ 4x_{k+1,\mathrm{s}}x_{k-1,\mathrm{s}}-4x^2_{k+1,\mathrm{s}} \end{matrix}\right]\\
&= \tfrac{1}{\mu_k^2}\begin{bmatrix} -6y^2_{k-1,k+1} & -4y_{k-1,\mathrm{s}}y_{k-1,k+1} & 4y_{k+1,\mathrm{s}}y_{k-1,k+1}\\ 6x_{k-1,k+1}y_{k-1,k+1} & 2x_{k-1,\mathrm{s}}y_{k-1,k+1}+2x_{k-1,k+1}y_{k-1,\mathrm{s}} & -2x_{k+1,\mathrm{s}}y_{k-1,k+1}-2x_{k-1,k+1}y_{k+1,\mathrm{s}}\\ -6x^2_{k-1,k+1} & -4x_{k-1,\mathrm{s}}x_{k-1,k+1} & 4x_{k+1,\mathrm{s}}x_{k-1,k+1} \end{bmatrix}.
\end{aligned}$$

For the next steps we recall

$$\mu_k = x_{k+1,\mathrm{s}}y_{k-1,\mathrm{s}} - x_{k-1,\mathrm{s}}y_{k+1,\mathrm{s}} \quad \text{and} \quad \underline{H}_k = \begin{bmatrix} 1 & 0 & 0\\ 0 & x_{k+1,\mathrm{s}} & y_{k+1,\mathrm{s}}\\ 0 & x_{k-1,\mathrm{s}} & y_{k-1,\mathrm{s}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0\\ 0 & f^\mathsf{T}_{k+1} & \\ 0 & f^\mathsf{T}_{k-1} & \end{bmatrix}.$$

This yields

$$\underline{F}_k(\hat{D}_2\hat{\Phi}_0)(\hat{a}_\mathrm{s})\underline{H}_k = \tfrac{1}{\mu_k^2}\begin{bmatrix} -6y^2_{k-1,k+1} & -4\mu_k y_{k-1,k+1} & 0\\ 6x_{k-1,k+1}y_{k-1,k+1} & 2\mu_k x_{k-1,k+1} & -2\mu_k y_{k-1,k+1}\\ -6x^2_{k-1,k+1} & 0 & 4\mu_k x_{k-1,k+1} \end{bmatrix},$$

$$\begin{aligned}
\underline{F}_{k+1}(\hat{D}_2\hat{\Phi}_2)(\hat{a}_\mathrm{s})\underline{H}_{k+1} &= \tfrac{1}{\mu_{k+1}^2}\begin{bmatrix} 6y^2_{k-1,\mathrm{s}} & -2y^2_{k-1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}}\\ -6x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}} & 2x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}}\\ 6x^2_{k-1,\mathrm{s}} & -2x^2_{k-1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}} \end{bmatrix}\\
&= \tfrac{1}{\mu_{k+1}^2}\begin{bmatrix} y^2_{k-1,\mathrm{s}}\\ -x_{k-1,\mathrm{s}}y_{k-1,\mathrm{s}}\\ x^2_{k-1,\mathrm{s}} \end{bmatrix}(c^k)^\mathsf{T},\\
\underline{F}_{k-1}(\hat{D}_2\hat{\Phi}_1)(\hat{a}_\mathrm{s})\underline{H}_{k-1} &= \tfrac{1}{\mu_{k-1}^2}\begin{bmatrix} 6y^2_{k+1,\mathrm{s}} & -2y^2_{k+1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}}\\ -6x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}} & 2x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}}\\ 6x^2_{k+1,\mathrm{s}} & -2x^2_{k+1,\mathrm{s}}f^\mathsf{T}_{k,\mathrm{s}} \end{bmatrix}\\
&= \tfrac{1}{\mu_{k-1}^2}\begin{bmatrix} y^2_{k+1,\mathrm{s}}\\ -x_{k+1,\mathrm{s}}y_{k+1,\mathrm{s}}\\ x^2_{k+1,\mathrm{s}} \end{bmatrix}(c^k)^\mathsf{T},\\
\underline{F}_{k+1}(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k+1})^\mathsf{T} &= \tfrac{1}{\mu_{k+1}^2}\begin{bmatrix} -2y_{k,\mathrm{s}}y_{k-1,\mathrm{s}}\\ x_{k-1,\mathrm{s}}y_{k,\mathrm{s}} + x_{k,\mathrm{s}}y_{k-1,\mathrm{s}}\\ -2x_{k,\mathrm{s}}x_{k-1,\mathrm{s}} \end{bmatrix}(b^k_{k+1})^\mathsf{T},\\
\underline{F}_{k-1}(\hat{D}_2\hat{\beta})(\hat{a}_\mathrm{s})(b^k_{k-1})^\mathsf{T} &= \tfrac{1}{\mu_{k-1}^2}\begin{bmatrix} -2y_{k+1,\mathrm{s}}y_{k,\mathrm{s}}\\ x_{k,\mathrm{s}}y_{k+1,\mathrm{s}} + x_{k+1,\mathrm{s}}y_{k,\mathrm{s}}\\ -2x_{k+1,\mathrm{s}}x_{k,\mathrm{s}} \end{bmatrix}(b^k_{k-1})^\mathsf{T}.
\end{aligned}$$